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A Note on the Qualitative Behaviour of some Second Order Nonlinear Differential Equations

Una Nota sobre el Comportamiento Cualitativo de algunas Ecuaciones Diferenciales No Lineales de Segundo Orden

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Abstract

In this paper we present two qualitative results concerning the solutions of the equation

$$(p(t)x')' + f(t,x)x' + g(t,x) = e(t).$$

The first result covers the boundedness of solutions while the second one discusses when all the solutions are in L^2.
Key words and phrases: Bounded, L^2-solutions, square-integrable, asymptotic behaviour.

Resumen

En este trabajo se presentan dos resultados cualitativos concernientes a las soluciones de la ecuación

$$(p(t)x')' + f(t,x)x' + g(t,x) = e(t).$$

El primer resultado cubrela acotación de las soluciones mientras que el segundo discute cuándo todas las soluciones están en L^2.
Palabras y frases clave: Acotado, L^2-solución, cuadrado integrable, comportamiento asintótico.

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1 Introduction

In this note we consider the equation

$$(p(t)x')' + f(t,x)x' + g(t,x) = e(t), \quad (1)$$

under the following conditions:

i) p is a continuous function on $I := [0,+\infty)$ such that $0 < p \le p(t) < +\infty$ and e is also a continuous and square-integrable function on I.

ii) f is a continuous functions on **IxR** satisfying $0 < f_0 \le f(t,x)$, and g is function of the class $C^{(0,1)}$ on **IxR** such that $\int_0^{\pm\infty} g(t,x)dx = \pm\infty$ uniformly in t and $x\frac{\partial g(t,x)}{\partial t} \le 0$.

We shall determine sufficient conditions for boundedness and L^2 properties of solutions of equation (1). Our approach differs from those of the earlier research as all they constructed energy or Liapunov Functions; so, our results differ significantly from those obtained previously, see some attempts in that sense in [11] and references cited therein.

The solutions of equation (1) are bounded if there exists a constant $K > 0$ such that $|x(t)| < K$ for all $t \ge T > 0$ for some T. By an L^2-solution, we mean a solution of equation (1) such that $\int_0^\infty x^2(t)dt < \infty$.

In the last forty decades, many authors have investigated the Liénard equation

$$x'' + f(x)x' + g(x) = 0. \quad (2)$$

They have examined some qualitative properties of the solutions. The book of Sansone and Conti [20] contains an almost complete list of papers dealing with these equation as well as a summary of the results published up to 1960. The book of Reissig, Sansone and Conti [16] updates this list and summary up to 1962. The list of the papers which appeared between 1960 and 1970 is presented in the paper of John R. Graef [6]. Among the papers which were published in the last years we refer to the following ones [2], [4], [7], [10], [13], [15], and [21-22].

If in (1) we make $p(t) \equiv 1$, $e(t) \equiv 0$, $f(t,x) = f(x)$ and $g(t,x) = g(x)$, it is clear that equation (1) becomes equation (2) so, every qualitative result for the equation (1) produces a qualitative result for (2).

We now state and prove a general boundedness theorem. Without loss of generality, we shall assume $t \ge 0$.

Theorem 1. *We assume that conditions i), ii) above holds. Then any solution x(t) of (1), as well as its derivative, is bounded as* $t \to \infty$ *and* $\int_0^\infty x'^2(t)dt$.

Proof. By standard existence theory, there is a solution of (1) which exists on $[0, T)$ for some $T > 0$. Multiply the equation (1) by x' and perform on integration by parts from 0 to t on the last term of the left hand side of (1) we obtain

$$p\frac{[x'(t)]^2}{2} + \int_0^t f(s, x(s))\,[x'(s)]^2\,ds + \int_{x(0)}^{x(s)} g(t, u)\,du - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial g(s, u)}{\partial s}\,du\,ds \le p\frac{[x'(0)]^2}{2} + \left(\int_0^t |e(s)x'(s)|\,ds\right). \quad (3)$$

Now if $x(t)$ becomes unbounded then we must have that all terms on the left hand side of (1) become positive from our hypotheses. By the Cauchy-Schwarz inequality for integrals on the right hand side of (3), we get

$$p\frac{[x'(t)]^2}{2} + \int_0^t f(s, x(s))\,[x'(s)]^2\,ds + \int_{x(0)}^{x(s)} g(t, u)\,du - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial g(s, u)}{\partial s}\,du\,ds$$

$$\le p\frac{[x'(0)]^2}{2} + \left(\int_0^t e^2(s)ds\right)^{\frac{1}{2}} \left(\int_0^t x'^2(s)ds\right)^{\frac{1}{2}}.$$

Now, let $H(t) = \left(\int_0^t x'^2(s)ds\right)^{\frac{1}{2}}$. Dividing both sides by $H(t)$ yields

$$H^{-1}(t)\Bigg(p\frac{[x'(t)]^2}{2} + \int_0^t f(s, x(s))\,[x'(s)]^2\,ds + \int_{x(0)}^{x(s)} g(t, u)\,du - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial g(s, u)}{\partial s}\,du\,ds\Bigg) \le H^{-1}(t)p\frac{[x'(0)]^2}{2} + \left(\int_0^t e^2(s)\,ds\right)^{\frac{1}{2}}. \quad (4)$$

Taking into account the positivity of left hand side of (4) if $x(t)$ increase without bound and that term $H^{-1}(t)f_0\int_0^t x'^2(s)\,ds = f_0\left(\int_0^t x'^2(s)\,ds\right)^{\frac{1}{2}}$ is bounded by the right hand side of equation (4) we obtain that x' is square integrable and is also abounded after we examine the first term of the left hand side of (4). However, the above implies that $|x(t)|$ must be bounded. Otherwise, the left hand side of (4) becomes infinite which is impossible. A standard argument now permits the solution to be extended on all t of I, see for example [1], [16] and [20]. The proof is thus completed. □

Remark 1. In [9] the author consider an oscillator described by the following equation

$$x'' + f(t)x' + g(t)x = 0, \tag{5}$$

where the damping and rigidity coefficients $f(t)$ and $g(t)$ are continuous and bounded functions. If in equation (1) we put $p(t) {\equiv} 1$, $e(t) {\equiv} 0$, $f(t,x) = f(t)$ and $g(t,x) = g(t)x$, then we improve the Theorem 1 of Ignatiev, since the assumption

$$\frac{1}{2}\frac{g'(t)}{g(t)} + f(t) > \alpha_2 > 0,$$

is not necessary, and

$$|f(t)| < M_1, |g(t)| < M_2, |g'(t)| < M_3,$$

is droped.

Under the above remarks, the Ignatiev's Corollary 1 is obvious.

Remark 2. If in (1) the functions involved are constants, $p(t) {\equiv} 1$, $e(t) \equiv 0$, $f(t,x) \equiv f_0$ and $g(t,x) = g_0 x$, from assumptions ii) and iii) of Theorem 1 we obtain

ii') $f_0 > 0$,
iii') $g_0 > 0$.

Then, that assumptions amount to the usual Routh-Hurwitz criterion (see [1]).

Remark 3. In [12] the author proved for the generalized Liénard equation (2) with restoring term h(t), the following result:

[12, Theorem 1] We assume that $g \in C(\mathbf{R})$, with limit at infinity and

$$g(-\infty) < g(x) < g(+\infty), \forall x \in \mathbf{R}.$$

In addition, either

$p \in V$, $g(-\infty) < p(t) < g(+\infty)$, or $p \in L^\infty(I)$, $g(-\infty) = -\infty, g(+\infty) = g(+\infty)$, where $V = \left\{ h \in L^\infty(I) : \ h_m = \underset{T\to\infty}{Lim} \int_\alpha^{\alpha+T} h(t)dt \text{ uniformly in } \alpha \right\}$, denoting with h_m the medium value of h. Then (2) has a solution in $W^{2,\infty}(\mathbf{R})$. Also, $\forall \gamma > 0$, $\exists \Gamma > 0$ such that for any solution $x(t)$ of (2) with $|x(t_0)| + |x'(t_0)| \leq \gamma$, for some $t \in I$, then $|x(t)|+|x\prime(t)| \leq \Gamma$, $t \geq t_0$.

This result is easily obtained from our Theorem 1.

Remark 4. Repilado and Ruiz [17-18] studied, the asymptotic behaviour of the solutions of the equation

$$x'' + f(x)x' + a(t)g(x) = 0, \tag{6}$$

under the following conditions:
a) f is a continuous and nonnegative function for all $x \in \mathbf{R}$,
b) g is also a continuous function with $xg(x) > 0$ for $x \neq 0$,
c) $a(t) > 0$ for all $t \in I$ and $a \in C^1$.
In particular, the following result is proved
[18, Theorem 2]. Under conditions
1.$\int_0^{+\infty} a(t)dt = +\infty$.
2. $\int_0^{+\infty} \frac{a'(t)_-}{a(t)} dt = +\infty$, $a'(t)_- = max\{-a'(t), 0\}$.
3. There exists a positive constant N such that $|G(x)| \leq N$ for $x \in (-\infty, \infty)$, where $G(x) = \int_0^x g(s)ds$, all solutions of equation (6) are bounded if and only if

$$\int_0^{+\infty} a(t) f\left[\pm k(t - t_0)\right] dt = \pm\infty, \tag{7}$$

for all $k \geq 0$ and some $t_0 \geq 0$.

The first result of this nature was obtained by Burton and Grimmer [3] when they showed that all continuable solutions of equations $x'' + a(t)f(x) = 0$ under condition b) and c) are oscillatory (and bounded) if and only if the condition (7) is fulfilled.

It is easy to obtain the sufficiency of the above result from our Theorem 1.

Remark 5. Taking into account the above remark and Theorem 1 of [8], raises the following open problem

Under which additional hypotheses, the assumption is a necessary and sufficient condition for boundedness of the solutions of equation (1)?.

This is not a trivial problem. The resolution implies obtaining a necessary and sufficient condition for completing the study of asymptotic nature of solutions of (1).

Remark 6. If in (1) we take $f(t, x) \equiv 0$, $e(t) \equiv 0$ and $g(t, x) = g(t)x$, our result becomes Theorem 1 of [14], referent to boundedness of $x(t)$ and $p(t)x'(t)$ for all $t \geq a$ with a some positive constant.

Remark 7. A. Castro and R. Alonso [5] considered the special case

$$x'' + h(t)x' + x = 0, \tag{8}$$

of equation (1) under condition $h \in C^1(I)$ and $h(t) \geq b > 0$. Further, they required that the condition $ah'(t) + 2h(t) \leq 4a$ be fulfilled, and obtained various results on the stability of the trivial solution of (8). It is clear that all assumptions of Theorem 1 are satisfied. Thus, we obtain a consistent result under milder conditions.

By imposing more stringent conditions on g(t,x) and p(t), all solutions become L^2-solutions. This case is covered by the following result.

Theorem 2. *Under hypotheses of Theorem 1, we suppose that $g(t,x)x > g_0 x^2$ for some positive constant g_0, and $0 < p < p(t) < P < +\infty$, then all the solutions of equation (1) are L^2-solutions.*

Proof. In order to see that $x \in L^2[0,\infty)$, we must first multiply equation (1) by x, the integration from 0 to t yields

$$x(p(t)x')) - \int_0^t p(s)x'^2(s)ds + \int_0^t f(t,x)x(s)x'(s)ds + \int_0^t x(s)g(s,x(s))ds =$$

$$= x(0)p(0)x'(0) + \int_0^t e(s)x(s)ds.$$

Next, let $\int_{x(0)}^{x(t)} zf(x^{-1}(z), z)dz = F(x)$. So, the above equation may be rewritten as

$$px(t)x'(t) - P\int_0^t x'^2(s)ds + F(x) + g_0\int_0^t x^2(s)ds \leq K, \tag{9}$$

where $K = P\,|x(0)x'(0)| + \left|\int_0^t e(s)x(s)ds\right|$. Notice that the term is bounded by $\left(\int_0^t e^2(s)ds\right)^{\frac{1}{2}} \left(\int_0^t x'^2(s)ds\right)^{\frac{1}{2}}$ by using the Cauchy-Scwharz inequality. Dividing the left hand side of (9) by $M(t)$ and using the hypotheses of Theorem 2 we obtain

$$M^{-1}(t)\left[px(t)x'(t) - P\int_0^t x'^2(s)ds + F(x)\right] + g_0\left(\int_0^t x^2(s)ds\right)^{\frac{1}{2}} \leq \frac{K}{M(t)}. \tag{10}$$

Since the right hand side of (9) is bounded and all the terms of the left hand side are either bounded or positive, the result follows because the left hand side cannot be unbounded. Here, we need that x is square integrable. □

Remark 8. This result complete those of Ignatiev referent to equation (5), see [9], with restoring term

$$x'' + f(t)x' + g(t)x = h(t), \tag{11}$$

Taking $h(t)$ continuous on I (in Ignatiev's results $h \equiv 0$) such that and $f(t) > f_0 > 0$, $g(t) > g_0 > 0$ with continuous nonpositive derivatives we have that all the solutions of (11), as well as their derivatives, are bounded and in $L^2(I)$.

Remark 9. Our results contains and improve those of [19] (obtained with $h{\equiv}0$) referent to the boundedness of the solutions of equation

$$x'' + f(t)x' + a(t)g(x) = h(t),$$

because the author used regularity assumptions on function $a(t)$.

Remark 10. Under assumptions $f(t,x){\geq}f_0 > 0$ for some positive constant f_0, the class of equation (1) is not very large, but if this condition is not fulfilled, we can exhibit equations that have unbounded solutions. For example

$$\left(e^{-\left(\frac{t^3}{3}+3t\right)}x'\right)' + 2(t^2+1)e^{-\left(\frac{t^3}{3}+3t\right)} = 0,$$

has the unbounded solution $x(t) = e^{2t}$.

Remark 11. In [11] the author discussed the boundedness and L^2 character of equation (1) with $f(t,x) = c(t)f(x)$ and $p(t) \equiv 1$. Thus, our results contains those of Kroopnick.

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