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Closed Congruences on Semigroups

Congruencias Cerradas en Semigrupos

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Abstract

A compatible equivalence relation on a semigroup S is called a *congruence*. Wallace proved that if R is a closed congruence defined on a compact topological semigroup S then S/R with the quotient topology and the standard multiplication is a topological semigroup. In this work Wallace's result is proved for a more general class of topological semigroups; namely, σ-compact and locally compact semigroups.
Key words and phrases: Congruence, topological semigroup, quotient map, saturated map, σ-compact, locally compact.

Resumen

Una una relación de equivalencia compatible en un semigrupo es llamada *congruencia*. Wallace probó que si R es una congruencia cerrada definida en un semigrupo topológico compacto S entonces S/R con la topología cociente y la multiplicación estándar es un semigrupo topológico. En este trabajo se prueba el resultado de Wallace para una clase más general de semigrupos topológicos, a saber los σ-compactos y localmente compactos.
Palabras y frases clave: Congruencia, semigrupo topológico, aplicación cociente, aplicación saturada, σ-compacto, localmente compacto.

1 Introduction

A relation R on a semigroup S is said to be *left* [*Right*] *compatible* if $(a, b) \in R$ and $x \in S$ imply $(xa, xb) \in R$ [$(ax, bx) \in R$], and *compatible* if it is both left and right compatible.

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A compatible equivalence relation on a semigroup S is called a *congruence*.

Observe that an equivalence R on a semigroup S is a congruence if and only if $(a,b) \in R$ and $(c,d) \in R$ imply $(ac,bd) \in R$.

Theorem 1. *Let S be a semigroup and let R be a congruence on S. Then S/R is a semigroup under the multiplication defined by $\pi(x)\pi(y) = \pi(xy)$ and $\pi\colon S \to S/R$ is a surmorphism.*

It is useful to observe that if R is a congruence on a semigroup S and m, m' are the multiplication on S and S/R respectively then m' is the unique multiplication on S/R such that the following diagram conmutes:

$$\begin{array}{ccc} (S/R) \times (S/R) & \xrightarrow{m'} & S/R \\ {\scriptstyle \pi\times\pi}\uparrow & & \uparrow{\scriptstyle \pi} \\ S \times S & \xrightarrow{m} & S \end{array}$$

If R is a congruence on S, which is closed on $S \times S$ we will call it a *closed congruence*.

If $f\colon X \to Y$ is a function from a space X to a space Y and $A \subset X$, then A is said to be f-*saturated* if $f^{-1}(f(A)) = A$.

If $f\colon X \to Y$ is surjective, then f is said to be a *quotient map* if W being open [closed] in Y is equivalent to $f^{-1}(W)$ being open [closed] in X. Observe that a quotient map is necessarly a continuous surjection. It is clear that each open continuous surjection from X onto Y is a quotient map.

If X is a space, Y is a set, and $f\colon X \to Y$ is a surjection, then we can define a topology on Y by declaring a subset $W \subset Y$ to be open if and only if $f^{-1}(W)$ is open in X. This topology is called the *quotient topology* on Y induced by f. Note that if Y is given the quotient topology induced by f then f is a quotient map.

If R is a congruence on a semigroup S, we give S/R the quotient topology induced by the natural map $\pi\colon S \to S/R$.

We want now to present an example of a topological semigroup S and a closed congruence R on S such that S/R is not a topological semigroup, since S/R fails to be Hausdorff.

Let S be a Hausdorff space which is not normal and define multiplication on S by $(x,y) \to x$. Then S is a topological semigroup. Let A, B be two disjoint closed subsets os S such that each open set containing A meets each open set containing B, and let $R = \Delta(S) \cup A \times A \cup B \times B$. Then R is a closed congruence on S but S/R is not a topological semigroup, since it fails to be Hausdorff. Observe that multiplication on S/R is continuous.

2 Results

We proceed to establish a sequence of lemmas which will be used to demonstrate that if S is a locally compact, σ-compact semigroup and R is a closed congruence on S, then S/R is a topological semigroup. This result was established for compact semigroups in [1].

Lema 1. *Let S be a topological semigroup and let R be a closed congruence on S such that $\pi \times \pi$: $S \times S \to S/R \times S/R$ ia a quotient map. Then S/R is a topological semigroup.*

Proof. Let m denote multiplication on S and m' multiplication on S/R. Then the diagram,

$$\begin{array}{ccc} (S/R) \times (S/R) & \xrightarrow{m'} & S/R \\ {\scriptstyle \pi\times\pi}\uparrow & & \uparrow{\scriptstyle \pi} \\ S \times S & \xrightarrow[m]{} & S \end{array}$$

commutes. Since $\pi \circ m$ is continuous and $\pi \times \pi$ is a quotient map we get that m' is continuous. Since $R = \pi \times \pi^{-1}(\Delta(S/R))$ is closed and $\pi \times \pi$ is a quotient map we have that $\Delta(S/R)$ is closed. Hence S/R is a Hausdorff topological space. □

Lema 2. *Let R be a closed equivalence on a space X and let K be a compact subset of X. Then $\pi^{-1}(\pi(X))$ is closed.*

Proof. Let π_1: $X \times K \to X$ denote the first projection. Then $\pi^{-1}(\pi(K)) = \pi_1((X \times K) \cap R)$ is closed. □

Lema 3. *Let X be a compact Hausdorff space and let R be a closed equivalence on X. Then:*

a) *π: $X \to X/R$ is closed.*

b) *if V is an open subset of X, then $V_R = \{x \in X : \pi^{-1}(\pi(x)) \subset V\}$ is open and π-saturated.*

c) *If A is a closed π-saturated subset of X and W is an open subset of X such that$A \subset W$, then there exists an open π-saturated subset U and a closed π-saturated subset K such that $A \subset U \subset K \subset W$.*

d) *X/R is Hausdorff.*

Proof. To prove a) let C be a closed subset of X. Then C is compact, and hence $\pi^{-1}(\pi(C))$ is closed. Since π is a quotient, $\pi(C)$ is closed, so π is closed.

To prove b), observe that $V_R = V - \pi^{-1}\pi(X - V)$. Since π is closed, we have that V_R is open. It is clear that V_R is π-saturated.

To prove c), observe that $A \subset W_R \subset W$, since A is π-saturated. Since A is normal and W_R is open, there exists an open set V such that $A \subset V \subset \overline{V} \subset W_R \subset W$. Let $U = V_R$ and $K = \pi^{-1}\pi(\overline{V})$. Then U and K satisfy the required conditions.

To prove d), let $\pi(x)$ and $\pi(y)$ be distinct points of X/R. Then $\pi^{-1}\pi(x)$ and $\pi^{-1}\pi(y)$ are disjoint closed subsets of X. Since X is normal, there exist disjoint open sets M and N containing $\pi^{-1}\pi(x)$ and $\pi^{-1}\pi(y)$ respectively. We obtain that $\pi(M_R)$ and $\pi(N_R)$are disjoint open subsets of X/R containing $\pi(x)$ and $\pi(y)$ respectively. □

Lema 4. *Let X be a locally compact σ-compact space. Then there exists a sequence $\{K_n\}$ of compact subsets of X such that $X = \bigcup_{n\in N} K_n$ and $K_n \subset K^\circ_{n+1}$ (the interior of K_{n+1} in X) for each $n \in N$.*

Proof. Let $X = \bigcup_{n\in N} C_n$, where C_n is compact for each $n \in N$. Let $K_1 = C_1$ and $\{K_i \in I\}$ a finite cover of K_1 by open sets of finite closure. Let $K_2 = \bigcup_{i\in I} \overline{K_i}$. Since $K_2 \cup C_2$ is compact there exists a finite cover $\{K_i^2 : i \in I\}$ of $K_2 \cup C_2$ by open sets with compact closures. Let $K_3 = \bigcup_{i\in I} \overline{K_i^2}$. Continuing recursively, let K_{n+1} be a compact subset such that $K_n \cup C_n \subset K^\circ_{n+1}$. □

The next result is due to Jimmy Lawson (see [5]). It was communicated to me while I was his student at Louisiana State University.

Theorem 2. *Let S be a locally compact σ-compact semigroup and let R be a closed congruence on S. Then S/R is a topological semigroup.*

Proof. We will establish that S/R is a topological semigroup by proving that $\pi \times \pi$: $S \times S \to S/R \times S/R$ is a quotient map and then applying Lemma 1.

Let Q be a subset of $S/R \times S/R$ such that $\pi \times \pi^{-1}(Q)$ is open in $S \times S$. We want to show that Q is open in $S/R \times S/R$. Let $(\pi(a), \pi(b)) \in Q$. By lemma 4, there exists a sequence $\{K_n\}$ of compact subsets of S such that $S = \bigcup_{n\in N} K_n$ and $K_n \subset K^\circ_{n+1}$, $\forall n \in N$.

Assume, without loss of generality that both a and b lie in K_1. Let $V_0 = \pi^{-1}(\pi(a)) \cap K_1$ and $W_0 = \pi^{-1}(\pi(b)) \cap K_1$. It is easy to see that both V_0 and W_0 are compact subsets of K_1. By Wallace theorem, there exist sets G and T which are open in K_1 such that $V_0 \times W_0 \subset G \times T \subset \pi \times \pi^{-1}(Q) \cap K_1 \times K_1$. Using lemma 3, we obtain sets V_1 and W_1 which are closed in K_1, and hence compact, and $\pi|_{K_1}$-saturated such that V_1 is a K_1-neigborhood of V_0, W_1 is a K_1-neigborhood of W_0, $V_1 \subset G$, and $W_1 \subset T$. By lemma 2 $\pi^{-1}(\pi(V_1))$ and $\pi^{-1}(\pi(W_1))$ are closed. Again by Wallace's theorem, there exist set M and N which are open in K_2 so that

$$[\pi^{-1}(\pi(V_1)) \cap K_2] \times [\pi^{-1}(\pi(W_1)) \cap K_2] \subset M \times N \subset \pi \times \pi^{-1}(Q) \cap K_2 \times K_2.$$

As above, there are sets V_2 and W_2 which are closed in K_2 (and hence compact), $\pi|_{K_2}$-saturated and such that V_2 is a K_2-neigborhood of $\pi^{-1}(\pi(W_1)) \cap K_2$, $V_2 \subset M$ and $W_2 \subset N$.

Define recursively a pair of towered sequences $\{V_n\}$ and $\{W_n\}$ such that V_n and W_n are closed in K_n (and hence compact) and $\pi|_{K_n}$-saturated such that V_n is a K_n-neigborhood of $\pi^{-1}(\pi(V_{n-1})) \cap K_n$, W_n is a K_n-neigborhood of $\pi^{-1}(\pi(W_{n-1})) \cap K_n$ and $V_n \times W_n \subset \pi \times \pi^{-1}(Q)$ for each $n \in N$.

Let $V = \bigcup_{n=1}^{\infty} V_n$ and $W = \bigcup_{n=1}^{\infty} W_n$. It is clear that $\pi^{-1}(\pi(a)) \subset V$ and $\pi^{-1}(\pi(b)) \subset W$. Observe that V is π-saturated since $V = \bigcup_{n=1}^{\infty} \pi^{-1}(\pi(V_n))$, and similarly W is π-saturated.

To see that V is open, let $x \in V$. Then $x \in V_n$ for some $n \in N$. Now V_{n+1} is a K_{n+1}-neighborhood of V_n and $V_n \subset K_{n+1}^{\circ}$. It follows that $V_{n+1} \cap K_{n+1}^{\circ}$ is a neigborhood of V_n. Observing that $V_n \subset V_{n+1} \cap K_{n+1}^{\circ} \subset V_{n+1} \subset V$, we see that V is a neigboprhood of x, and hence it is open.

A similar argument proves that W is open. Finally, $(\pi(a), \pi(b)) \in \pi(V) \times \pi(W) \subset Q$ and $\pi(V) \times \pi(W)$ is open since V and W are open π-saturated subsets of S. □

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