Divulgaciones Matemáticas Vol. 12 No. 1(2004), pp. 87–94

Banach Lattices and Infinite Rigid Sets

Retículos de Banach y Conjuntos Rígidos Infinitos

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Abstract

We investigate the existence of infinite dimensional Banach spaces having rigid sets with an infinite number of elements. Among Banach lattices, examples are provided by infinite dimensional AM-spaces as well as for some abstract L_p-spaces, $1 \leq p < \infty$.
Key words and phrases: rigid sets, lattices, Banach spaces.

Resumen

Se investiga la existencia de espacios de Banach que contengan conjuntos rígidos con infinitos elementos. Entre los retículos de Banach se proveen ejemplos mediante espacios AM de dimensión infinita así como algunos espacios L_p abstractos, $1 \leq p < \infty$.
Palabras y frases clave: conjuntos rígidos, retículos, espacios de Banach.

1 Introduction

According to [1] a compact subset A of a real normed space X is said to be *rigid* if A is the closure of a sequence $\{x_n\}$ satisfying

$$||x_{n+k} - x_n|| = ||x_{1+k} - x_1||,$$

for all $n, k \in \mathbb{N}$. In [9] it was proved that any finite dimensional space with a polyhedral norm has only finite rigid sets. In particular, this is true for l_1^n, the vector space $\{(x_1, \ldots, x_n),\ x_i \in \mathbb{R}, i = 1, \ldots, n\}$ with the l_1-norm, and for the space l_∞^n (the same vector space as before with the l_∞-norm).

Received 2004/02/06. Revised 2004/05/07. Accepted 2004/05/08.
MSC (2000): Primary 46B42, Secondary 46B25.

A natural problem would be to investigate if there are other finite dimensional normed spaces with this property. Among the l_p^n-spaces $(1 < p < \infty)$ we know that this is not true if p is a positive even integer (see [1]). For example, if $p = 2$, the normed space l_2^2 has an infinite rigid set given by $A = \overline{\{x_n\}}$, with $x_n = (\cos n\theta, \sin n\theta)$, θ irrational. The above problem is not easy. However, if we turn our interest into finding infinite dimensional Banach spaces that have an infinite rigid set, then several things can be said. It is clear from the definition of rigid sets that if X and Y are normed spaces such that X has an infinite rigid set and Y contains an isometric copy of X, then Y contains an infinite rigid set. By exploiting this observation when X is the space c_0 (it has an infinite rigid set, see [3]) or it is the euclidean plane, we are able to show that each abstract AM-space and some abstract L_p-spaces contain infinite rigid sets.

This paper is organized as follows. In Section 2, we review some facts from the theory of Banach lattices. In Section 3, we deal with infinite dimensional AM-spaces and show that they admit infinite rigid sets. In Section 4 we see that this is also true for some abstract L_p-spaces, $1 \leq p < \infty$.

2 Banach lattices

Let recall some notations and results from [7] and [8]. A *Riesz space* is a partially ordered real vector space E which in addition is a lattice, i.e., any two elements $x, y \in E$ have a least upper bound, denoted by $x \vee y = \sup\{x, y\}$, and a greatest lower bound, denoted by $x \wedge y = \inf\{x, y\}$. For every $x \in E$, let $x^+ = x \vee 0$, $x^- = (-x) \wedge 0$ and $|x| = x \vee (-x)$ be the positive part, the negative part and the absolute value of x, respectively. We have the identities $x = x^+ - x^-$ and $|x| = x^+ + x^-$. The set $E^+ = \{x : x \geq 0\}$ is called the *positive cone* of E and its elements are called *positive*. Two vectors x and y in E are said to be *disjoint*, written $x \perp y$, if $|x| \wedge |y| = 0$. A set M of vectors is pairwise disjoint if $x, y \in M$ and $x \neq y$ imply $|x| \wedge |y| = 0$. We have the following characterization for disjoint vectors.

Lemma 2.1. *Let $x, y \in E$. The following are equivalent:*

(i) $x \perp y$;

(ii) $|x + y| = |x - y|$;

(iii) $|x + y| = |x| \vee |y|$.

Proof. (i) $\Leftrightarrow$ (ii) follows from $|x| \wedge |y| = \frac{1}{2}\big|\, |x + y| - |x - y|\, \big|$; the equivalence (ii) $\Leftrightarrow$ (iii) comes from $|x| \vee |y| = \frac{1}{2}(|x + y| + |x - y|)$. Finally, (iii) $\Leftrightarrow$ (i),

$|x+y| = |x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|)$ implies $|x+y| = |x-y|$ and so we get $|x| \wedge |y| = \frac{1}{2}\big|\,|x+y| - |x-y|\,\big| = 0$. □

Corollary 2.2. *If $\{x_1, \ldots, x_n\}$ is a finite pairwise disjoint set of vectors, then*

$$|\sum_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i| = \bigvee_{i=1}^{n} |x_i|.$$

Proof. Recalling $a + b = a \vee b + a \wedge b$, we have $x_1 \perp x_2$ implies

$$|x_1 + x_2| = |x_1| \vee |x_2| = |x_1| + |x_2| - |x_1| \wedge |x_2| = |x_1| + |x_2|.$$

Since $x_n \perp (x_1 + \ldots + x_{n-1})$, the statement follows by induction on n. □

A Riesz space E is said to be *Archimedean*, if $x, y \in E$ and $nx \leq y$ for all $n \geq 1$ imply $x \leq 0$.

Proposition 2.3 ([7]). *Let E be an Archimedean Riesz space. If each collection of nonzero pairwise disjoint elements in E is finite, then E is finite dimensional.*

A norm $||\cdot||$ on a Riesz space is a *lattice norm* whenever $|x| \leq |y|$ implies $||x|| \leq ||y||$. A Riesz space equipped with a lattice norm is known as a normed Riesz space. If a normed Riesz space is also norm complete, then it is referred to as a Banach lattice. We observe that the positive cone $E^+ = \{x \in E : x^- = 0\}$ of a normed Riesz space E is closed as the inverse image of the closed set $\{0\}$ under the continuous map $x \to x^-$. Therefore, in particular, E is Archimedean. In fact, if $x, y \in E$ and $nx \leq y$ for all $n \geq 1$, then $x - n^{-1}y \in -E^+$ for all $n \geq 1$, which implies $x \in -E^+$, or $x \leq 0$, since $-E^+$ is closed.

Examples of Banach lattices are:

1. $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, the Banach space of equivalence classes of real measurable functions on (X, Σ, μ) whose p-th power is integrable (resp. which are essentially bounded if $p = \infty$). The norm is defined by $||f||_p = (\int_X |f(x)|^p d\mu)^{1/p}$ (resp. by esssup $|f(x)|$ if $p = \infty$). We write $L_p(0,1)$ when (X, Σ, μ) is the usual Lebesgue measure space on $[0,1]$, and $l_p = L_p(\mathbb{N}, \Sigma, \mu)$ (resp. $l_p^n = L_p(\{1, \ldots, n\}, \Sigma, \mu)$ when μ is the counting measure on the set $\mathbb{N}$ (resp. $\{1, \ldots, n\}$).
2. c, the subspace of l_∞ consisting of convergent sequences, and c_0 the subspace of l_∞ of all sequences convergent to zero.
3. $C(K)$, the Banach space of all continuous scalar-valued functions on the compact Hausdorff space K with the supremum norm.

3 AM-spaces and rigid sets

A Banach lattice E is called an *abstract* M-space, also AM-space, if $||x+y|| = \max\{||x||, ||y||\}$ whenever $x \wedge y = 0$. In an AM-space $||x \vee y|| = \max\{||x||, ||y||\}$ for $x, y \in E^+$.

Theorem 3.1. *Every infinite dimensional AM-space E contains rigid sets with an infinite number of elements.*

Proof. The statement will follow showing the existence of an isometry from c_0 into E, i.e. a linear map $c_0 \to E$ that satisfies $||Tx|| = ||x||$ for all $x \in c_0$. By Proposition 2.3, there exists a disjoint sequence $\{x_n\}$ of nonzero vectors in E. Replacing each x_n by $|x_n|/||x_n||$, we can assume $x_n \geq 0$ and $||x_n|| = 1$ for each n. Let $\{\xi_1, \ldots, \xi_n\}$ be arbitrary scalars. By Corollary 2.2

$$|\sum_{i=1}^{n} \xi_i x_i| = \sum_{i=1}^{n} |\xi_i| x_i = \bigvee_{i=1}^{n} |\xi_i x_i|,$$

and since E is an AM-space we get

$$||\,|\sum_{i=1}^{n} \xi_i x_i|\,|| = ||\sum_{i=1}^{n} \xi_i x_i|| = ||\bigvee_{i=1}^{n} |\xi_i x_i|\,|| = \max_{1\leq i\leq n} \{|\xi_i|\,||x_i||\} = \max_{1\leq i\leq n} |\xi_i|.$$

Hence

$$||\sum_{i=1}^{n} \xi_i x_i|| = ||\sum_{i=1}^{n} |\xi_i| x_i|| = \max_{1\leq i\leq n} |\xi_i|.$$

Let $\xi = \{\xi_1, \xi_2, \ldots\} \in c_0$. By the above equality it follows that $\sum_{i=1}^{\infty} \xi_i x_i$ is norm convergent, $|\sum_{i=1}^{\infty} \xi_i x_i| = ||\xi||_\infty$, and the map $T : c_0 \to E$, $T(\{\xi_1, \xi_2, \ldots\}) = \sum_{i=1}^{\infty} \xi_i x_i$, is an isometry. □

The map $T : c_0 \to E$ defined in the previous proof is actually a lattice embedding of c_0 into E. This means that T is a lattice homomorphism (i.e. $T(x \vee y) = T(x) \vee T(y)$, for all $x, y \in c_0$) and that T is an embedding (i.e. there exists two positive constants K and M such that $m||x|| \leq ||T(x)|| \leq M||x||$, for all $x \in c_0$). This fact is a consequence of the next result that characterizes the embedding of c_0 into a Banach lattice.

Theorem 3.2. [2] *The Banach lattice c_0 is lattice embeddable in a Banach lattice E if and only if there exists a disjoint sequence $\{x_n\}$ of E^+ such that* *(i)* $\{x_n\}$ *is not convergent in norm to zero;*

(ii) $\{x_n\}$ *has a norm bounded sequence of partial sums, i.e. there exists some* $M > 0$ *such that* $||\sum_{i=1}^{n} x_i|| \leq M$ *for all* n.

Corollary 3.3. *The Banach lattices* $C(K)$, c, $L_\infty(\mu)$ *and* l_∞ *have infinite rigid sets.*

Proof. Immediate from Theorem 3.1 because they are AM-spaces. □

Remark 3.4. A lattice isometry from c_0 into $C[0,1]$ can be explicitly constructed as follows. For each n choose a function $0 \leq f_n \in C[0,1]$ such that $||f_n||_\infty = 1$ and $f_n(t) = 0$ for every $t \notin [\frac{1}{n+1}, \frac{1}{n}]$. Then the linear operator $T : c_0 \to C[0,1]$, $T(\{\xi_1, \xi_2, \ldots\}) = \sum_{n=1}^{\infty} \xi_n f_n$ gives the statement.

Corollary 3.3 can be used to produce another example of an infinite dimensional Banach space with infinite rigid sets.

Proposition 3.5. *The Banach space* $\mathcal{B}(l_2)$ *of all bounded linear operators from* l_2 *to* l_2 *has infinite rigid sets.*

Proof. This follows from the fact that $\mathcal{B}(l_2)$ contains an isometric copy of l_∞. The map $\varphi : l_\infty \to \mathcal{B}(l_2)$, defined by $\varphi(\{\xi_i\})(\{x_i\}) = \{\xi_i x_i\}$ for $\{x_i\} \in l_2$, is in fact an isometry. It is enough to show that $||\{\xi_i\}||_\infty = 1$ implies $||\varphi(\{\xi_i\})||_{\mathcal{B}(l_2)} = 1$. If $||x||_2 = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2} = 1$, then

$$||\{\xi_i x_i\}||_2 = (\sum_{i=1}^{\infty} |\xi_i|^2 |x_i|^2)^{1/2} \leq ||\{\xi_i\}||_\infty (\sum_{i=1}^{\infty} |x_i|^2)^{1/2}.$$

Thus $||\varphi(\{\xi_i\})||_{\mathcal{B}(l_2)}$ has norm at most 1. On the other hand, choosing $x = e_n = \{0, \ldots, 0, 1, 0, \ldots\}$, (1 in the n^{th}-entry), we have $||\varphi(\{\xi_i\})|| \geq |\xi_n|$, for all n. Therefore $||\varphi(\{\xi_i\})||_{\mathcal{B}(l_2)} \geq 1$. □

4 Abstract L_p-spaces and rigid sets

A Banach lattice E is called an *abstract* L_p-space, for some $1 \leq p < \infty$, whenever its norm is p-additive, i.e., whenever $||x+y||^p = ||x||^p + ||y||^p$ holds for all $x, y \in E$ with $x \wedge y = 0$. Examples of abstract L_p-spaces are the Banach lattices $L_p(\mu)$, $1 \leq p < \infty$. There is the following representation Theorem due to Kakutani.

Theorem 4.1 ([6]). *An abstract L_p-space E, $1 \leq p < \infty$, is lattice isometric to an $L_p(\mu)$-space for a suitable measure space (X, Σ, μ).*

Let recall some facts about $L_p(\mu)$-spaces. The support of a function $f \in L_p(\mu)$ is the set $\{x : f(x) \neq 0\}$. We say that two functions $f, g \in L_p(\mu)$ are disjointly supported if the sets $\{f \neq 0\}$ and $\{g \neq 0\}$ are disjoint, that is $f \cdot g = 0$. A sequence $\{f_n\}$ in $L_p(\mu)$ is called a basic sequence if it is a basis for its closed linear span $[f_n]$.

Lemma 4.2. *Let $\{f_n\}$ be a sequence of disjointly supported nonzero vectors of $L_p(\mu)$, $1 \leq p < \infty$. Then $\{f_n\}$ is a basic sequence in $L_p(\mu)$ and $[f_n]$ is isometric to l_p.*

Proof. The linear span of $\{f_n\}$ is unaffected if we replace each element f_n by $f_n/||f_n||_p$. Thus, we may assume that each f_n has norm one. Next, since the f_k are disjointly supported, we have

$$||\sum_{k=n}^{m} a_k f_k||_p^p = \int_X |\sum_{k=n}^{m} a_k f_k|^p d\mu = \sum_{k=n}^{m} |a_k|^p \int_X |f_k|^p d\mu = \sum_{k=n}^{m} |a_k|^p,$$

for any scalars $\{a_k\}$. This tells us that $\sum_{n=1}^{\infty} a_n f_n$ converges in $L_p(\mu)$ if and only if $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Since the linear span of the e_n's is dense in l_p, this fact tells us that the map $T(e_n) = f_n$ extends linearly to an isometry from l_p onto $[f_n]$. In particular, it follows that $\{f_n\}$ is a basic sequence in $L_p(\mu)$. □

Theorem 4.3. *Abstract L_{2m}-spaces, $m \geq 1$, have infinite rigid sets.*

Proof. Theorem 4.1, Lemma 4.2 and the fact that the space l_{2m}^2 is an isometric subspace of l_{2m} will give the statement. □

Recall that a subset S of a Riesz space E is called solid if $|y| \leq |x|$ and $x \in S$ imply $y \in S$. A solid vector subspace of E is called an ideal of E. The ideal generated by S is the intersection of all ideals that include S. An element $z \in E^+$ is called an *atom* for a Riesz space E if the ideal generated by z is one-dimensional.

For a separable abstract L_p-space E with no atoms, the representation given in Theorem 4.1 can be realized in a more concrete manner.

Theorem 4.4. [8] *Let $1 \leq p < \infty$. Every separable abstract L_p-space E without atoms is isometrically lattice isomorphic to $L_p(0, 1)$.*

As a consequence we have the following result.

Theorem 4.5. *Let $1 \leq p < \infty$. Every separable abstract L_p-space E without atoms has infinite rigid sets.*

Proof. This follows from the previous Theorem and the fact that l_2^m is an isometric subspace of $L_p(0,1)$, for all $1 \leq p < \infty$ and $m \geq 2$, as the next result says (*see*[4]):

"Let $1 \leq p < \infty$, $1 \leq r \leq \infty$ and $m \in \mathbb{N}$, $m \geq 2$. Then l_r^m is an isometric subspace of $L_p(0,1)$ if and only if one of the following assertions holds: (i) $p \leq r < 2$; (ii) $r = 2$; (iii) $m = 2$ and $p = 1$." □

Recall that a Banach lattice is called a Hilbert lattice if it is n Hilbert space with respect to the same norm.

Proposition 4.6. *Every Hilbert lattice E has infinite rigid sets.*

Proof. This follows from Theorem 4.1 since E is an abstract L_2-space. In fact, if $x, y \in E$ are disjoint, then Lemma 2.1, (ii), implies $||x+y||^2 = ||x-y||^2$. Since E is a Hilbert space, this shows that $< x, y >= 0$ and hence $||x+y||^2 = ||x||^2 + ||y||^2$. □

Remark 4.7. The Lebesgue spaces $L_p(\mathbb{R})$, $L_p(a,b)$, $L_p(a,\infty)$ are all isometric to $L_p(0,1)$, for $1 \leq p < \infty$, and so they have infinite rigid sets.

Remark 4.8. By the Riesz-Fischer Theorem, every infinite dimensional separable Hilbert space is isometrically isomorphic to l_2. Therefore such spaces admit infinite rigid sets. Similarly, every euclidean space of dimension $n \geq 2$ is isometrically isomorphic to l_2^n and so also have infinite rigid sets.

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