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Fermat Numbers in the Pascal Triangle

Números de Fermat en el Triángulo de Pascal

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Abstract

For any positive integer m let $F_m = 2^{2^m} + 1$ be the mth Fermat number. In this short note we show that the only solutions of the diophantine equation $F_m = \binom{n}{k}$ are the trivial ones, i.e., those with $k = 1$ or $n - 1$.
Key words and phrases: Fermat numbers, Pascal triangle, diophantine equations.

Resumen

Para cualquier entero positivo m sea $F_m = 2^{2^m}+1$ el m-simo número de Fermat. En esta breve nota demostramos que las únicas soluciones de la ecuación diofántica $F_m = \binom{n}{k}$ son las triviales, es decir aquellas para las cuales $k = 1$ o $n - 1$.
Palabras y frases clave: Números de Fermat, triángulo de Pascal, ecuaciones diofánticas.

1 Introduction

For any integer $m \geq 0$ let $F_m = 2^{2^m} + 1$ be the mth Fermat number. A triangular number is a positive integer of the form $\binom{n}{2}$ for some positive integer n. In [2], Krishna showed that the only triangular Fermat number is $F_0 = 3 = \binom{3}{2}$. This result appears also in Radovici-Mărculescu [4]. Both proofs are immediate and are based on modular arguments. In this note, we extend the above result. Our main theorem is the following.

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Theorem 1. *If*

$$F_m = \binom{n}{k}, \quad \textit{for some } n \geq 2k \geq 2, \tag{1}$$

then $k = 1$.

Notice that the condition $n \geq 2k$ is not really restrictive because of the symmetry of the binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}. \tag{2}$$

The above result can be interpreted by saying that the Fermat numbers sit in the Pascal triangle only in the trivial way.

A connection of the Fermat numbers with the Pascal triangle was pointed out in the paper of Hewgill [1]. We mention that several other diophantine equations involving the Pascal triangle and Fermat numbers were previously investigated. For example, in [3] we determined all binomial coefficients which can be numbers of sides of regular polygons which can be constructed with the ruler and the compass.

2 Proof of Theorem 1

Assume that equation (1) has a solution with $k > 1$. Notice that in this case $m > 4$ because F_m is prime for $m = 0, 1, \ldots, 4$. We first show that $k < 2^m$. Indeed, assume that $k \geq 2^m$. Since $m \geq 5$, it follows that $k \geq 2^5 = 32$. One can easily check that

$$k! < \left(\frac{k}{2.2}\right)^k \qquad \text{for all } k \geq 10. \tag{3}$$

Indeed, inequality (3) follows from Stirling's formula. Equation (1) and inequality (3) now imply that

$$2^{2^m} + 1 = F_m = \binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} > \frac{(n-k)^k}{k!}$$

$$> \left(\frac{2.2(n-k)}{k}\right)^k \geq (2.2)^k \geq (2.2)^{2^m},$$

or

$$1 + \frac{1}{2^{2^m}} > \left(\frac{2.2}{2}\right)^{2^m} = \left(1 + \frac{1}{10}\right)^{2^m} > 1 + \frac{2^m}{10},$$

or

$$10 > 2^{m+2^m},$$

which is certainly impossible for $m \geq 5$. Thus, $k < 2^m$.

At this point, we recall the following well-known result due to Lucas. Assume that p is a prime and write

$$n = n_0 + n_1 p + \cdots + n_t p^t \quad \text{for some } n_i \in \{0, 1, \ldots, p-1\}, \text{ with } n_t \neq 0, \tag{4}$$

and

$$k = k_0 + k_1 p + \cdots + k_t p^t \quad \text{for some } k_i \in \{0, 1, \ldots, p-1\}. \tag{5}$$

Then,

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_t}{k_t} \pmod{p}. \tag{6}$$

Assume that

$$n = \prod_{p|n} p^{\alpha_p}, \tag{7}$$

and let

$$A = \{p : p|n \text{ and } p \equiv 1 \pmod{2^{m+1}}\}. \tag{8}$$

Finally, let $n = n_1 d$, where

$$n_1 = \prod_{p \in A} p^{\alpha_p}. \tag{9}$$

We now show that $d|k$. This is clear if $d = 1$. Assume that $d > 1$ and choose a prime number $q|d$. Since all the prime divisors of F_m are congruent to 1 modulo 2^{m+1}, it follows that $q \not| F_m$. But since $q|d|n$, it follows that if one writes n in base q according to formula (4), then one gets that $n_0 = 0$. If $q \not| k$, then $k_0 > 0$ and now formula (6) would imply that

$$F_m \equiv \binom{n}{k} \equiv \binom{0}{k_0} \cdot \binom{n_1}{k_1} \cdot \cdots \cdot \binom{n_t}{k_t} \equiv 0 \pmod{q},$$

which is impossible because q does not divide F_m. Hence, every prime divisor of d divides k as well. To show that $d|k$, we need to show that if $q^\alpha \mid\mid d$ for some $\alpha \geq 1$, then $q^\alpha | k$. Assuming that this were not so, it follows that $q^\beta \mid\mid k$ for some $\beta < \alpha$. But in this case, $n_\beta = 0$ and $k_\beta \neq 0$ which, via formula (6), would imply again that q divides F_m, which is impossible.

Hence, $d|k$. In particular, since $k < 2^m$, it follows that $d < 2^m$ as well. We now notice that since n_1 is a product of primes from A, it follows that

$n_1 \equiv 1 \pmod{2^{m+1}}$. This implies that $n \equiv d \pmod{2^{m+1}}$. However, since $d \le k < 2^m < 2^{m+1}$, Lucas's theorem for the prime $p = 2$ implies that

$$F_m = \binom{n}{k} \equiv \binom{d}{k} \pmod 2. \tag{10}$$

Since F_m is odd and $d \le k$, formula (10) implies that $d = k$. Thus, $k|n$. We may now write equation (1) as

$$F_m = \frac{n}{k} \cdot \binom{n-1}{k-1}, \tag{11}$$

where $\frac{n}{k}$ is an integer. At this point, one should notice that the relevant feature of the preceeding argument was based only on the shape of the prime divisors of F_m. Hence, one can iterate the above argument to get that $(k-i)|(n-i)$ for all $i = 0, 1, \ldots, k-1$. This is equivalent to

$$n \equiv i \pmod{k-i} \equiv k \pmod{k-i}, \quad \text{for all } i = 0, 1, \ldots, k-1. \tag{12}$$

Let

$$N = \text{lcm}(1,\ 2,\ \ldots,\ k) = [1,\ 2,\ \ldots,\ k]. \tag{13}$$

From formula (12), we get that $n \equiv k \pmod N$. Write $n = k + aN$ for some positive integer a. Now equation (1) implies that

$$F_m = \binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \cdots \cdot \frac{n-k+1}{1} = \prod_{i=0}^{k-1} \Big(1 + a\frac{N}{k-i}\Big). \tag{14}$$

Let $N_i = \frac{N}{k-i}$ for $i = 1, 2, \ldots, k-1$. Notice that exactly one of the numbers N_i is odd and all the other ones are even. Indeed, the only odd number N_i corresponds to $i = k - 2^\mu$, where 2^μ is the largest power of 2 less than or equal to k. We now look again at equation (14) and we write it as

$$2^{2^m} + 1 = \prod_{i=0}^{k-1} (1 + aN_i) = 1 + aS_1 + a^2S_2 + \ldots + a^kS_k, \tag{15}$$

where S_j is the jth fundamental symmetric polynomial in the N_i's. Since eactly one of the numbers N_i is odd, it follows that S_1 is odd and that S_j is even for all $j \ge 1$. Equation number (15) can now be written as

$$2^{2^m} = a(S_1 + aS_2 + \ldots + a^{k-1}S_k). \tag{16}$$

From formula (16), one can see right away that the factor $S_1+aS_2+...+a^{k-1}S^k$ is odd and larger than 1 (here is where $k > 1$ is really used), so it cannot divide the power of 2 from the left hand side of equation (16).
The Theorem is therefore proved.
Remark. One can mimic the above arguments to show that is $a > 1$ is any positive integer and $F_m(a) = a^{2^m} + 1$ is the mth generalized Fermat number, then the equation

$$F_m(a) = \binom{n}{k}, \qquad \text{for some } n \geq 2k \text{ and } k \geq 2, \tag{17}$$

has only finitely many computable solutions. That is, there exists a constant $C(a)$ depending only on a such that all solutions of equation (17) satisfies $m < C(a)$. The fact that there are sometimes non-trivial solutions of (17) is illustrated by the example

$$F_1(3) = \binom{5}{2}.$$

A generalization of the result from the present paper to diophantine equations involving equal values of binomial coefficients and members of Lucas sequences will be given in a forthcoming paper.

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