Asymptotic expansions of canards with poles. Application to the stationary unidimensional Schrödinger equation.

E. Benoît

Abstract

The central topic of this paper is the problem of turning points. The paradigm is the stationary unidimensional Schrödinger equation, with various potentials. The first step is to transform the linear equation of second order into a Riccati equation. The non standard analysis and the theory of *canards* allow to compute the first eigenvalue and the corresponding solution. With a change of variables, it is possible to reduce the problem of the n-th energy level to the $(n-1)$-th. The first result (already proved by others methods) of the paper is an algorithm to compute the asymptotic expansion of the n-th energy level in powers of the small parameter $\hbar$. The second (new) result is an algorithm to compute an expansion of the corresponding solution. This expansion is a fraction so that the singularity is *resolved*. For example it is possible to determine the zero of the eigenfunctions of the Schrödinger operator up to any power of $\hbar$. The algorithms are given with *Maple* programs, and illustrated with a double symmetrical well as potential.

1 Introduction

A classical problem is the stationary unidimensional Schrödinger's equation

$$-\hbar^2\psi'' \;+\; V(q)\,\psi \;=\; E\,\psi \tag{1}$$

where E is the energy, and $V(q)$ the potential. The Planck's constant $\hbar$ is small, and in this paper it will be supposed infinitely small, in the sense of Non-Standard Analysis. The usual question is to find the values of the parameter E for which there are bounded solutions (or L^2 solutions). The problem is studied for various potentials, for example :

- The simple well $V(q) = q^2$ give a harmonic oscillator.
- All the functions $V(q)$ with a unique quadratic minimum give almost the same behaviour.

- The double symmetric well as $V(q) = (q^2 - 1)^2$, is studied for the exponential splitting of the energy levels (see [14, 18]).
- All the potentials $V(q)$ with quadratic or degenerated minima, and such that $V(\infty) = +\infty$.

Various methods can be used (see [12, 13, 8, 7, 16, 17],...) : the WKB-method gives asymptotic formal divergent expansions in powers of $\hbar$. We can sum this expansion with the Borel summation and we can define an exact solution with the theory of resurgents functions. Other asymptotic methods are also developed around the problem of turning points.

In [4], J.L. Callot transform the second order linear equation (1) into a first order Riccati equation

$$\varepsilon\dot{x} \;=\; -x^2 \;+\; V(t) \;-\; E \tag{2}$$

$$\text{where} \qquad \psi\prime = \frac{1}{\varepsilon}x\psi \qquad \hbar = \varepsilon \qquad q = t \qquad \dot{} = \frac{d}{dt}$$

If ψ is in $L^2(\mathbb{R})$, it is obvious that $x = \varepsilon\psi'/\psi$ has the sign of $(-t)$ outside a given interval. In [4], J.L. Callot gave a weaker constraint on the wanted solutions : he searches the functions ψ such that, the maximum of ψ is reached in the interior of some standard given, not too small, interval. Such functions are called "visibles" and they match the canards (with or without poles) of equation (2) (see [4]). In the case of a simple well as potential, the work of J.L. Callot shows with non standard methods (see [2, 10],...) the existence of the energy levels, their exponentially small thickness, and he computes the first term of their asymptotic expansion (see [4, 14]).

In this paper, I will give an effective algorithm (and his program in *Maple* language) to compute the asymptotic expansion of the n-th energy level, for any potential with quadratic minima. Moreover, I will compute an expansion of the corresponding solution x of equation (2). This expansion will be not a series of powers of ε. It will look like a fraction, and this structure allows to understand the turning point, better as the classical ones : the singularity at the minimum of the potential will be resolved, and we could for example, compute an expansion in powers of $\varepsilon^{1/2}$ of the zeroes of the solution ψ.

One of the known question in the study of slow-fast vector fields in $\mathbb{R}^2$ is the existence of canards* in one parameter family

$$\varepsilon\dot{x} \;=\; \varepsilon\frac{dx}{dt} \;=\; f(x,t,e) \qquad \varepsilon \simeq 0 \qquad \varepsilon > 0$$

where the slow curve $\mathcal{L}(\,^\circ f(x,t,e) = 0)$ has a singularity. In the generic case, this singularity is the transverse intersection of two branches, as in figure† 1.

*A *canard* is a solution of a slow-fast vector field which first go along an attractive branch of the slow curve, next go along a repulsive branch (see [2, 10]).

†In figures 1 to 4, I took one specific equation which is interesting only for the pictures. The reader has to look for the qualitative behaviour of the curves and for the order of magnitude of the distances. The horizontal axis is the t-axis, and the vertical one is the x-axis.

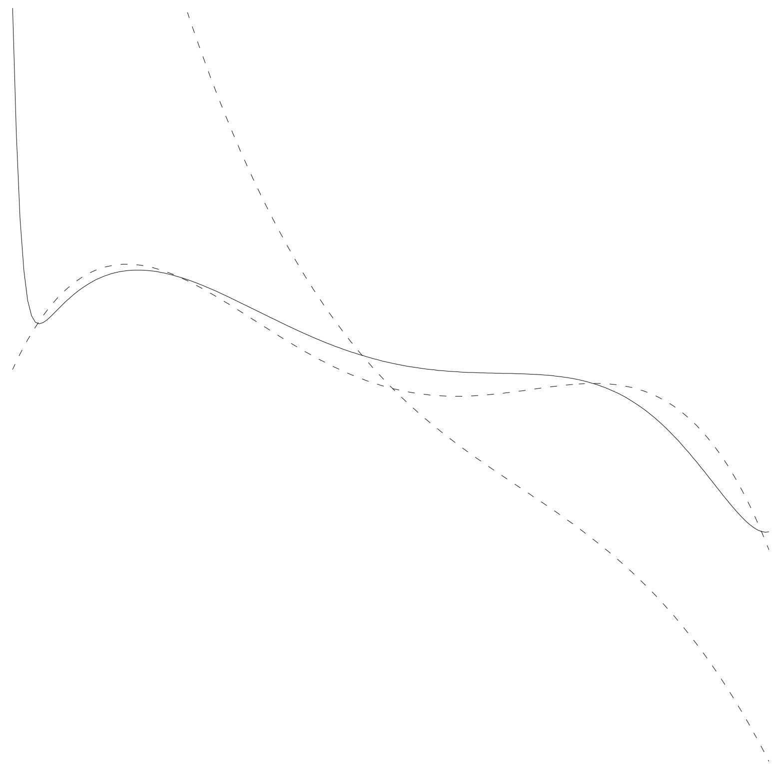

Figure 1: A canard without poles. (The dashed line is the slow curve).

The two following theorems are proved :

- There exist values of the parameter e called "canards-values" for which the equation has canards solutions. See [2, 11]
- The canards values and the canards solutions have ε-shadows expansions and there are algorithms to compute this expansions. See [9, 15, 6].

With non standard methods, we prove the theorems, and after we can compute the ε-shadows expansions with formal identifications. With resurgents methods, we first compute the formal expansions, we prove that this expansions are Gevrey, and the Borel-summation of the series are the exact canards solutions.

If $f(x, t, e)$ is a polynomial of degree 2 in x, the equation is a Riccati-equation, the point $x = \infty$ is a regular point : the change of variables $x = 1/u$ moves this point to the origin $u = 0$ and the vector field in u is C^∞ even at the point $u = 0$. In fact the vector fields in x and in u are two charts of a vector field on the cylinder $\mathbb{R} \times S^1$. Therefore, it is allowed to follow a solution when it goes to infinity, and the solutions can have poles. I will generalize the two theorems above. For that, I will answer the two questions below (see fig. 2) :

- Do exist some canards-values $\bar{e}_n$ of the parameter e with canards-solutions with n poles near the bottom of the well ?

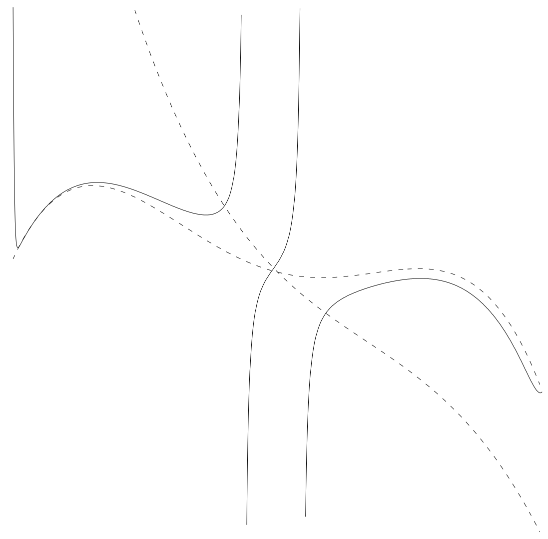

Figure 2: A canard with 2 poles.

- Do exist ε-shadows expansions of $\bar{e}_n$ and of the corresponding canards-solutions ? There exist some algorithm to compute them ?

J.L. Callot answer the first one (see [3]) in a particular case with the local model of Riccati-Hermite equation :

$$\varepsilon \dot{x} \;=\; -tx \,+\, ex^2 \,+\, \varepsilon$$

It is easy to generalize his results for all Riccati equations with a slow curve constituted into two transversal branches (see [1]). With the proof of Callot, we have also the first term of the ε-shadow expansion of $\bar{e}_n$.

In this paper, I will give the full expansion of $\bar{e}_n$ in powers of ε. I will also give the expansion of the corresponding canards-solutions. However, this expansion is not a series in powers of ε. It is written with fractions because we have to take into account the n poles in the $\sqrt{\varepsilon}$-neighborhood of the bottom of the well. Moreover, I will not use the proofs of Callot : I will prove his results easier and in the way of my own proofs.

The semi-classical method to solve the problem with poles is more difficult : The series one have to sum have singularity at the bottom of the well. So one can not use a Borel summation, and one have to use all the details of the theory of resurgents functions (see [7, 8]).

2 Notations-Hypotheses

I will study a family of one parameter e of Riccati equations :

$$\varepsilon \dot{x} \;=\; c\,(x-a)\,(x-b) \,+\, \varepsilon d \tag{3}$$

where ε is a positive infinitesimal fixed real number, a, b, c and d are functions of the time t and the parameter e. I will consider these equations on the domain

$\mathcal{D} =]t_-, t_+[\times]e_-, e_+[$ where t_- and t_+ are two standard real numbers. Here is the list of the assumptions on the functions a, b, c, d :

- On $\mathcal{D}$, the functions a, b, c and d are of class C^∞ (with respect to the variables t and e). It is possible to consider that these functions are only of class C^n, with a given n (not necessarily the same for all functions), but, for simplicity I will not do it.

- On $\mathcal{D}$, the functions a, b, c, d, $\dot{a}$ and $\dot{b}$ are S^0 (i.e. they have a continuous shadow). They have a ε-shadows expansion. It will be still possible to suppose that these expansions are valid only until a given order, but for simplicity, I will not do it.

- The function c is appreciable on all the domain $\mathcal{D}$. Its sign is constant. (Geometrically, this hypothesis shows that the slow curve doesn't intersect the axe of infinity).

  If necessary, we make the change of variables $x \to -x$, so that, on $\mathcal{D}$ we have $c > 0, c \not\simeq 0$.

- For each e in $]e_-, e_+[$, there exists a unique t_0 in $]t_-, t_+[$ such that $a(t_0, e) = b(t_0, e)$ (The two branches $\mathcal{L}_a$ ($x = {}^{\circ}a$) and $\mathcal{L}_b$ ($x = {}^{\circ}b$) of the slow curve have a unique intersection with abscissa ${}^{\circ}t_0$).

- Moreover, $\dot{a}(t_0) \not\simeq \dot{b}(t_0)$ (So this intersection is transverse).

- $c(t_0)\dot{a}(t_0) > c(t_0)\dot{b}(t_0)$, so $\mathcal{L}_a$ is attractive for ${}^{\circ}t < {}^{\circ}t_0$ and repulsive for ${}^{\circ}t > {}^{\circ}t_0$. A canard solution will go along $\mathcal{L}_a$.

Definition 1 *Let n be a natural nonnegative standard number. A solution $\bar{x}$ of the equation* (3) *is a* canard with n poles *if and only if there exist standard numbers t_e and t_s such that :*

- $t_e \lesssim_{\not\simeq} t_0 \lesssim_{\not\simeq} t_s$
- *If ${}^{\circ}t \in [t_e, t_s]$ and ${}^{\circ}t \neq {}^{\circ}t_0$, then $\bar{x}(t) \simeq a(t)$.*
- *The function $\bar{x}$ has n poles in the halo of t_0.*

Definition 2 *The* index *of the equation* (3) *is the real number*

$$k = \frac{d(t_0) - \dot{a}(t_0)}{\dot{a}(t_0) - \dot{b}(t_0)}$$

Remark The polynomial $c(x-a)(x-b) + \varepsilon d$, of degree two in x is given with four coefficients a, b, c and d. That is one more than necessary. Only c, $c(a+b)$ and $cab + \varepsilon d$ are useful. For example, the equation

$$\varepsilon \dot{x} = x^2 + (t+\varepsilon)x + \varepsilon e$$

can be written in the two forms

$$\varepsilon \dot{x} = (x+\varepsilon)(x+t) + \varepsilon(e-t) \quad \text{or} \quad \varepsilon \dot{x} = x(x+t+\varepsilon) + \varepsilon e$$

However, I have a bent for the functions a, b, c and d which allows more elegant computations.

There is some consequences of this remark : the same equation has several forms. The properties of the functions a, b, c, and d will be called *intrinsic* if they do not depend on the choice of the form. For example, the properties of ${}^{\circ}a$ and ${}^{\circ}b$ are intrinsic. The point t_0 is not intrinsic, but ${}^{\circ}t_0$ is. The hypothesis $\dot{a}(t_0) \not\simeq \dot{b}(t_0)$ is intrinsic only because we have supposed that $\dot{a}$ and $\dot{b}$ are S^0. The index is not intrinsic, but his standard part is.

3 Main propositions

3.1 Statements and proofs

Proposition 1 *Suppose that the function $d - \dot{a}$ is not infinitesimal in the domain $\mathcal{D}$. Then, the change of variables*

$$x \;=\; a \,-\, \varepsilon\, \frac{d-\dot{a}}{c(y-b)} \tag{4}$$

transform the equation

$$\varepsilon \dot{x} \;=\; c\,(x-a)\,(x-b) \,+\, \varepsilon d \tag{3}$$

into the other equation of the same type

$$\varepsilon \dot{y} \;=\; c\,(y-a_1)\,(y-b) \,+\, \varepsilon d_1 \tag{5}$$

where the functions a_1 and d_1 are given by

$$d_1 \;=\; d \,+\, (\dot{b}-\dot{a})$$

$$a_1 \;=\; a \,-\, \frac{\varepsilon}{c}\left(\frac{\dot{d}-\ddot{a}}{d-\dot{a}} \,-\, \frac{\dot{c}}{c}\right)$$

Proof It is a straight forward computation which substitute (4) in (3). ∎

Remark The hypothesis "$d - \dot{a}$ is not infinitesimal in $\mathcal{D}$" is not intrinsic. But the weaker hypothesis $d(t_0) - \dot{a}(t_0) \not\simeq 0$ is. And this weaker hypothesis is enough to prove the existence of a standard domain around t_0 where the change of variable is valid.

Proposition 2 *Let $n \geq 1$. A solution $\bar{x}(t)$ of* (3) *is a canard with n poles if and only if it is transformed by the change of variables* (4) *into a canard $\bar{y}(t)$ with $n-1$ poles of* (5)*.*

Proof
- Let $\bar{y}$ a canard of (5) with $n-1$ poles. Let $\bar{x}$ the corresponding solution of (3). The formula (4) shows that the poles of $\bar{x}$ are the t where $\bar{y}(t) = b(t)$. We will count this values with a qualitative geometric study (see figure 2) of the equation (5) in the interval $[t_e, t_s]$:

- The points t such that $\bar{y}(t) = b(t)$ are all in the halo of t_0.
- At this points, the function $\dot{\bar{y}} - \dot{b} = d_1 - \dot{b} = d - \dot{a}$ has always the same sign. So the curves $\bar{y}(t)$ and $b(t)$ have transversal intersections always in the same direction.
- For geometrical elementary reasons (see figure 2), there is exactly n intersections of the curves $\bar{y}(t)$ et $b(t)$.

- Conversely, if $\bar{x}$ is a canard with n poles, $\bar{y} - b$ is vanishing exactly n times, and $\bar{y}$ has exactly $n-1$ poles.

∎

3.2 Geometrical meaning

The change of variables (4) is a sequence of elementary geometrical change of variables I will explain. All are homographic so that all the equations are Riccati equations.

- First we use a magnifying glass around the branch $\mathcal{L}_a$ of the slow curve :

$$x = a + \varepsilon u \qquad \text{which gives us}$$

$$\varepsilon \dot{u} = c(a-b)u + d - \dot{a} + \varepsilon c u^2 \tag{6}$$

The slow curve of this new Riccati equation is $u = {}^\circ\!\left(\frac{d-\dot{a}}{c(b-a)}\right)$. She has one simple pole at $t = {}^\circ t_0$. If $\bar{x}$ is a canard with n poles, the corresponding solution $\bar{u}$ of (6) go along the slow curve and has exactly n poles in the halo of t_0.

- Now we are going to angular coordinate φ (modulo π) on the cylinder of the Riccati equation :

$$u = \tan\varphi \qquad \text{gives} \qquad \varepsilon\dot{\varphi} = c(a-b)\sin\varphi\cos\varphi + (d-\dot{a})\cos^2\varphi + \varepsilon c \sin^2\varphi$$

We have now a slow curve with two branches : the first is $\varphi = {}^\circ\!\arctan\frac{d-\dot{a}}{c(b-a)}$, and the second $\varphi = \pi/2$. They intersect at ${}^\circ t_0$. If $\bar{x}$ is a canard with n poles, the corresponding solution $\bar{\varphi}$ is a canard. It intersect n times the axis $\varphi = \pi/2$ (the poles of $\bar{x}$). At this points, the sign of $\dot{\varphi} = c$ is always the same. When $\varphi = 0$, the sign of $\dot{\varphi} = d - \dot{a}$ is always the same, so we have exactly $n-1$ intersections between $\bar{\varphi}$ and the axis $\varphi = 0$, all of them in the halo of t_0.

- Now we put $\varphi = \psi + \pi/2$ to rotate the cylinder with an angle of $\pi/2$. It is possible to write directly with the variable u :

$$u = -1/v \qquad \text{gives} \qquad \varepsilon\dot{v} = -c(a-b)v + (d-\dot{a})v^2 + \varepsilon c$$

The slow curve has two branches : $v = 0$ and $v = {}^\circ\!\left(\frac{c(a-b)}{d-\dot{a}}\right)$ the canards are going along. The solution $\bar{v}$ corresponding to $\bar{x}$ is a canard with $n-1$ poles.

- For aesthetic reasons, we use one more change of variables, so that the slow curve has the same equation as initially :

$$v = \frac{c(y-b)}{d-\dot{a}}$$

The change of variables (4) of the proposition 1 is the composition of all the changes of variables above.

3.3 Index

Lemma 1 *From the equation* (3) *to the equation* (5)*, the standard part of the index is decreasing of 1.*

Proof It is a straight forward computation :

$$\begin{aligned} {}^{\circ}\!\left(\frac{d_1(t_0)-\dot{a}_1(t_0)}{\dot{a}_1(t_0)-\dot{b}(t_0)}\right) &= {}^{\circ}\!\left(\frac{(d(t_0)+\dot{b}(t_0)-\dot{a}(t_0))-\dot{a}(t_0)}{\dot{a}(t_0)-\dot{b}(t_0)}\right) \\ &= {}^{\circ}\!\left(\frac{d(t_0)-\dot{a}(t_0)}{\dot{a}(t_0)-\dot{b}(t_0)}\right) - 1 \end{aligned}$$

■

4 Theorems

4.1 Canards without poles

In addition to the hypothesis of paragraphs 2 and 3, we suppose that the standard part of the index, as a function of the parameter e, has a simple zero at e_0. The reason we put such a hypothesis is that the equation has to depend of the parameter.

To study the solutions without poles, we can now ignore that the equation (3) is of Riccati type. We apply non standard techniques and theorems (developed in [2, 9, 5]) to prove

Theorem 1 *If the equation* (3) *has a canard solution* $\bar{x}$ *without poles for some value* $\bar{e}$ *of the parameter, then :*

1. *The index* k *of* (3) *is infinitesimal, so* $\bar{e} \simeq e_0$.

2. *The real number* $\bar{e}$ *has an* ε*-shadow expansion, i.e. there exist standard real numbers* e_0, e_1,$\ldots e_p$, $\ldots$ *such that, for any standard integer* p,

$$\bar{e} \;=\; e_0 + e_1\varepsilon + e_2\varepsilon^2 + \ldots + e_p\varepsilon^p + \phi\varepsilon^p$$

where ϕ *is an infinitesimal real number.*

3. *The function $\bar{x}(t)$ has an ε-shadow expansion, i.e. there exist standard functions $x_0(t)$, $x_1(t)$,... $x_p(t)$, ... such that, for any standard integer p,*

$$\bar{x}(t) \; = \; x_0(t) + x_1(t)\varepsilon + x_2(t)\varepsilon^2 + \ldots + x_p(t)\varepsilon^p + \phi\varepsilon^p$$

where ϕ is an infinitesimal function of t.

4. *It is possible to compute the two expansions above with an identification in the equation* (3) *where e and x are substituted by formal series.*

Conversely, there exist some values of parameter e for which equation (3) *has canards without poles.*

Remark I put emphasis on the feasibility of the computation of the expansions (see [6] and below, paragraph 5.2)

4.2 Canards with poles, expansion of canard-values

Let n be a standard positive fixed integer.

Suppose, as before, that the standard part of $k - n$, as a function of e, has a unique simple zero in the studied domain.

Theorem 2 *If the equation* (3) *has a canard $\bar{x}_n(t)$ with n poles, for some value $\bar{e}_n$ of the parameter e, then :*

1. *The index k of* (3) *satisfy $^\circ k - n \simeq 0$.*

2. *The value $\bar{e}_n$ has an ε-shadow expansion.*

3. *One can compute this expansion with a formal identification of series.*

Conversely, there exist some values of parameter e for which equation (3) *has canards with n poles.*

Proof The idea is : do n times the change of variables (4); apply theorem 1 of canards without poles to the result; use the lemma 1 to watch the index.

It would be enough if the hypothesis on $d(t_0) - \dot{a}(t_0)$ is satisfied at each step. This hypothesis can be written $^\circ k_i \neq 0$ where k_i is the index of the i-th equation. To prove that $^\circ k_i \neq 0$, we will prove successively :

- If the index k is negative, non infinitesimal, there is no canard without poles (it is a corollary of theorem 1).

- If the index k is negative, non infinitesimal, there is even no canard with n poles. We can easily prove that by induction with propositions 1 and 2 and lemma 1.

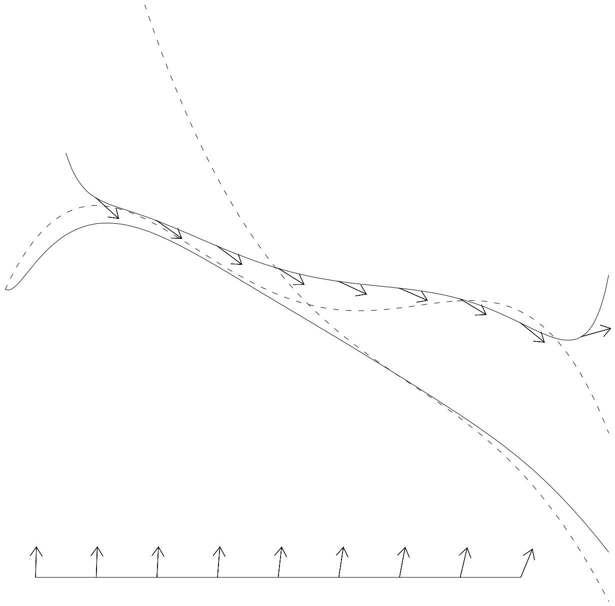

Figure 3: The trajectory y_0 is caught by $\bar{y}$ and $y = -\infty$.

- If $k \simeq -1/2$, one will prove that a solution y_0 which go along the attractive branch $\mathcal{L}_a$ for $t \lessapprox t_0$ has no pole, even in the halo of t_0 :

  We fix the parameter e (with index k near $-1/2$), and we introduce a new parameter h to study equation

$$\varepsilon \dot{y} = c(y-a)(y-b) + \varepsilon(d + (\dot{a}(t_0) - \dot{b}(t_0))h) \tag{7}$$

  The index is $k+h$. With theorem 1, we find a canard without pole $\bar{y}$ and the corresponding value $h_0 \simeq 1/2$ of equation (7).

  We can suppose that the initial condition of $\bar{y}$ is standard, between the two branches of the slow curve (see figure[‡] 3). It is easy to see that, in some standard neighborhood of t_0, the curves $y = \bar{y}(t)$ and $y = -\infty$ make a trap, so that y_0 has no pole.

- The curve y_0 above intersect the curve $y = b$ at most one time in the halo of t_0. That is because in the halo of t_0, when the function $y_0 - b$ vanish, his derivative $d - \dot{b}$ has always the same sign.

- If $k \simeq 1/2$, a solution x_0 which go along the attractive branch $\mathcal{L}_a$ for $t \lessapprox t_0$ has at most one pole in the halo of t_0. The reason is that, after the change of variables (4), the corresponding solution y_0 will satisfy the above conditions, and the poles of x_0 are the zeros of $y_0 - b$.

- If k is infinitesimal, we will prove that there is no canard with poles.

  Let $\bar{x}$ be a canard, solution for $h = 0$ of equation

$$\varepsilon \dot{x} = c(x-a)(x-b) + \varepsilon(d + (\dot{a}(t_0) - \dot{b}(t_0))h) \tag{8}$$

  with an initial point in the halo of $\mathcal{L}_a$. Let x_0 be the solution of equation (8) for $h = 1/2$, with the same initial point. According to the paragraphs above,

[‡] In the case $c < 0$, one have to make the figures upside down

x_0 is not a canard and has at most one pole in the halo of t_0. At a point t where $\bar{x}(t) = x_0(t)$, we have $\dot{\bar{x}} - \dot{x_0} < 0$, so x_0 is a trap for $\bar{x}$ (see figure 4). Because $\bar{x}$ is a canard, we can see that he has no pole.

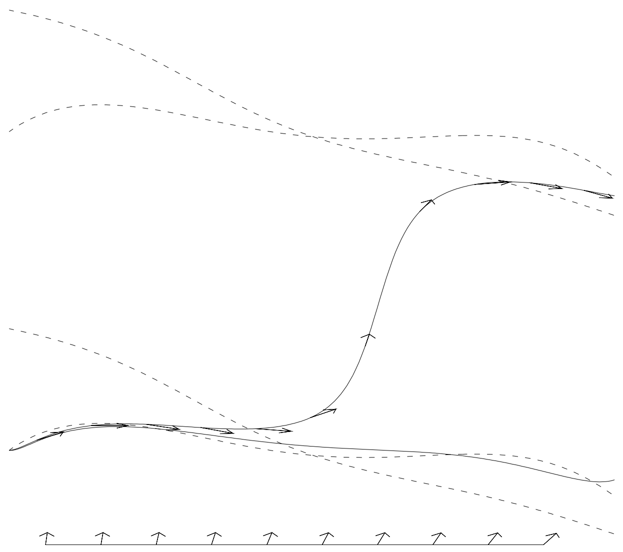

Figure 4: The trajectory $\bar{x}$ is caught by x_0. In fact, the trajectories are plotted on the covering of the cylinder.

The last assertion above prove that, when there is a canard with pole, it is possible to do the change of variables (4). So the idea of the proof, given first is valid. ∎

4.3 Expansion of canards with poles

Theorem 3 *If the equation* (3) *has a canard $\bar{x}_n(t)$ with n poles, for some value $\bar{e}_n$ of the parameter e, the canard with poles has an expansion :*

$$\bar{x}_n = a + \cfrac{-\varepsilon(d-\dot{a})/c}{-b + a_1 + \cfrac{-\varepsilon(d_1 - \dot{a}_1)/c}{-b + a_2 + \cfrac{-\varepsilon(d_2-\dot{a}_2)/c}{-b + \ddots + \cfrac{-\varepsilon(d_{n-1} - \dot{a}_{n-1})/c}{-b + z}}}} \tag{9}$$

- *where the a_i and d_i are given by induction by :*

$$a_{i+1} = a_i - \frac{\varepsilon}{c}\left(\frac{\dot{d}_i - \ddot{a}_i}{d_i - \dot{a}_i} - \frac{\dot{c}}{c}\right)$$

$$d_{i+1} = d_i + (\dot{b} - \dot{a}_i)$$

- *where the expansion of theorem* 2 *is substituted to e,*

- *where z is a ε-shadow expansion computed with an identification of formal series.*

Remark All the coefficients of the series above are regular at ${}^\circ t_0$; the poles of the canard are readable on the explicits quotients.

Proof It is only an application of the preceding theorems : one use n times the change of variables (4); in the result, one compute with formal identification the expansion of the canard without poles z; one use the reverse change of variables. ■

Remark If we make the formal divisions in the formula (9), we obtain an expansion of $\bar{x}_n$ of this type :

$$\bar{x}_n \;=\; {}^\circ a + x_1\varepsilon + x_2\varepsilon^2 + \ldots + x_p\varepsilon^p + \not{\phi}\varepsilon^p \tag{10}$$

but the $x_i(t)$ have poles at ${}^\circ t_0$.

In fact, this expansion (10) may be obtained by direct identification of formal series. But, first it is not possible to characterize the canards-values $\bar{e}_n$ by this method, second, the expansion doesn't give any approximation on the solutions in neighborhood of t_0 of size of order $\sqrt{\varepsilon}$.

5 Examples and effective computations

5.1 The simplest example

In this paragraph, I will only illustrate the theory above on a very well known example : the Hermite equation

$$\frac{d^2X}{dT^2} - T\frac{dX}{dT} + eX \;=\; 0$$

and the asymptotic behaviour of the solutions at infinity.

To move the problem from infinity to visible domain, we use a *macroscope* on the variables T and X. To have a Riccati equation, we do the usual change of variable. So $T = t/\sqrt{\varepsilon}$ and $x = \frac{X}{dX/dt}$ give the slow-fast Riccati equation[§]

$$\varepsilon\dot{x} \;=\; ex^2 - tx + \varepsilon \;=\; e(x - t/e)x + \varepsilon \tag{11}$$

We put

$$a = t/e \qquad b = 0 \qquad c = e \qquad d = 1$$

This functions satisfy all the hypothesis of the paragraphs above, provided that e is appreciable. An easy computation give $e-1$ for index. The inductive computations of a_i and d_i are here exceptionally simple and give

$$a_i = t/e \qquad d_i = 1 - i/e$$

[§]The change of variable $x = y + t/2e$ gives the equation $\varepsilon\dot{y} = ey^2 - t^2/e + \varepsilon(1 - t/2e)$ which is the Ricatti equation related to the quadratic-potential Schrödinger equation.

To have a canard solution with n poles of equation (11), we know that the index must satisfy $e \simeq n+1$. The n change of variables give

$$\varepsilon \dot{z} = e(z - t/e)z + \varepsilon(1 - n/e)$$

The general method of this paper consist now to substitute formal series to e and z and to identify the formal series. Here, we have an exceptional case : the expansions have a unique nonzero term : for any standard integer p, we have :

$$\bar{e}_n = n + 1 + \phi\varepsilon^p \qquad \bar{z}_n = \frac{t}{n+1} + \phi\varepsilon^p$$

Of course, we can see directly that $e = n+1$, $z = t/(n+1)$ is a canard without poles, but we have to remember that there are others canards, exponentially near this one (see [2]).

With the reverse change of variables of (4), we obtain the expansions of the canards with n poles of (11) :

$$\bar{x}_n = \frac{t}{\bar{e}_n} + \cfrac{-\varepsilon\left(1 - \frac{1}{\bar{e}_n}\right)/\bar{e}_n}{\frac{t}{\bar{e}_n} + \cfrac{-\varepsilon\left(1 - \frac{2}{\bar{e}_n}\right)/\bar{e}_n}{\frac{t}{\bar{e}_n} + \cfrac{-\varepsilon\left(1 - \frac{3}{\bar{e}_n}\right)/\bar{e}_n}{\frac{t}{\bar{e}_n} + \ddots + \cfrac{-\varepsilon\left(1 - \frac{n}{\bar{e}_n}\right)/\bar{e}_n}{\bar{z}_n}}}}$$

We can make it easier readable :

$$\bar{e}_n\bar{x}_n = t + \cfrac{-\varepsilon(\bar{e}_n - 1)}{t + \cfrac{-\varepsilon(\bar{e}_n - 2)}{t + \cfrac{-\varepsilon(\bar{e}_n - 3)}{t + \ddots + \cfrac{-\varepsilon(\bar{e}_n - n)}{\bar{e}_n\bar{z}_n}}}}$$

Substituting $\bar{e}_n$ and $\bar{z}_n$ with their expansions :

$$(n + 1 + \phi\varepsilon^p)\bar{x}_n = t + \cfrac{-\varepsilon(n + \phi\varepsilon^p)}{t + \cfrac{-\varepsilon(n - 1 + \phi\varepsilon^p)}{t + \cfrac{-\varepsilon(n - 2 + \phi\varepsilon^p)}{t + \ddots + \cfrac{-\varepsilon(1 + \phi\varepsilon^p)}{t + \phi\varepsilon^p}}}}$$

The particular solution

$$\bar{e}_n = n+1 \qquad (n+1)\bar{x}_n \;=\; t + \cfrac{-\varepsilon n}{t + \cfrac{-\varepsilon(n-1)}{t + \cfrac{-\varepsilon(n-2)}{t + \cfrac{\ddots}{+\cfrac{-\varepsilon}{t}}}}}$$

correspond to the Hermite polynomial of degree $n+1$.

5.2 Algorithms

In the general case, it is very hard to do the computation by hand. So I have written a *Maple*-program to compute the expansions of theorems 1, 2 and 3, as soon as the functions a, b, c, and d are given by explicit C^∞ formulas. The program is given below in appendix. I hope it is readable, and I know it is not the most speedy.

To show the complexity of the obtained formulas, I will give the results in the case of the double symmetric well $V(t) = (t^2-1)^2$, for the third energy-value (canards with two poles). The terms of magnitude ε^3 will be neglected in the approximations.

$$e \;=\; 10 - \frac{19}{2}\varepsilon \;-\; \frac{555}{32}\varepsilon^2 \;+\; \phi\varepsilon^2$$

$$\begin{aligned} z(t) \;&=\; -\,(t^2-1) \;-\; \frac{9t^2-86t+145}{(t-5)(3t-5)(t+1)}\,\varepsilon \;- \\ &\quad -\; \frac{399t^6 - 4218t^5 + 14285t^4 - 4252t^3 - 89031t^2 + 188918t - 159925}{4(3t-5)^2(t+1)^3(t-5)^3}\,\varepsilon^2 \;+ \\ &\quad +\, \phi\varepsilon^2 \end{aligned}$$

$$x(t) \;=\; -(t^2-1) + \varepsilon \frac{-e+2t}{-2(t^2-1) + \varepsilon\dfrac{2}{-e+2t} + \varepsilon\dfrac{-e+6t+4\dfrac{\varepsilon}{(-e+2t)^2}}{-t^2+1+z(t)}}$$

We have to remember that e and $z(t)$ will be substituted by the series above.

Figures 5 and 6 show how good the approximations are :

The figure 5 is the graph of the function $x(t)$ above, where the infinitesimal functions ϕ are replaced by 0, and $\varepsilon = 1/25$. The value of e is then 9.59225.

On the figure 6, I have plotted a numerical solution of the same equation, with the arbitrary initial condition $(-0.55, 0)$. The parameter e is selected with a dichotomic process so that this numerical trajectory has two poles and go at best along the slow curve. The value of e is then 9.587227...

On the figure 5 we can see that the truncated expansion of x is a very good approximation of the solution in a standard neighborhood of the studied point $t = 1$. But, we can also see that for $t < -1/2$ the approximation becomes very bad. In the

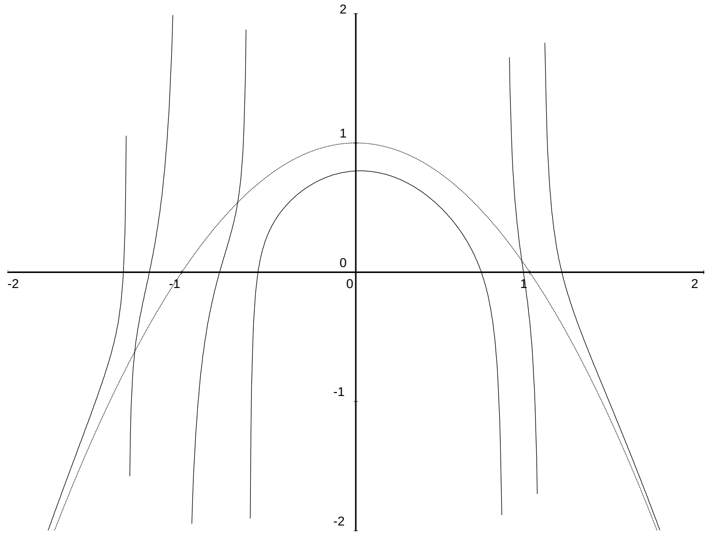

Figure 5: Numerical plot of the expansion of a canard with poles.

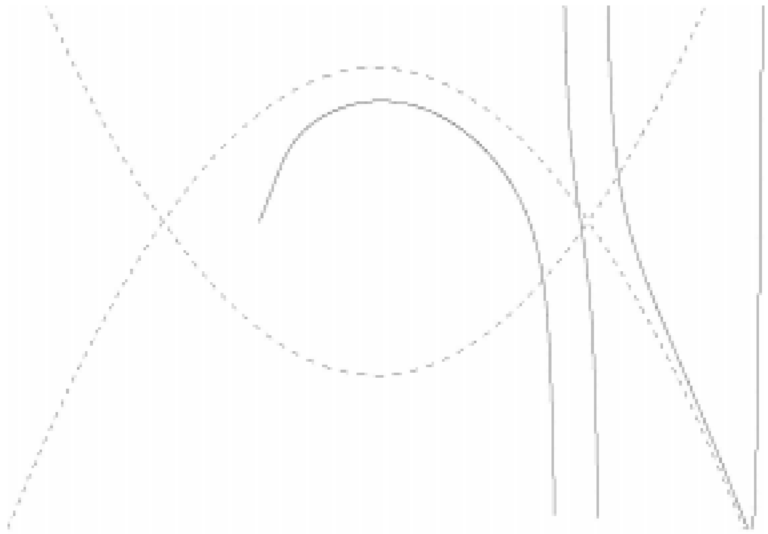

Figure 6: Numerical plot of a canard with poles.

change of variable of the main proposition, we supposed that $d - \dot{a}$ didn't vanish. It is true near the studied point $t = 1$, but far from it, we have zeroes of this function, at some level of the recursive computations. This zeroes can give poles in the expansion of x.

However, the trajectory in the figure 6 is drawed only for $t > -0.55$, and for $t \simeq -1$ we should have poles, and the trajectory is almost surely not a canard.

6 Conclusion

6.1

If the functions a, b, c or d are not analytic, the series computed in this paper are not Gevrey. But the results here are still available, although the theory of resurgents functions is not available.

If the functions a, b, c or d are only C^r, the same computations are available up to an order p. It is possible to determine p as in [15]. We obtain the "Matkowski conditions".

6.2

An open problem is now to understand the computations above when the number of poles n is non limited, of order $1/\varepsilon$. All the formulas stay true, but the expansions are not easily understandable because the d_i are not limited, the a_i are not all infinitely near of a.

This is connected with the problem of the determination of canards in the equation

$$\varepsilon\dot{x} = \alpha(t)\,x^2 + \beta(t)\,x + \gamma(t)$$

where $\beta^2 - 4\alpha\gamma$ can be negative appreciable.

Appendix : The *Maple*-program

#

```
# This is a program to compute the eps-shadow expansion of the
# canard with n poles in the Riccati equation
#
#       eps dx/dt  =  c ( x - a ) ( x - b )  +  eps d
#
# where a , b , c , d are functions of the time t and the energy e
#
#SYNTAX :
#     a , b , c , d , are the functions in the equation
#     n  is the number of poles
#     p  is the order of the computed expansions
#     t0 is the intersection of the two branchs of the slow curve,
```

```
#       where we are looking for the canards.
#


#The result has the following form:
#   ee[0],ee[1],ee[2],... gives the expansion of the energy
#   zz[0],zz[1],zz[2],... gives the expansion of the canard
#                         without poles
#   xxx will be the expansion of the canards with poles
#   yyy will be the meromorphic expansion

##############      Some examples of data : ###############

#EXEMPLE : Hermite
# a:=t/e;   b:=0;   c:=e;   d:=1;   t0:=0;   n:=4;   p:=5;

#EXEMPLE : Double symmetric well :
  a:=(-t*t+1);   b:=t*t-1;   c:=-1;   d:=-e;   t0:=1;  n:=2;  p:=2;

################## Values of epsilon to plot ##############
 epsN := .04;

######################### Index ##########################
bp:=diff(b,t) : ap:=diff(a,t):
indice:=normal(subs(t=t0,(d-ap)/(ap-bp)));


###########################################################

a[0]:=a : d[0]:=d :             #a[] will be the sequence of a.i
bp:=diff(b,t) : cp:=diff(c,t) : #the "p" shows the derivation

############### Change of variables #######################

for i from 0 to n-1 do
  ap := diff(a[i],t) :
  d[i+1] := d[i] + bp - ap :
  a[i+1] := a[i] - (eps/c)*( diff(d[i]-ap,t)/(d[i]-ap) - cp/c ) :
  print(‘I did ‘,i+1,‘changes of variable‘):
od:

################### Identification of series ###############
##################### to compute z(t) #####################

zz[0](t) := subs(eps=0,e=ee[0],a[n]) :

zero:=subs(e=convert([’ee[i]*eps^i’ $ ’i’=0..0] , ‘+‘),
   z(t)=convert([’zz[i](t)*eps^i’ $ ’i’=0..1] , ‘+‘ ),
   eps*diff(z(t),t) -( c*(z(t)-a[n])*(z(t)-b) + eps*d[n])):
otherzero:=coeff(convert(taylor(zero,eps,2),polynom),eps,1):
ee[0]:=-coeff(expand(subs(t=t0,otherzero)),ee[0],0)/
```

```
   coeff(expand(subs(t=t0,otherzero)),ee[0],1):
print(e = sum('ee[j]*eps^j', 'j'=0..0)+infinitesimal*eps^0):

for i from 0 to p-1 do
  zz[i+1](t):=factor(-coeff(expand(otherzero),zz[i+1](t),0)/
     coeff(expand(otherzero),zz[i+1](t),1)):
  zero:=subs(e=convert(['ee[i]*eps^i' $ 'i'=0..i+1] , `+`),
     z(t)=convert(['zz[i](t)*eps^i' $ 'i'=0..i+2] , `+` ),
     eps*diff(z(t),t) -( c*(z(t)-a[n])*(z(t)-b) + eps*d[n])):
  otherzero:=coeff(convert(taylor(zero,eps,i+3),polynom),eps,i+2):
  ee[i+1]:=-coeff(expand(subs(t=t0,otherzero)),ee[i+1],0)/
     coeff(expand(subs(t=t0,otherzero)),ee[i+1],1):
  print(e = sum('ee[j]*eps^j', 'j'=0..i+1)+infinitesimal*eps^(i+1)):
od:

print(`I am computing x(t)`):
x:=z:
for i from n-1 by -1 to 0 do
   x:=a[i]+(-eps*(d[i]-diff(a[i],t))/c)/(-b+x):
od:

print(`I make the pictures`):

for i from 0 to p do
  zzz[i]:=convert(['zz[j](t)*eps^j' $ 'j'=0..i] , `+`):
  eee[i]:=convert(['ee[j]*eps^j' $ 'j'=0..i] , `+`):
  xxx[i]:=subs(e=eee[i],z=zzz[i],x):
  yyy[i]:=convert(taylor(xxx[i],eps,i+1),polynom):
od:

print('e'=subs(eps=epsN,['eee[j]' $ 'j'=0..p])):
interface(plotdevice=postscript,plotoutput=`fig5.ps`):
#plot({subs(eps=epsN,zzz[p]),zzz[0]},-2..2,-2..2,numpoints=500,style=LINE);
plot({subs(eps=epsN,xxx[p]),zzz[0]},-2..2,-2..2,numpoints=1000,style=LINE);
#plot({subs(eps=epsN,yyy[p]),zzz[0]},-2..2,-2..2,numpoints=500,style=LINE);
quit;
\#
```

References

[1] E. Benoît. Canards et enlacements. *Publications de l'Inst. des Hautes Etudes Scient.*, 72:63–91, 1990.

[2] E. Benoît, J.L. Callot, F. Diener, and M. Diener. Chasse au canard. *Collectanea Mathematica*, 31–32(1–3):37–119, 1981.

[3] J.L. Callot. Bifurcation du portrait de phase pour des équations différentielles linéaires du second ordre ayant pour type l'équation d'Hermite. Thèse d'état, I.R.M.A., 7, rue R. Descartes F67084 Strasbourg Cedex, 1981.

[4] J.L. Callot. Solutions visibles de l'équation de Schrödinger. In M. Diener and G. Wallet, editors, *Mathématiques finitaires et Analyse Non Standard*, pages 105–119. Publications mathématiques de l'Université Paris 7, 1985. volume 31(1), (publié en 1989).

[5] M. Canalis-Durand. Solution formelle Gevrey d'une équation singulièrement perturbée. Prépublications de l'Université de Nice, Mathématiques volume 277, juin 1990.

[6] M. Canalis-Durand, F. Diener, and M. Gaetano. Calcul des valeurs à canard à l'aide de Macsyma. In M. Diener and G. Wallet, editors, *Mathématiques finitaires et Analyse Non Standard*, pages 153–167. Publications mathématiques de l'Université Paris 7, 1985. volume 31(1), (publié en 1989).

[7] E. Delabaere and H. Dillinger. Contribution à la résurgence quantique. résurgence de Voros et fonction spectrale de Jost. Thèse, Université de Nice, 1991.

[8] E. Delabaere, H. Dillinger, and F. Pham. Développements semi-classiques exacts des niveaux d'énergie d'un oscillateur à une dimension. *Comptes-Rendus de l'Académie des Sciences de Paris, série I*, 310:141–146, 1990.

[9] F. Diener. Développements en ε-ombres. In I.D. Landau, editor, *Outils et modèles mathématiques pour l'automatique, l'analyse des systèmes et le traitement du signal*, pages 315–328. Editions du C.N.R.S., 1983. volume 3.

[10] F. Diener and G. Reeb. *Analyse Non Standard.* Collection Enseignement des Sciences. Hermann, Paris, 1989.

[11] M. Diener. Etude générique des canards. Thèse d'état, I.R.M.A., 7, rue R. Descartes F67084 Strasbourg Cedex, 1981.

[12] R.B. Dingle. *Asymptotic Expansions: Their Derivation and Interpretation.* Academic Press, London and New York, 1973.

[13] J. Ecalle. Les fonctions résurgentes. Publications mathématiques d'Orsay, Université de Paris-Sud, Département de Mathématiques, bat 425, 91425 Orsay, France, 1985.

[14] A. Gaignebet. Equation de Schrödinger unidimensionnelle stationnaire. Quantification dans le cas d'un double puits de potentiel symétrique. *Comptes-Rendus de l'Académie des Sciences de Paris, série I*, 315:113–118, 1992.

[15] V. Gautheron and E. Isambert. Finitely differentiable ducks and finite expansions. In E. Benoît, editor, *Dynamic Bifurcations*, pages 40–56. Springer Verlag, 1991. Lecture Notes in Mathematics, volume 1493.

[16] J. Grasman and B.J. Matkowsky. A variational approach to singularly perturbed boundary value problems for ordinary and partial differential equations with turning points. *SIAM Journal on Applied Mathematics*, 32(3):588–597, May 1977.

[17] B.J. Matkowsky. On boundary layer problems exhibiting resonance. *SIAM Review*, 17(1):82–100, 1975.

[18] T.F. Pankratova. Quasimodes and exponential splitting of eigenvalues. *Problemy Math. Phys. vyp.*, 11:167–177, 1986.

Département de Mathématiques,
Université de la Rochelle,
avenue Marillac,
17042 LA ROCHELLE Cedex France