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Barycentric-sum problems: a survey

Problemas sobre sumas baricéntricas: una revisión

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Abstract

Let G be a finite abelian group. A sequence in G is barycentric if it contains one element "average" of its terms. We give a survey of results and open problems concerning sufficient conditions for the existence of barycentric sequences. Moreover values and open problems on the k-barycentric Davenport constant $BD(k, G)$, the barycentric Davenport constant $BD(G)$, the strong k-barycentric Davenport constant $SBD(k, G)$ and barycentric Ramsey numbers $BR(H, G)$ for some graphs H are presented. These constants are related to the Davenport constant $D(G)$.
Key words and phrases: barycentric sequence, Davenport constant, k-barycentric Davenport constant, barycentric Davenport constant, strong k-barycentric Davenport constant, barycentric Ramsey number.

Resumen

Sea G un grupo abeliano finito. Una sucesión en G es baricéntrica si contiene un elemento el cual es "promedio"de sus términos. En este artículo, se presenta una revisión de resultados y problemas abiertos sobre condiciones suficientes para la existencia de sucesiones baricéntricas. Además se dan valores y problemas abiertos sobre la constante k-baricéntrica de Davenport $BD(k, G)$, la constante baricéntrica de Davenport $BD(G)$, la constante fuerte k-baricéntrica de Davenport $SBD(k, G)$ y el número Ramsey baricéntrico $BR(H, G)$ para algunos

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grafos H. Estas constantes están relacionadas con la constante de Davenport $D(G)$.
Palabras y frases clave: sucesión baricéntrica, constante de Davenport, constante k-baricéntrica de Davenport, constante baricéntrica de Davenport, constante fuerte k-baricéntrica de Davenport, número Ramsey baricéntrico.

1 Introduction

Let G be a finite abelian group. Then $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}$, $1 < n_1|....|n_r$, where $n_r = exp(G)$ is the exponent of G and r is the rank of G. Let $M(G) = \sum_{i=1}^{r}(n_i - 1) + 1$. In this paper, we denote by p a prime number.

Definition 1 ([12])**.** *Let A be a finite set with $|A| \geq 2$ and G an abelian group. A sequence $f : A \to G$ is* barycentric *if there exists $a \in A$ such that $\sum_A f = |A|f(a)$. The element $f(a)$ is called its barycenter.*

The word sequence is used to associate the set A with the set $\{1, 2, \ldots, |A|\}$. When $|A| = k$ we shall speak of a *k-barycentric sequence.* Moreover when f is injective the word *barycentric set* is used instead of barycentric sequence. The condition $|A| \geq 2$ avoids the trivial realization of equality $\sum_A f = |A|f(a)$ when $A = \{a\}$.

The history of barycentric sequences is short, it dates back to 1995 [10]. The works of Hamidoune [20, 21] on weighted sequences was the inspiration, in the regular seminar on Combinatoria held at the LaTecS Laboratory, ISYS Center, Universidad Central de Venezuela, to introduce barycentric sequences and barycentric constants.

Let f be a sequence in G. An obvious sufficient condition for the existence of a barycentric subsequence is that $|A| > |G|$ since this implies the existence of two distinct elements a, a^1 with $f(a) = f(a^1)$. Then $f(a)f(a^1)$ is a $2-$barycentric subsequence of f. Moreover, $|A| > (k-1)|G|$ implies the existence of a k-barycentric subsequence of f.

Notice that $a_1, a_2, \cdots, a_k$ is a k-barycentric sequence of barycenter a_j if and only if $a_1 + a_2 + \cdots + (1-k)a_j + \cdots + a_k = 0$. That is to say, a weighted zero-sum sequence with $w_i = 1$ for all $i = 1, \cdots, k$ excepting $w_j = 1 - k$. Therefore, the barycentric-sum problem can be located among the so called zero-sum problems.

A weighted sequence is constituted by terms of the form $w_i a_i$, where $a_i \in G$ and w_i are positive integers. The weighted sequences with zero-sum are studied in [18],[19], [20], [21] and [23].

The following theorem is the starting point of zero-sum problems.

Theorem 1 ([14]). *Let G be a finite abelian group of order n. Then every sequence of length $2n-1$ has a subsequence of length n with zero-sum.*

Caro in 1966 [3] gives a nice structured survey on zero-sum problems, where the following conjecture due to Caro was formulated:

Conjecture 1. *Let G be a finite abelian group of order n. Let $w_1, w_2, \ldots, w_k$ be positive integers such that $w_1 + w_2 + \ldots + w_k = 0 \pmod{n}$. Let $a_1, a_2, \ldots, a_{n+k-1}$ in G not necessarily distinct. Then there exist k distinct indices $i_1, \ldots, i_k$ such that $w_1 a_{i_1} + w_2 a_{i_2} + \ldots + w_k a_{i_k} = 0 \pmod{n}$.*

In the context of weighted sequence, Grynkiewicz in [19] proves the veracity of this conjecture giving the following theorem:

Theorem 2 ([19]). *Let m, n and $k \geq 2$ be positive integers. If f is a sequence of $n+k-1$ elements from a nontrivial abelian group G of order n and exponent m, and if $W = \{w_i\}_{i=1}^{k}$ is a sequence of integers whose sum is zero modulo m, then there exists a rearranged subsequence $\{b_i\}_{i=1}^{k}$ of f such that $\sum_{i=1}^{k} w_i b_i = 0$. Furthermore, if f has an k-set partition $A = A_1, \cdots, A_k$ such that $|w_i A_i| = |A_i|$ for all i, then there exists a nontrivial subgroup H of G and an k-set partition $A^1 = A_1^1, \cdots, A_k^1$ of f with $H \subseteq \sum_{i=1}^{k} w_i A_i^1$ and $|w_i A_i^1| = |A_i^1|$ for all i.*

Theorem 2 extends the Erdős-Ginzburg-Ziv, which is the case when $k = n$ and $w_i = 1$ for all i.

Recently Gao and Geroldinger present a survey on zero-sum problems [16], updating the Caro survey [3].

The following remark establishes a relationship between the zero-sum problem and the barycentric-sum problem.

Remark 1 ([25]). *Let A be a set in a finite abelian group G. Let $a \in A$, then A contains a barycentric set with barycenter a if and only if $A - a \setminus \{0\}$ contains a zero-sum set.*

Definition 2 ([9]). *Let G be a finite abelian group. The* Davenport constant *$D(G)$ is the least positive integer d such that every sequence of length d in G contains a non-empty subsequence with zero-sum.*

It is clear that $M(G) \leq D(G) \leq |G|$ [17]. It is well known that $D(\mathbb{Z}_n) = n$. Moreover for noncyclic groups we have:

Theorem 3 ([27]). *Let G be a finite noncyclic group of order n then $D(G) \leq \lceil \frac{n+1}{2} \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x.*

Moreover for p-groups we have:

Lemma 1 ([26]). *Let $G = \mathbb{Z}_{p^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{\alpha_k}}$ be a $p-$group. Then we have $D(G) = M(G)$.*

We have the following results:

Theorem 4 ([22]). *Let G be an abelian group. Let $f : A \to G$ be a sequence with $k \leq |A| \leq 2k - 1$ and $|\{\sum_{x \in S} f(x) : S \subseteq A : |S| = k\}| \leq |A| - k$. Then f contains a k-barycentric or a $(k+1)$-barycentric sequence.*

Theorem 5 ([20]). *Let G be a finite abelian group of order $n \geq 2$ and $f : A \to G$ a sequence with $|A| \geq n+k-1$. Then there exists a k-barycentric subsequence of f. Moreover, in the case $k \geq |G|$ the condition $|A| \geq k + D(G) - 1$ is sufficient for the existence of a k-barycentric subsequence of f.*

This result shows the existence of the following constant:

Definition 3 ([11]). *Let G be an abelian group of order $n \geq 2$. The k-*barycentric Davenport constant *$BD(k, G)$ is the minimal positive integer t such that every t-sequence in G contains a k-barycentric subsequence.*

Hence by Theorem 5 we have $BD(k, G) \leq n + k - 1$. Notice that by Theorem 2 this constant is also assured.

The following two theorems are the algebraic background used in [12], in order to establish in Theorem 8, Theorem 9 and Corollary 1 conditions for the existence of barycentric subsequences in a given sequence with prescribed length.

Theorem 6 ([13]). *Let H be a subset of $\mathbb{Z}_p$. Let d be a positive integer such that $2 \leq d \leq |H|$.*

$$\textit{Set } \bigwedge^d H = \{\sum_{x \in S} x \ : \ S \subset H, |S| = d\}.$$

Then $|\bigwedge^d H| \geq \min\{p, d(|H| - d) + 1\}$.

Theorem 7 ([8]). *Let A and B be subsets in $\mathbb{Z}_p$. Then $|A+B| \geq min\{p, |A| + |B| - 1\}$.*

Theorem 8 ([12]). *Let s, d be integers ≥ 2 such that $p \geq d+2+\frac{1}{d-1}$. Let A be a set with $s+d$ elements, and $f : A \to \mathbb{Z}_p$ a sequence with $|f(A)| \geq \frac{p-1}{d}+d+1$. Then f contains an s-barycentric subsequence.*

The following theorem improves Theorem 8, under the additional condition $s > \lceil \frac{p-1}{d} \rceil$.

Theorem 9 ([12]). *Let s, d be integers ≥ 2 such that $s > \lceil \frac{p-1}{d} \rceil$. Let A be a set with $|A| = s+d$. Let $f : A \to \mathbb{Z}_p$ be a sequence such that $|f(A)| \geq \lceil \frac{p-1}{d} \rceil + d$. Then there exists an s-barycentric subsequence of f.*

Corollary 1 ([12]). *Let $f : A \to \mathbb{Z}_p$ be a sequence with $|A| = p+2$ and $|f(A)| \geq \frac{p+3}{2}$. Then f contains a p-subsequence with zero-sum.*

The following problem is still open:

Problem 1 ([12]). *Let A be a subset of size k in $\mathbb{Z}_p$. If there are no barycentric sequences of size $\leq t$ in A, what can be said about the minimum number $F(k, d, t)$ of sums of d different terms in A when it is less than p?*

The case $t = 2$ is described by Hamidoune and Dias da Silva in Theorem 6: $F(k, d, 2) = d(k-d)+1$.

As an example, we easily see that $F(4, 2, 3) = 5 = F(4, 2, 2)$, and that the function has the symmetry $F(k, d, t) = F(k, k-d, t)$. It seems that $F(5, 2, 3) = 9 > F(5, 2, 2) = 7$.

In order to present another barycentric constant, we have the following definition:

Definition 4 ([10],[17],[29]). *Let G be a finite abelian group. The Olson constant, denoted $O(G)$, is the least positive integer d such that every subset $A \subseteq G$, with $|A| = d$ contains a non-empty subset with zero-sum.*

It is clear that $O(G) \leq D(G)$. Moreover we have the theorem:

Theorem 10 ([17]). *Let $G = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}_n^{s+1}$ with $r \geq 0$, $s \geq 0$, $1 < n_1|....|n_r|n$ and $n_r \neq n$. If G is a $p-$group and $r + \frac{s}{2} \geq n$, then $O(G) = M(G) = D(G)$.*

Theorem 11 ([11, 29]). $O(\mathbb{Z}_2^s) = s+1$ *for* $s \geq 1$ *and* $O(\mathbb{Z}_3^s) = 2s+1$ *for* $s \geq 3$.

The following constant is introduced and studied in [12].

Definition 5 ([12]). *Let G be a finite abelian group. The* barycentric Davenport constant $BD(G)$ *is the least positive integer m such that every m-sequence in G contains a barycentric subsequence of length ≥ 2.*

If f is not injective, then there is a 2-barycentric subsequence. In the injective case, if $|G| \neq 1$, then using pairs of distinct elements it is easy to show that $BD(G) \geq 3$. Hence, we have the following alternate definition of $BD(G)$:

Definition 6 ([12]). *Let G be a finite abelian group with $|G| \geq 3$; $BD(G)$ is the least positive integer d such that every subset $A \subset G$, with $|A| = d$ contains a barycentric subset B.*

By Remark 1 and Definition 6 we have that $BD(G) \leq O(G) + 1$.

We have the following results, conjecture and open problem:

Theorem 12 ([12]). $BD(\mathbb{Z}_p) \leq \left\lceil \sqrt{4p+1} \right\rceil - 2$ *for* $p \geq 5$.

Theorem 13 ([12]). $BD(\mathbb{Z}_2^s) = s + 2$ *for* $s \geq 1$.

Theorem 14 ([12, 25]). *For $s \geq 2$ we have $2s + 1 \leq BD(\mathbb{Z}_3^s) \leq 2s + 2$. Moreover $BD(\mathbb{Z}_3^s) = 2s + 1$, for $1 \leq s \leq 5$.*

At present there is no known value of s for which the upper bound $2s+2$, in Theorem 14, is attained. Then the following conjecture is formulated:

Conjecture 2. $BD(\mathbb{Z}_3^s) = 2s + 1$ *for* $s \geq 2$.

Problem 2. *The groups G and their values or upper bounds known up now of $O(G)$ and $BD(G)$ are those given in* [12]. *Since $BD(G) \leq O(G) + 1$, in the measure that $O(G)$ is determined for specific G then we have an upper bound for $BD(G)$. To enlarge the groups and their values or upper bounds for both constants is an open problem.*

In [28] *the strong barycentric Davenport constant* $SBD(k, G)$ is introduced as the minimum positive integer t such that any t-set in G contains a k-barycentric set, provided such an integer exists. Moreover in [28], the existence of $SBD(k, G)$ are established and some values or bounds are given. In general there is no known algebraic background to calculate $SBD(k, G)$. The action of the group $G_n = \{f_{a,b} : \mathbb{Z}_n \to \mathbb{Z}_n, f_{a,b}(x) = ax + b, a, b \in \mathbb{Z}_n, (a, n) = 1\}$ on the set $X_n^k = \{\{x_1, x_2, \ldots, x_k\} : x_i \in \mathbb{Z}_n\}$ partitions it in equivalence classes or orbits. If $\{x_1, \cdots, x_k\}$ is k-barycentric then all elements of its orbit $\theta(\{x_1, \cdots, x_k\})$ are k-barycentric sets. This fact allowed in [28] give the existence and then to calculate $SBD(k, \mathbb{Z}_n)$ for some n and k in particular for $3 \leq n \leq 12$ and $3 \leq k \leq n$. For example the following results are establish:

Theorem 15 ([28]). *$SBD(3, \mathbb{Z}_n) = 5$ for $n = 6, 8, 9, 10, 13$ and $SBD(3, \mathbb{Z}_4) = 3$.*

We discuss now another barycentric constant:

Definition 7 ([11]). *Let G be an abelian group of order $n \geq 2$ and let H be a graph with $e(H) = k$ edges. The* barycentric Ramsey number *of the pair (H, G), denoted by $BR(H, G)$, is the minimum positive integer r such that any coloring $c : E(K_r) \to G$ of the edges of K_r by elements of G yields a copy of H, say H_0, with an edge e_0 such that the following equality holds:*

$$\sum_{e \in E(H_0)} c(e) = kc(e_0) \tag{1}$$

In this case H is called a barycentric *graph.*

The barycentric Ramsey number theory introduced in [11] can be traced back in the Ramsey number $R(H, n)$ and in the Ramsey-zero-sum number $R(H, G)$.

The *Ramsey number* $R(H, n)$ is the smallest integer t such that for any coloring of the edges of K_t with n colors there exists a monochromatic copy of H.

Let G be a finite abelian group of order n. Let H be a graph where its edges satisfy $e(H) = 0 \pmod n$, the *Ramsey zero-sum number* $R(H, G)$ is defined as the minimal positive integer s such that any coloring $c : E(K_s) \to G$ of the edges of the complete graph K_s by elements of G yields a copy of H, say H_0 with

$$\sum_{e \in E(H_0)} c(e) = 0, \tag{2}$$

where 0 is the zero element of G. The necessity of the condition $e(H) =$ (mod n) for the existence of $R(H, G)$ is clear, it comes from the monochromatic coloration of the edges of H.

The Ramsey zero-sum number was introduced by Bialostocki and Dierker in [1] when $e(H) = n$ and the concept is extended to $e(H) = 0 \pmod n$ by Caro in [4]. Notice that when $e(H) = 0 \pmod n$ then $R(H, G) \leq R(H, n)$ and $R(H, 2) \leq R(H, G)$ when $e(H) = n$.

It is clear that $BR(H, G) \leq R(H, |G|)$, then $BR(H, G)$ always exists.
Besides this introduction that provides the history and tools on barycentric sequences, this paper contains two main sections dedicated to discuss the k-barycentric Davenport constant and the barycentric Ramsey number respectively.

2 k-barycentric Davenport constant

Let G be an abelian group of order n. In general there is no known algebraic method to calculate $BD(k,G)$. In [11] $BD(k,\mathbb{Z}_p)$ is calculated for some prime p. In [28] some $BD(k,\mathbb{Z}_n)$ for $3 \le n \le 12$ and $3 \le k \le n$ is derived from $SBD(k,\mathbb{Z}_n)$. For example $BD(3,\mathbb{Z}_4) = 5$ and $BD(3,\mathbb{Z}_6) = 6$ are obtained from $SBD(3,\mathbb{Z}_4) = 3$ and $SBD(3,\mathbb{Z}_6) = 5$ respectively.

In [11], the following inequality are used to calculate $BD(k,G)$:

$$BD(k,G) \le n + k - 1. \tag{3}$$

For example from (3) we have:

Proposition 1 ([11]). $BD(2,G) = n + 1$.

Proposition 2 ([11]). $BD(k,\mathbb{Z}_2) = 2\left\lfloor \frac{k}{2} \right\rfloor + 1$.

Proposition 3 ([12]). $BD(k,\mathbb{Z}_3) = \begin{cases} k+1 & \text{if } k \ne 0(\text{mod}3), \\ k+2 & \text{if } k = 0(\text{mod}3) \end{cases}$

The following theorem is derived from the Dias da Silva-Hamidoune theorem.

Theorem 16 ([11]). $BD(3,\mathbb{Z}_p) \le 2\lceil \frac{p}{3} \rceil + 1$ *for* $p \ge 5$.

In particular we have:

Corollary 2 ([11]). $BD(3,\mathbb{Z}_5) = 5$, $BD(3,\mathbb{Z}_7) = 7$, $BD(3,\mathbb{Z}_{11}) = BD(3,\mathbb{Z}_{13}) = 9$.

For certain values of p, the inequality (3) can be improved:

Theorem 17 ([11]). $BD(k,\mathbb{Z}_p) \le p + k - 2$ *for* $4 \le k \le p - 1$.

Problem 3. *Derive from Theorem 17 exact values of* $BD(k,\mathbb{Z}_p)$ *for* $4 \le k \le p-1$. *Moreover, find for which* $4 \le k \le p-1$ *it is verified* $BD(k,\mathbb{Z}_p) = p+k-2$.

Related to Problem 3, we have the following corollary and theorem:

Corollary 3 ([11]). $BD(p-1,\mathbb{Z}_p) = 2p - 3$ *for* $p \ge 5$.

However, we have:

Theorem 18 ([11]). $BD(4,\mathbb{Z}_7) = 8$.

The following theorem is used to derive a result (Theorem 20) similar to Theorem 17 for $k > p$.

Theorem 19 ([11])**.** *Let G be a group of order n, and $k > n$.*

- *If $BD(k-n, G) \geq n-1$, then $BD(k, G) \leq n + BD(k-n, G)$.*
- *If $BD(k-n, G) \leq n-1$, then $BD(k, G) \leq 2n-1$.*

Theorem 20 ([11])**.** *Let $p \geq 5$, $k > p$ and the remainder of the division of k by p is in $\{4, \ldots, p-1\}$, then $BD(k, \mathbb{Z}_p) \leq p + k - 2$. Moreover when the remainder is $p-1$ we have $BD(k, \mathbb{Z}_p) = p + k - 2$.*

Finally we have the following two theorems and problem.

Theorem 21 ([11])**.** $BD(3, \mathbb{Z}_2^s) = 2^s + 1$.

Theorem 22 ([11])**.**

s	1	2	3	4
$BD(4, \mathbb{Z}_3^s) = BD(3, \mathbb{Z}_3^s)$	5	9	19	41

Problem 4. *In papers* [11] *and* [28] *the orbit technique was used to calculate $SBD(k, \mathbb{Z}_n)$ and $BD(k, \mathbb{Z}_n)$ for some n and k. Using this technique, we propose to extend the list of known exact values or bounds of $SBD(k, \mathbb{Z}_n)$ and $BD(k, \mathbb{Z}_n)$ presented in both papers.*

3 Barycentric Ramsey numbers

Let G be an abelian group of order n and let H be a graph with $e(H)$ edges. In this section we summarize the values or bounds of $BR(H, \mathbb{Z}_n)$ for stars, paths, circuits and matching. In particular for $2 \leq n \leq 5$ *and* $2 \leq e(H) \leq 4$. We use the following notations: the stars are the complete bipartite graphs $K_{1,k}$, P_k are paths with k vertices and $k-1$ edges, C_k are circuits with k vertices and mK_2 an m matching, i.e. m disjoint edges. At present there is no known algebraic background to calculate the upper bound values of $BR(H, \mathbb{Z}_n)$ for $e(H) \not\equiv 0 \pmod{n}$, so that it is only possible to compute them manually by cases or by computer. For lower bounds it is sufficient to find an ad hoc decomposition of a complete graph in edges disjoint subgraphs, colored in order to avoid some particular barycentric graph. Moreover, in some cases the following remark gives a lower bound:

Table 1: Barycentric graphs coloring

$e(H)$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$
2	monochromatic	monochromatic	monochromatic	monochromatic
3	any coloring	a, b, c	a, b, c	a, b, c
		monochromatic	$a, a, a+2$	monochromatic
			monochromatic	
4	a, a, b, b	a, a, b, c	$a, a, a+2, a+2$	a, a, b, c
	monochromatic	a, a, a, b	$a, a, a+1, a+3$	monochromatic
		monochromatic	monochromatic	

Remark 2. *If a graph H is not barycentric with any 2-coloring, then $R(H, 2) \leq BR(H, G)$.*

The following remark is useful to establish an upper bound of $BR(H, \mathbb{Z}_n)$:

Remark 3 ([15])**.** *Let H be a graph with $2 \leq e(H) \leq 4$ edges colored by elements of $\mathbb{Z}_n$ $(2 \leq n \leq 5)$. Table 1 shows the possible coloring for H to be barycentric. For example, in case $e(H) = 3$ and the edges colored by elements from $\mathbb{Z}_4$, H is barycentric when the edges are colored with three different colors a, b, c or the edges are colored by $a, a, a+2$ for any color a or the edges are colored monochromatically.*

The following remark and theorem, allow to establish $BR(H, \mathbb{Z}_2)$:

Remark 4 ([11])**.** *Let H be a graph and $e(H)$ the number of its edges. Then:*

$$BR(H, \mathbb{Z}_2) = \begin{cases} |V(H)| & \text{if } e(H) \text{ is odd,} \\ R(H, \mathbb{Z}_2) & \text{if } e(H) \text{ is even} \end{cases}$$

Theorem 23 ([6])**.** *Let H be a graph on h vertices and an even number of edges. Then:*

$$R(H, \mathbb{Z}_2) = \begin{cases} h+2 & \text{if } H = K_h, h = 0, 1(\pmod 4), \\ h+1 & \text{if } H = K_p \cup K_q, \binom{p}{2} + \binom{q}{2} = 0(\pmod 2), \\ h+1 & \text{if all the degrees in } H \text{ are odd,} \\ h & \text{otherwise} \end{cases}$$

3.1 Barycentric Ramsey numbers for stars

The barycentric Ramsey numbers for stars is obtained in the following way: the upper bound is derived from the inequality $BR(K_{1,k}, G) \leq BD(k, G) + 1$:

for any vertex in $K_{BD(k,G)+1}$ there is a barycentric star centered on this vertex.

We have the following theorem:

Theorem 24 ([2, 4]). *Let $K_{1,m}$ be the star on m edges with $m = 0 \pmod{n}$. Then*

$$BR(K_{1,m}, \mathbb{Z}_n) = R(K_{1,m}, \mathbb{Z}_n) = \begin{cases} m+n-1 & \text{if } m = n = 0 \pmod{2} \\ m+n & \text{otherwise} \end{cases}$$

The following theorem and its corollaries allow to obtain a particular coloring of a complete graph avoiding the existence of a barycentric $K_{1,k}$. That is to say, we derive lower bounds of $BR(K_{1,k}, \mathbb{Z}_n)$ by decomposing a complete graph into edge-disjoint subgraphs.

Theorem 25 ([24]). *Let K_n be a complete graph of n vertices. Then:*
K_n, with n odd, is the edge-disjoint union of $\frac{n-1}{2}$ hamiltonian cycles.
K_n, with n even, is the edge-disjoint union of $\frac{n-2}{2}$ hamiltonian cycles and one perfect matching. Hence K_n can be decomposed in $n-1$ perfect matching.

Corollary 4. *Let K_n be a complete graph of n vertices, with n odd. Then K_n can be decomposed into two complete graphs $K_{\frac{n+1}{2}}$ sharing a vertex and a bipartite complete graph $K_{\frac{n-1}{2},\frac{n-1}{2}}$.*

Corollary 5. *Let K_n be a complete graph of n vertices, with n even. Then K_n can be decomposed into two vertex-disjoint complete graphs $K_{\frac{n}{2}}$, the remaining $K_{\frac{n}{2},\frac{n}{2}}$ into one perfect matching and one $(\frac{n}{2}-1)$-regular graph.*

Therefore with the above considerations, the following results for stars were proved in [11]:

Theorem 26. $BR(K_{1,3}, \mathbb{Z}_{13}) = 10$.

Theorem 27. $BR(K_{1,p-1}, \mathbb{Z}_p) = 2p-2$.

Theorem 28. $BR(K_{1,4}, \mathbb{Z}_7) = 9$.

Theorem 29. $BR(K_{1,9}, \mathbb{Z}_5) = 13$.

Theorem 30. $BR(K_{1,tp+1}, \mathbb{Z}_p) = (t+1)p$ *for $p \geq 3$ and t positive integer.*

Theorem 31. $BR(K_{1,5t+2}, \mathbb{Z}_5) = 5(t+1)$.

3.2 Barycentric Ramsey numbers for matching

For an m-matching, the following two theorems are established:

Theorem 32 ([15]). *Let G be an abelian group of order $n \geq 2$. Then $BR(2K_2, G) = n + 3$.*

Theorem 33 ([5, 2]). $BR(mK_2, \mathbb{Z}_n) = R(mK_2, \mathbb{Z}_n) = 2m + n - 1$ *for* $m = 0 \pmod{n}$.

In [15] the following values for $BR(mK_2, \mathbb{Z}_n)$ with $m = 2$ and $n = 3, 4, 5$, $m = 3$ and $n = 4, 5$, $m = 4$ and $n = 3, 5$ are given.

Theorem 34 ([15]). $BR(2K_2, \mathbb{Z}_3) = 6$*,* $BR(2K_2, \mathbb{Z}_4) = 7$*,* $BR(2K_2, \mathbb{Z}_5) = BR(3K_2, \mathbb{Z}_4) = BR(3K_2, \mathbb{Z}_5) = 8$*,* $BR(4K_2, \mathbb{Z}_3) = 8$ *and* $BR(4K_2, \mathbb{Z}_5) = 11$*.*

3.3 Barycentric Ramsey numbers for paths and circuits

The following lemma was used in [15] to establish for $3 \leq n \leq 5$, the values of $BR(P_m, \mathbb{Z}_n)$ for $m = 3, 4, 5$ and $BR(C_m, \mathbb{Z}_n)$ for $m = 3, 4$.

Lemma 2 ([1]). *If the edges of K_n where $n \geq 5$, are colored by at least three different colors, then there exists a path on three differently colored edges.*

Theorem 35 ([3]). $BR(P_4, \mathbb{Z}_3) = BR(P_5, \mathbb{Z}_4) = 5$*.*

We have then the following theorems:

Theorem 36 ([15]).

- $BR(P_3, \mathbb{Z}_3) = BR(P_3, \mathbb{Z}_4) = 5$ *and* $BR(P_3, \mathbb{Z}_5) = 7$*.*
- $BR(P_4, \mathbb{Z}_4) = BR(P_4, \mathbb{Z}_5) = 5$*.*
- $BR(P_5, \mathbb{Z}_3) = BR(P_5, \mathbb{Z}_5) = 5$*.*

We have the following results:

Theorem 37 ([7]). $BR(C_3, \mathbb{Z}_3) = 11$*.*

Theorem 38 ([3]). $BR(C_4, \mathbb{Z}_4) = 6$*.*

Theorem 39 ([15]). $51 \leq BR(C_3, \mathbb{Z}_5) \leq 126$*.*

Problem 5. *Determine the exact value of $BR(C_3, \mathbb{Z}_5)$ or improve the bounds given in Theorem 39.*

Problem 6. *The computation of $BR(H, \mathbb{Z}_n)$ for $n \geq 6$ and the same graph H treated here, is an open problem.*

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