Noncommutative Algebraic Geometry: from pi-algebras to quantum groups

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Abstract

The main purpose of this paper is to provide a survey of different notions of algebraic geometry, which one may associate to an arbitrary noncommutative ring R. In the first part, we will mainly deal with the prime spectrum of R, endowed both with the Zariski topology and the stable topology. In the second part we focus on quantum groups and, in particular, on schematic algebras and show how a noncommutative site may be associated to the latter. In the last part, we concentrate on regular algebras, and present a rather complete up to date overview of their main properties.

Introduction.

The main purpose of this paper is to present a survey of the subject commonly known as "noncommutative algebraic geometry". The first two sections treat the prime spectrum of a noncommutative ring, endowed with its canonical structure sheaf. This approach is useful for algebras with enough prime ideals, like algebras satisfying a polynomial identity (pi-algebras for short). Allowing for a more general topology (induced by Artin-Rees ideals) provides a geometry for rings with the so-called second layer condition. However, many interesting algebras fall outside the scope of these techniques. They arise naturally in the study of quantum groups, an introduction to which is given in section 3.

The last two sections are devoted to projective noncommutative geometry. The central object of study here is the quotient category $Proj$, which, for a commutative algebra, is equivalent to the category of quasi-coherent sheaves on its projective variety. There is a large class of graded algebras, the so-called schematic algebras,

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which allow the construction of a generalized categorical topological space such that the same property holds for their *Proj*. This is the content of section 4. The last section studies regular algebras, i.e., graded algebras which satisfy a strong homological condition. For instance, if an algebra is commutative and regular, then it is a polynomial algebra. Therefore, regular algebras of global dimension d are considered as noncommutative ("quantum") $\mathbb{P}^{d-1}$'s. Regular algebras of dimension d not bigger than 3 have been classified using a tight connection between their defining relations and a subvariety of $\mathbb{P}^{d-1}$. Recently, one has discovered that this connection is more loose if $d \geq 4$. The account in section 5 is a short survey of the results so far in this exciting subject.

Lack of time and space prevented us from being complete – we do apologize to authors whose efforts have not been reviewed. In any case, we have tried to provide the reader with a complete reference list.

1 Rings with polynomial identity

Throughout this text, we fix a field k, which we assume to be algebraically closed and of characteristic zero, for simplicity's sake. All rings are supposed to be algebras over this field k.

1 In the previous decades, most attempts to construct what should be referred to as "noncommutative algebraic geometry" find their origin in Grothendieck's innovative ideas about classical (commutative) algebraic geometry, which amount, up to a certain level, to constructing a dictionary between ring theory and algebraic geometry.

This is realized by associating to any commutative ring R the affine scheme $(Spec(R), \mathcal{O}_R)$, where the set $Spec(R)$ of all prime ideals of R is endowed with the Zariski topology and where $\mathcal{O}_R$ is the structure sheaf over it, canonically associated to R and, conversely, associating to any scheme $(X, \mathcal{O}_X)$, the ring of global sections $\Gamma(X, \mathcal{O}_X)$. Problems concerning R are thus translated to equivalent problems about the affine scheme $Spec(R)$ and may thus be tackled by applying geometric methods.

We should also point out that for any ring R, the category $R-\mathbf{mod}$ of modules over R is equivalent to the category of quasi-coherent sheaves of $\mathcal{O}_R$-modules on $Spec(R)$, which allows to apply geometric and cohomological methods to the study of R-modules as well.

Since this translation from ring theory to algebraic geometry (and vice versa) has proven to be extremely fruitful, obvious attempts were (and are) made, to develop an analogue for noncommutative rings of Grothendieck's approach to algebraic geometry.

Of course, aiming at such a construction, some non-obvious choices have to be made, if we naively wish to view a geometric object associated to a noncommutative ring R as a triple $(X, \mathcal{T}_X, \mathcal{O}_X)$, where X is a space of "points", $\mathcal{T}_X$ a topology on X and $\mathcal{O}_X$ a sheaf of rings on the topological space $(X, \mathcal{T}_X)$.

Let us already point out here that defining geometric objects in this way is only one of many alternative points of view, as we will see below.

2 Although several alternative approaches have been considered, cf. [17, 23, 25, 52, 73], e.g., we will restrict in the first part of this text to the choice of associating to any ring R its spectrum $Spec(R)$, which consists of all two-sided prime ideals of R, i.e., two-sided ideals P of R with the property that $xRy \subseteq P$ implies $x \in P$ or $y \in P$, for any pair of elements $x, y \in R$.
The motivation for this is twofold. First of all, we should stress that, at this point, it seems unlikely that one might succeed in developing a "useful" noncommutative algebraic geometry for *arbitrary* rings. If one expects noncommutative algebraic geometry to be of any help to the study of noncommutative rings, by allowing methods similar to those in the commutative set-up, one should restrict to the geometric study of rings, which are not *too noncommutative*. The present text does not aim to define what should be meant precisely by this term. Let us just mention that most rings encountered in real-life applications are of this type, including pi-algebras or, more generally, fully bounded noetherian (fbn) rings, group rings, enveloping algebras and even most quantum groups. These rings have the property of possessing a sufficiently large prime spectrum, allowing a geometric treatment, as well as the possibility of proving local-global results.

3 The second reason for studying $Spec(R)$ stems from the fact that the prime spectrum arises rather naturally within the framework of representation theory.
Usually a representation of dimension n of a k-algebra R is defined to be a k-algebra map $\pi : R \to M_n(k)$, where $M_n(k)$ is the ring of $n \times n$ matrices over k. One calls π *irreducible*, if it is surjective. Of course, the kernel $Ker(\pi)$ is then a maximal ideal of R.
Two representations π and π' of R (of the same dimension n) are equivalent, if they differ by an inner automorphism of $M_n(k)$. If the representations are irreducible, this is easily seen to be equivalent to $Ker(\pi) = Ker(\pi')$. In this way, the set $Max(R)$ of maximal ideals of R may be decomposed into a disjoint union

$$Max(R) = Max_1(R) \cup \ldots \cup Max_n(R) \cup \ldots \cup Max_\infty(R),$$

where, for each positive integer n, the maximal ideals in $Max_n(R)$ are those corresponding to equivalence classes of irreducible representations of dimension n and where $Max_\infty(R)$ consists of the remaining maximal ideals.
Typically, if

$$R = \begin{pmatrix} k[X] & k[X] \\ (X) & k[X] \end{pmatrix}$$

then $Max(R) = Max_1(R) \cup Max_2(R)$, where $Max_1(R) = \{M_-, M_+\}$, with

$$M_- = \begin{pmatrix} k[X] & k[X] \\ (X) & (X) \end{pmatrix} \text{ resp. } M_+ = \begin{pmatrix} (X) & k[X] \\ (X) & k[X] \end{pmatrix}$$

and where $Max_2(R)$ consists of all $M_\alpha = (X - \alpha)R$, for $0 \neq \alpha \in k$.
Of course, at the other extreme, it may occur that for all positive integers n the set $Max_n(R)$ is empty, the Weyl algebra

$$A_1(k) = k\{X, Y\}/(YX - XY - 1)$$

being an obvious example.

4 More generally, consider a ring homomorphism $\alpha : R \to M_n(K)$, where K is a field. It is easy to see that $\alpha(R)$ generates $M_n(K)$ over K if and only if for any field $L \supseteq K$ the induced representation

$$\alpha_L : R \xrightarrow{\alpha} M_n(K) \hookrightarrow M_n(L)$$

is irreducible. We call such representations *absolutely irreducible* (of degree n). Absolutely irreducible representations $\alpha : R \to M_n(K)$ and $\beta : R \to M_m(L)$ are said to be *equivalent* if $m = n$, and if there exists an extension field H of K and L, such that the induced representations $\alpha_H, \beta_H : R \to M_n(H)$ coincide up to an H-automorphism of $M_n(H)$.
It is fairly easy to see (cf. [51, 73]) that the kernel $Ker(\alpha)$ of any absolutely irreducible representation is a prime ideal of R, and that two absolutely irreducible representations α and β of R are equivalent if and only if $Ker(\alpha) = Ker(\beta)$. In this way, there is, just as for maximal ideals, a bijective correspondence between equivalence classes of absolutely irreducible representations of R and those prime ideals P of R for which R/P is a pi-algebra.
Actually, just as before, this leads to a decomposition of $Spec(R)$ of the form

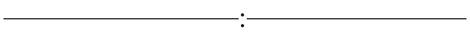
$$Spec(R) = Spec_1(R) \cup \ldots \cup Spec_n(R) \cup \ldots \cup Spec_\infty(R),$$

where the prime ideals P of $Spec_n(R)$ correspond to absolutely irreducible representations of degree n (equivalently, such that R/P has pi-degree n), and where $Spec_\infty(R)$ contains the remaining prime ideals.

5 We endow $Max(R)$ with the *Zariski topology*, whose closed sets are the

$$V(S) = \{M \in Max(R);\ S \subseteq M\},$$

for some subset $S \subseteq R$ (which may obviously be assumed to be a two-sided ideal of R), and with open sets $D(S) = Max(R) - V(S)$. It has been proved by Artin [5] that for any positive integer n the corresponding $Max_n(R)$, with the induced topology, possesses the structure of an ordinary algebraic variety and is locally closed in $Max(R)$. (For example, if $R = k\{X, Y\}$, the free algebra in two variables, then $Max_n(R)$ is an algebraic variety of dimension $n^2 + 1$.)
If one wishes to study $Max(R)$, one thus essentially has to know how these algebraic varieties fit together. For the example given in 3, it appears that topologically $Max(R)$ is just the usual affine line $\mathbb{A}^1_*$ over k with split origin:



Of course, in other examples, the gluing of the separate components $Max_n(R)$ may be much more complex and will thus necessitate the use of structure sheaves (which, by the way, are also needed in order to differentiate between the "affine" variety $Max(R)$ with R as before, and the "non-affine" (non-separated !) variety $\mathbb{A}^1_*$).

6 Besides the basic obstruction that sometimes $Max(R) = Max_\infty(R)$ (and that the previous method thus does not work!), another problem to be dealt with is that, although the structure of each of the components $Max_n(R)$ may be studied individually, the fact that there may be an infinite number of them for arbitrary rings makes it hardly possible to provide an easy description of $Max(R)$ in terms of the $Max_n(R)$. In order to remedy this, one restricts to rings with only a finite number of non-empty $Max_n(R)$. Actually, starting from an arbitrary ring R and a positive integer d, it is fairly easy to canonically construct a ring R_d such that $Max(R_d) = \bigcup_{n=1}^{d} Max_n(R)$.

Indeed, let us first recall that there are certain polynomial identities, which are satisfied by $d \times d$ matrices over k. Indeed, one may prove that $S_{2d} = 0$ is such an identity for $M_d(k)$, where for any positive integer p, we put

$$S_p(X_1, \ldots, X_p) = \sum_{\sigma \in \mathcal{S}_p} (-1)^{\sigma} X_{\sigma(1)} \ldots X_{\sigma(p)},$$

where $\mathcal{S}_n$ is the permutation group on n elements. For example, for $d = 1$, we thus obtain the identity

$$S_2(X_1, X_2) = X_1 X_2 - X_2 X_1 = 0,$$

which just expresses the commutativity of the base field k.

One may show that $S_{2d} = 0$ is *minimal* for $M_d(k)$, in the sense that $S_p = 0$ is not an identity for $M_d(k)$ if $p < 2d$.

Denote by $I_d(R)$ the two-sided ideal of R generated by all substitutions of elements of R in S_{2d} and put $R_d = R/I_d(R)$. It is then obvious that there exist no irreducible representations of R_d into $M_n(k)$ for $n > d$, as S_{2d} vanishes on R_d and hence also on any homomorphic image of R_d. It follows that

$$Max(R_d) = \bigcup_{n=1}^{d} Max_n(R_d),$$

and since $Max(R_n)$ may be identified with the closed subset $V(I_n(R))$ of $Max(R)$ consisting of all maximal ideals of R containing $I_n(R)$, also that

$$Max(R_d) = \bigcup_{n=1}^{d} Max_n(R).$$

Passing from R to the quotient R_d thus truncates the possibly infinite collection of $Max_n(R)$ up to $Max_d(R)$. Moreover, R_d satisfies the identities of $d \times d$ matrices, hence is what is usually referred to as a pi-algebra.

Of course, the same technique works for the prime spectrum of R, where one obtains that

$$Spec(R_d) = \bigcup_{n=1}^{d} Spec_n(R).$$

2 Structure sheaves

1 Let us work over an arbitrary algebra R for a moment. Endowing $Spec(R)$ with its Zariski topology (with open sets $D(S)$ consisting of all prime ideals P with $S \not\subset P$), the techniques expounded in the previous section allow to study R

geometrically by decomposing $Spec(R)$ into its components $Spec_n(R)$, each of these having a (commutative!) scheme structure. Of course, one of the main problems one is faced with here, is to see how these components fit together within $Spec(R)$. In order to study this, one is obviously lead to endow $Spec(R)$ with a structure sheaf, which permits to glue together local information into global one.

A complete treatment of these constructions over pi-algebras may be found in [73], so we will not go into details here, preferring to provide some information about constructions that permit to tackle more general rings.

2 Assume R to be an arbitrary left noetherian ring and suppose R to be prime, i.e., (0) to be a prime ideal of R. In this case, it is well known that R possesses a classical ring of fractions $Q(R)$, i.e., the set of regular elements of R is an Ore-set, and the ring of fractions $Q(R)$ associated to it is simple and artinian, cf. [26, 43] for details. To any two-sided ideal I of R, one may associate a subring $Q_I(R) \subseteq Q(R)$, consisting of all $q \in Q(R)$ which may be multiplied into R by some positive power I^n of I. If R is commutative and $f \in R$, then it is easy to see that $Q_f(R)$ is just the usual localization R_f at the multiplicative set generated by f.

Using "abstract localization", such as introduced by Gabriel [19] (see also [62] for a complete survey), it was proved in [46, 70] that associating to any open subset $D(I)$ of $Spec(R)$ the ring $Q_I(R)$ defines a sheaf of rings $\mathcal{O}_R$ on $Spec(R)$ (endowed with its Zariski topology), whose ring of global sections $\Gamma(Spec(R), \mathcal{O}_R)$ reduces to the ring R.

3 The reason why "abstract localization" comes into the picture in the present context, is that any two-sided ideal I of R also defines a so-called idempotent kernel functor σ_I in $R-\mathbf{mod}$ (cf. [15, 22, 24, 71, et al]) by letting for any left R-module M the submodule $\sigma_I M$ consist of all $m \in M$ annihilated by some power of I. To any such idempotent kernel functor σ in $R-\mathbf{mod}$ (a left exact subfunctor σ of the identity in $R-\mathbf{mod}$ such that $\sigma(M/\sigma M) = 0$ for any left R-module M), one may canonically associate a "localization functor" Q_σ (see again [15, 22, 24, 71, et al]) and it appears that for $\sigma = \sigma_I$, one has $Q_\sigma(R) = Q_I(R)$.

Abstract localization theory also permits to calculate the stalks of the above structure sheaf. Actually, to any prime ideal P of R, one may associate an idempotent kernel functor σ_{R-P} defined by letting $\sigma_{R-P}M$ consist for any $M \in R-\mathbf{mod}$ of all $m \in M$ with the property that $Im = 0$ for some two-sided ideal $I \not\subset P$. Let us denote by Q_{R-P} the localization functor associated to σ_{R-P}. Then one easily verifies that the stalk of $\mathcal{O}_R$ at P is given by $\mathcal{O}_{R,P} = Q_{R-P}(R)$. This generalizes the commutative case, as one may verify that if R is commutative, then $Q_{R-P}(R)$ is just the usual localization R_P of R at P.

4 Although the previous construction possesses several nice features, it suffers from the fact of only being applicable to prime rings (and, in particular, not to arbitrary left R-modules) and of not behaving functorially. In order to remedy this, alternative constructions have been developed, based on ideals I satisfying the so-called *(left) Artin-Rees condition*, which says (in one of its many forms) that for any left ideal L there exists some positive integer n such that $I^n \cap L \subseteq IL$, cf. [15, 43].

Although this is somewhat hidden in many proofs, the reason why constructions in the commutative, noetherian case work so well is that, due to Krull's Lemma, all ideals satisfy the Artin-Rees condition, in this case.

Unfortunately, in the noncommutative case, this is no longer valid in general. It thus makes sense to modify the Zariski topology in the noncommutative (noetherian) case, by only considering open subsets of the form $D(I)$, where I is a two-sided ideal of R satisfying the Artin-Rees condition. Since it is clear that finite sums and products of Artin-Rees ideals again satisfy the Artin-Rees condition, it is rather easy to check that this thus defines a topology on $Spec(R)$, indeed, the so-called *Artin-Rees topology* $\mathcal{T}(R)$.

It has been verified in [15] that associating for any left R-module M to $D(I) \in \mathcal{T}(R)$ the localization $Q_I(M)$ of M at σ_I defines a sheaf $\mathcal{O}_M$ on $(Spec(R), \mathcal{T}(R))$, with the property that $\Gamma(Spec(R), \mathcal{O}_M) = M$.

5 It is clear that the efficiency of representing rings and modules over them by sheaves over the topological space $(Spec(R), \mathcal{T}(R))$ is highly dependent upon working over a topology "sufficiently close" to the Zariski topology, i.e., we want the base ring R to have "many" Artin-Rees ideals. For this reason, as expounded in [15], our methods work best over rings satisfying the so-called *second layer condition*.

As this follows outside of the scope of the present text, we refer to the literature for precise definitions and properties of this notion (cf. [13, 15, 26, 28, 43]). Let us just mention that the class of rings satisfying this condition is extremely vast, and includes such rings as fully bounded noetherian (fbn) rings (e.g., noetherian pi-rings), artinian rings, principal ideal rings, hereditary noetherian prime (HNP) rings with enough invertible ideals, group rings RG and enveloping algebras $R \otimes U(\mathfrak{g})$, where R is a commutative noetherian ring, G a polycyclic-by-finite group, $\mathfrak{g}$ a solvable finite dimensional Lie (super)algebra. The class also includes Ore extensions (cf. 4) of the form $R[x; id_R, \delta]$, $R[x; \alpha, 0]$ (and $R[x, x^{-1}; \phi, 0]$), where R is a commutative noetherian ring, most quantum groups and the group-graded and skew-enveloping analogues of the previous types of rings. Finally, let us also mention Letzter's result [36], which says that for any pair of rings $R \subseteq S$ such that S is a left and right finitely generated R-module, S satisfies the second layer condition, whenever R does.

6 Let $\phi : R \to S$ be an arbitrary ring homomorphism. Clearly, in general $Q \mapsto \phi^{-1}(Q)$ does not necessarily induce a map $Spec(S) \to Spec(R)$. Indeed, it suffices to consider for example the inclusion

$$\phi : k \times k = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \hookrightarrow \begin{pmatrix} k & k \\ k & k \end{pmatrix},$$

and to note that $\phi^{-1}((0)) = (0)$ is not prime in $k \times k$.

In order to remedy this, one has to restrict to particular types of ring homomorphisms, such as *centralizing extensions* (or more generally, strongly normalizing extensions, cf. [15, 45]). These are ring homomorphisms $\phi : R \to S$, which have the property that S is generated as an R-module by S^R, the set of all $s \in S$ with the property that $\phi(r)s = s\phi(r)$, for any $r \in R$.

It may be shown that any such ϕ induces a map ${}^a\phi : Spec(S) \to Spec(R)$, which is continuous for the Zariski topology. If we assume, moreover, both R and S to satisfy the second layer condition, then the map ${}^a\phi$ is also continuous for the Artin-Rees topology and actually induces a morphism of ringed spaces

$$(Spec(S), \mathcal{T}(S), \mathcal{O}_S) \to (Spec(R), \mathcal{T}(R), \mathcal{O}_R).$$

The proof of this fact, which establishes the announced dictionary between (a vast class of) rings and geometric objects is rather tricky, and may be found in [15].

3 Quantum groups

1 Although the constructions in the previous section are highly satisfactory, they fail to work well for rings without sufficiently many Artin-Rees ideals or not satisfying the second layer condition, examples of which may even be found in the class of so-called quantum groups.

Most examples of quantum groups arise as deformations of commutative Hopf algebras. In fact, it is well known that there is a bijective correspondence between affine algebraic groups and commutative Hopf algebras, given by associating to any commutative Hopf algebra R the affine scheme $Spec(R)$, the comultiplication of R (and its defining properties) canonically inducing the structure of algebraic group on $Spec(R)$. There are many examples (as we will see below) of commutative Hopf algebras, which deform into a family of noncommutative Hopf algebras or, more precisely, which occur at $q = 0$ say, within a family of Hopf algebras, depending upon a (continuous or discrete) parameter q.

Since, in contrast with the commutative case, it is not clear how to canonically associate to noncommutative Hopf algebras an "algebraic group", one usually prefers to continue working with these noncommutative Hopf algebras themselves, referring to them as "quantum groups".

2 Let us illustrate by some basic examples the notion of quantum group, a more precise description of which may be found in the literature.

Recall that the coordinate ring of the affine plane $\mathbb{A}_k^2$ is just the ring $k[x, y]$ in two variables. The addition in $\mathbb{A}_k^2$ corresponds to the comultiplication

$$\Delta : k[x, y] \to k[x, y] \otimes k[x, y],$$

defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ resp. $\Delta(y) = y \otimes 1 + 1 \otimes y$. The ring $k[x, y]$ may be viewed as the quotient $k\{x, y\}/(yx - xy)$, where $k\{x, y\}$ is the free k-algebra over the (noncommuting) variables x and y. One now defines for any $q \in k$ the *quantum plane* as

$$k_q[x, y] = k\{x, y\}/(yx - qxy).$$

Clearly, $k_1[x, y] = k[x, y]$, whereas $k_q[x, y]$ is noncommutative for $q \neq 1$.

3 In order to study the ring structure of $k_q[x, y]$, let us first recall some background on Ore extensions.

Let R be an arbitrary algebra and consider an endomorphism α on R. A k-linear endomorphism δ of R is said to be an *α-derivation* of R, if

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b,$$

for any $a, b \in R$. In particular, it then easily follows that $\delta(1) = 0$.

Consider the free left R-module $R[t]$ generated by $\{1, t, t^2, \ldots, t^n, \ldots\}$, for some free variable t, i.e., elements of $R[t]$ are "left polynomials" of the form $P = a_n t^n + a_{n-1}t^{n-1} + \ldots + a_0$. The degree $deg(P)$ of P is defined to be n if $a_n \neq 0$ and, by convention, we put $deg(P = 0) = \infty$.

The question of extending the algebra structure on R to $R[t]$ is answered by the next result:

3.1 Proposition [30, 43] *If $R[t]$ possesses an algebra structure extending that of R and such that $deg(PQ) = deg(P)deg(Q)$ for any $P, Q \in R[t]$, then R has no zero-divisors and there exist an injective endomorphism α of R and an α-derivation δ of R, such that*

$$(*)\ ta = \alpha(a)t + \delta(a),$$

for any $a \in R$. Conversely, if R has no zero-divisors, if α is an injective endomorphism of R and if δ is an α-derivation of R, then there exists a unique algebra structure on $R[t]$ extending that of R and such that (*) holds for any $a \in R$.

4 The algebra defined in the previous result is usually denoted by $R[t; \alpha, \delta]$ and referred to as an algebra of *skew polynomials* over R or an *Ore extension of* R with respect to the automorphism α and the α-derivation δ.

Of course, it may well happen that α or δ is trivial. In particular, $R[t; id_R, 0]$ is just the ring of polynomials in the central variable t over R, whereas for $\delta \neq 0$, clearly $R[t; id_R, \delta]$ is the corresponding ring of polynomial differential operators.

To mention another elementary example, let $R = k[x]$, the ring of polynomials over the central variable x and let $\delta = \frac{d}{dx}$, the usual derivation with respect to x. In $R[t; id_R, \delta]$, we then have $ta = at + \delta(a)$, where $\delta(ab) = a\delta(b) + \delta(a)b$, for any $a, b \in R$. In particular, since $\delta(x) = 1$, we obtain that $tx - xt = 1$, the Heisenberg relation, so $R[t; id_R, \delta]$ is just the first Weyl algebra $A_1(k)$ over k.

5 Ore extensions share many properties with ordinary rings of polynomials. For example, it easily follows from 3.1 that (still with R without zero-divisors and α injective), $R[t; \alpha, \delta]$ has no zero divisors either and that it is free, with basis $\{1, t, t^2, \ldots\}$, both as a left and a right R-module.

Let us also point out the following analogue of Hilbert's basis theorem: if R is (left or right) noetherian, then so is $R[t; \alpha, \delta]$.

Defining *iterated Ore extensions* in the obvious way, the previous remarks thus clearly extend to the latter.

As a first corollary, let us mention:

6 Corollary. *The quantum plane $k_q[x, y]$ is (left and right) noetherian and has no zero-divisors. Moreover, it is free over k with basis $\{x^p y^q\}_{p,q}$.*

Proof Define an automorphism α on $k[x]$ by $\alpha(x) = qx$. Then it is clear that $k_q[x, y]$ may be identified with the Ore extension $k[x][y; \alpha, 0]$. The result now trivially follows from the above construction and remarks. ∎

7 Since the quantum plane does not appear to be "too noncommutative", one is tempted to study it from the geometric point of view, mimicking the usual set-up in the commutative case, i.e., through its prime spectrum. However, this approach is rather disappointing.

Indeed, if q is not a root of unity, then a straightforward calculation shows that $Spec(k_q[x,y])$ consists of the primitive (maximal) ideals $(x-a,y)$ and $(x,y-a)$, with $a \in k$, and the only remaining prime ideals are the zero-ideal and the ideals (x) and (y). It thus follows that (excluding the dense point (0)), the prime spectrum of $k_q[x,y]$ consists just of the two intersecting affine lines corresponding to the x-axis and the y-axis. It also follows that simple $k_q[x,y]$-modules thus have dimension 1 or are infinite dimensional. We will see in section 3 how to overcome this problem.

In order to see what happens in the other case (q a root of unity), one first determines the center of $k_q[x,y]$. A straightforward calculation shows that $Z(k_q[x,y]) = k$, if q is not a root of unity, and that $Z(k_q[x,y]) = k[x^n, y^n]$ if q is a primitive n-th root of unity. In the latter case, $k_q[x,y]$ appears to be a pi-algebra, being generated by the $x^p y^q$, with $1 \leq p, q \leq n-1$. The geometric study of $k_q[x,y]$ may thus be realized using the methods described in the previous sections. Let us also point out that in this case there exist simple modules both of dimension one and two.

8 Note. Without entering into details here, let us mention that the previous construction may be extended to higher dimensions.

Actually, one defines the *quantum (affine) space* of dimension n to be the ring $k_q[x_1,\ldots,x_n]$ with relations $x_j x_i = q x_i x_j$ for any $i < j$. Again, it is fairly easy to see that $k_q[x_1,\ldots,x_n]$ is noetherian and generated over k by the basis consisting of all $x_1^{i_1}\ldots x_n^{i_n}$. The prime ideals of the quantum space of dimension n are exactly the ideals $(y_1,\ldots,y_r)$ (with $\{y_1,\ldots,y_r\} \subseteq \{x_1,\ldots,x_n\}$) and $(x_1,\ldots,x_i - a,\ldots,x_n)$ for some $1 \leq i \leq n$.

9 The quantum plane permits us to introduce *quantum matrices* in a natural way. Indeed, assume a, b, c and d to be variables commuting with the generators x and y of the quantum plane and define the elements x', y' resp. x'', y'' through the relations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

resp.

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a\,c \\ b\,d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Assume that $q^2 \neq -1$, then it is fairly easy to see that (x',y') and (x'',y'') are "points" of $k_q[x,y]$, i.e., that $y'x' = qx'y'$ and $y''x'' = qx''y''$ if and only if (a,b,c,d) satisfies the following relations:

$$\begin{aligned} ba &= qab \\ ca &= qac \\ db &= qbd \\ dc &= qcd \\ bc &= cb \\ ad - da &= (q^{-1} - q)bc. \end{aligned}$$

It thus makes sense to define $M_q(2)$, the algebra of *quantum* (2×2) *matrices* as the quotient of the free algebra $k\{a, b, c, d\}$ by the ideal J_q generated by $ba - qab, \ldots$ (the elements corresponding to the above relations). Let us note that, as most quantum groups, $M_q(2)$ is graded and that for $q = 1$, we find that $M_q(2) = M(2)$, the generic algebra of 2×2 matrices.

The main ring-theoretic properties of $M_q(2)$ are given by:

3.2 Proposition *The ring $M_q(2)$ is noetherian and has no zero-divisors.*

Proof It clearly suffices to show $M_q(2)$ to be an iterated Ore extension of k. Let us consider the chain of rings

$$k = A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4 = M_q(2),$$

where

$$\begin{aligned} A_1 &= k[a] \\ A_2 &= k\{a, b\}/(ba - qab) \\ A_3 &= k\{a, b, c\}/(ba - qab, ca - qac, cb - bc) \end{aligned}$$

It is trivial that A_1 is an Ore extension of A_0 and that $A_2 = A_1[b; \alpha_1, 0]$, where the automorphism α_1 is defined by putting $\alpha_1(a) = qa$. Defining the automorphism α_2 of A_2 by $\alpha_2(a) = qa$ and $\alpha_2(b) = b$, it is easy to see that $A_3 = A_2[c; \alpha_2, 0]$. Finally, define the automorphism α_3 on A_3 by $\alpha_3(a) = a$ and by $\alpha_3(b) = qb$ resp. $\alpha_3(c) = qc$. A straightforward calculation shows that one may define an α_3-derivation on A_3 by putting $\delta(b^j c^k) = 0$ and

$$\delta(a^i b^j c^k) = (q - q^{-1}) \frac{1 - q^{2i}}{1 - q^2} a^{i-1} b^{j+1} c^{k+1},$$

if $i \neq 0$, and that $A_4 = M_q(2) = A_3[d; \alpha_3, \delta]$. This proves the assertion. ∎

Of course, it also follows from the previous proof that the set of all $a^i b^j c^k d^l$ is a basis of $M_q(2)$ over k.

10 Define the *quantum determinant* as

$$det_q = ad - q^{-1}bc = da - qbc \in M_q(2).$$

It is easy to see that $det_q \in Z(M_q(2))$, the center of $M_2(q)$. Actually, a rather technical calculation shows that if q is not a root of unity, then $Z(M_q(2)) = k[det_q]$. More generally, for any algebra R, let us define an *R-point* of $M_2(q)$ to be a matrix

$$\mathbf{m} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(R),$$

whose components satisfy the relations $BA = qAB, \ldots$ defining the algebra $M_q(2)$. It is thus clear that R-points of $M_q(2)$ are in bijective correspondence with the algebra homomorphisms $M_q(2) \to R$. The quantum determinant of an R-point $\mathbf{m}$ is then defined to be

$$Det_q(\mathbf{m}) = AD - q^{-1}BC = DA - qBC.$$

The quantum determinant shares many properties with its traditional counterpart. In particular, if $\mathbf{m}$ and $\mathbf{n}$ are R-points, then so is their product $\mathbf{mn}$ and we have

$$Det_q(\mathbf{mn}) = Det_q(\mathbf{m})Det_q(\mathbf{n}).$$

11 Note. For arbitrary $n \geq 2$, the *algebra of quantum matrices* $M_q(n)$ is defined similarly as the quotient of the free algebra $k\{X_{ij}; 1 \leq i, j \leq n\}$, by the relations necessary for any $i < j$ and $k < l$ to make the canonical map

$$M_q(2) \to k\{X_{ik}, X_{il}, X_{jk}, X_{jl}\} : (a, b, c, d) \mapsto (X_{ik}, X_{il}, X_{jk}, X_{jl})$$

into a ring isomorphism.

The quantum determinant may also be generalized to this setting. Indeed, denote by $\mathcal{S}_n$ the symmetric group on n elements and for any $\sigma \in \mathcal{S}_n$, let $\ell(\sigma)$ be the length of σ, i.e., the minimal number of transpositions into which σ decomposes. We then put

$$det_q = \sum_{\sigma \in \mathcal{S}_n} (-q)^{\ell(\sigma)} X_{1,\sigma(1)} \dots X_{n,\sigma(n)}.$$

It is fairly easy to verify that det_q (and its generalization to R-points, for any algebra R) behaves in a similar way as its two-dimensional analogue. Note also that det_q generates the center of $M_q(n)$ over k, when q is not a root of unity.

12 One may endow $M_q(2)$ with a bialgebra structure by defining

$$\Delta : M_q(2) \to M_q(2) \otimes M_2(2)$$

resp.

$$\varepsilon : M_q(2) \to k$$

by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c \\ \Delta(b) &= a \otimes b + b \otimes d \\ \Delta(c) &= c \otimes a + d \otimes c \\ \Delta(d) &= c \otimes b + d \otimes d \end{aligned}$$

resp.

$$\begin{aligned} \varepsilon(a) &= \varepsilon(d) = 1 \\ \varepsilon(b) &= \varepsilon(c) = 0. \end{aligned}$$

Putting

$$\begin{aligned} \Delta_A(x) &= a \otimes x + b \otimes y \\ \Delta_A(y) &= c \otimes z + d \otimes y, \end{aligned}$$

then defines a map

$$\Delta_A : k_q[x,y] \to M_q(2) \otimes k_q[x,y],$$

which makes the quantum plane $k_q[x,y]$ into a $M_q(2)$-comodule-algebra.
Of course, $M_q(2)$ is not a Hopf algebra. However, let us define

$$GL_q(2) = M_q(2)[t]/(tdet_q - 1)$$

resp.

$$SL_q(2) = M_q(2)/(det_q - 1) = GL_q(2)/(t-1).$$

Then it is easy to see that Δ and ε induce a comultiplication and a counit on $GL_q(2)$ and $SL_q(2)$, making both into bialgebras. Moreover, $GL_q(2)$ and $SL_q(2)$ are now Hopf algebras, if one endows them with the antipode S defined in matrix form by

$$\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} = det_q^{-1} c \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

Note that S is not an involution, in general. Actually, if q is a primitive root of unity, then S has order n.
Note also that the $M_q(2)$-comodule-algebra structure on $k_q[x,y]$ induces an $SL_q(2)$-comodule-algebra structure in the obvious way.

4 Schematic algebras

1 In the previous section, we have essentially only considered the affine structure of a noncommutative ring R. However, as may be seen in the above examples, most quantum groups have a natural graded structure. It thus makes sense to use this graded structure and to study the quotient category *Proj R* as a basic object within noncommutative algebraic geometry.
Let us briefly recall its definition before we explain the recent interest in this category. From now on, we let R be a *connected* graded k-algebra, i.e., $R = \bigoplus_{n\in\mathbb{N}} R_n$ and $R_0 = k$. Suppose, moreover, that R is generated by R_1 and that R_1 is a finite dimensional vector space. We denote the positive cone $\sum_{n>0} R_n$ by R_+.

2 Let R be any noetherian connected k-algebra. Define a category *Proj R* with the same objects as $R-\mathbf{gr}$, the category of graded R-modules. We will write $\pi(M)$ when considering the graded R-module M as an object of *Proj R*. Morphisms in *Proj R* are given by:

$$Hom_{Proj\,R}(\pi(M), \pi(N)) = \varinjlim_{M'} Hom_{R-\mathbf{gr}}(M', N/\kappa_+(N)),$$

where M' runs through the submodules of M such that M/M' is torsion. Consequently, π is an exact functor from $R-\mathbf{gr}$ to *Proj R*. Moreover, π has a right adjoint $\omega : Proj\,R \to R-\mathbf{gr}$, in the sense that for all $\mathcal{N} \in Proj\,R$

$$Hom_{R-\mathbf{gr}}(M, \omega(\mathcal{N})) \cong Hom_{Proj\,R}(\pi(M), \mathcal{N}).$$

This $Proj\ R$ is in fact the quotient category of $R-\mathbf{gr}$ with respect to the idempotent kernel functor κ_+ which associates to any graded R-module M its graded submodule consisting of all $m \in M$ which are annihilated by some positive power of R_+. The functor $\omega \circ \pi$ is left exact and maps M to

$$\varinjlim_n \mathrm{Hom}_R(R_+^n, M).$$

This module $\omega\pi(M)$ has the property that

$$\bigoplus_i \mathrm{Hom}_{R-\mathbf{gr}}(R_+^n, \omega\pi(M[i])) \cong \omega\pi(M)$$

for all n. Here the shifted module $M[i]$ is just M as an R-module, but with gradation given by $M[i]_j = M_{i+j}$ for any integer j.

3 The interest in $Proj\ R$ is raised by its realization as the category of quasi-coherent sheaves on the projective scheme associated to R if R is commutative. We give a brief survey since the theory of schematic algebras is aiming at a similar description of $Proj\ R$ for a vast class of algebras. However, since many interesting algebras do not posses enough prime ideals (like the so-called Sklyanin-algebras [48, 49, 54, 55]), the schematic algebras need a rather unusual description of the projective scheme of a commutative algebra, not stressing the prime ideals but the complementary multiplicatively closed sets.

To any *commutative* algebra R one associates the couple $(X = Proj(R), \mathcal{O}_X)$, where $Proj(R)$ consists of all homogeneous prime ideals of R not containing R_+ and $\mathcal{O}_X$ is the sheaf of *graded* rings on X canonically associated to R. Each homogeneous element f of R defines an affine open set $D(f)$. Open sets of this kind form a basis and a finite number of them suffices to cover X. There is a functor F from $R-\mathbf{gr}$ to the category of quasi-coherent sheaves on X such that $F(R) = \mathcal{O}_X$ and such that for any graded R-module M the module of sections $\Gamma(D(f), F(M))$ is just the localization of M at the multiplicatively closed set generated by f. The global-sections functor G maps a quasi-coherent sheaf $\mathcal{F}$ to its sections on the total space X, i.e., $G(\mathcal{F}) = \Gamma(X, \mathcal{F})$. The composition $F \circ G$ is the identity, but the functor $\Gamma = G \circ F$ is only left exact. An important theorem of Serre's [53] states that these two functors induce an equivalence between the category of quasi-coherent sheaves on X and the quotient category $Proj\ R$. The functor $\Gamma = G \circ F$ is thus precisely the functor $\omega \circ \pi$. Hence Serre's Theorem motivates the use of $Proj\ R$ as a basic object of study in noncommutative algebraic geometry.

If M is graded R-module, then $F(M)$ being a sheaf implies that $\Gamma(M)$ may be described as the inverse limit of the sections of $F(M)$ on a cover of X. In particular, if $f_1, \ldots, f_n$ are homogeneous elements of R such that $\bigcup_i D(f_i) = X$, then $\Gamma(M)$ consists of the $(\frac{m_i}{f_i^{n_i}})_i \in \bigoplus_{i=1}^n M_{f_i}$ with the property that

$$\frac{f_j^{n_i} m_i}{(f_j f_i)^{n_i}} = \frac{f_i^{n_j} m_j}{(f_i f_j)^{n_j}}$$

within $M_{f_i f_j} = \Gamma(D(f_i) \cap D(f_j), F(M))$.

4 Let us now assume R to be *noncommutative*. If one wants a similar local description of the objects in $Proj\, R$, then one has to confine to algebras possessing "enough" Ore-sets; these are the so-called schematic algebras introduced in [75].

Let us say that R is *schematic* if there exists a finite number of two-sided homogeneous Ore-sets $S_1, \ldots, S_n$ with $S_i \cap R_+ \neq \emptyset$, such that for all $(s_i)_{i=1,\ldots,n} \in \prod_{i=1}^n S_i$, we may find some positive integer m with $R_+^m \subseteq \sum_{i=1}^n Rs_i$.

The origin of this definition lies in the commutative case: the "Ore-sets" generated by homogeneous elements f_i of R satisfy the above property exactly if $\bigcup_i D(f_i) = X$. Besides the commutative algebras, many interesting graded algebras are schematic: algebras which are finite modules over their center, homogenizations of enveloping algebras and Weyl-algebras, 3-dimensional Sklyanin-algebras and several algebras of quantum-type (like $k_q[x,y]$ and $M_q(2)$), cf. [74]. Finding counterexamples is easy after noting that for a schematic algebra R all $Ext_R^n(k_R, R_R)$ are torsion, cf. [76]. For instance, as pointed out in [61], the subalgebra S of $k\{x,y\}/(yx - xy - x^2)$ generated by y and xy is not schematic since $Ext_S^1(k_S, S_S)$ is not torsion.

Even if R is schematic, then it is *not* true that

$$\Gamma(M) \cong \{(\frac{m_i}{s_i})_i \in \bigoplus_i S_i^{-1}M;\ \frac{m_i}{s_i} = \frac{m_j}{s_j} \text{ in } (S_i \vee S_j)^{-1}M\}$$

where $S_i \vee S_j$ is the Ore-set generated by S_i and S_j. The reason is that two consecutive Ore-localizations do not necessarily commute, i.e., $S_i^{-1}R \otimes_R S_j^{-1}R$ is not necessarily isomorphic to $S_j^{-1}R \otimes_R S_i^{-1}R$ in general. The solution to this problem is a refinement of the inverse system. Indeed, one may show that $\Gamma(M)$ is isomorphic to the set of those tuples $(\frac{m_i}{s_i})_i$ in $\bigoplus_i S_i^{-1}M$ such that for any i,j we have

$$1 \otimes \frac{m_i}{s_i} = \frac{1}{s_j} \otimes \frac{m_j}{1} \text{ in } S_j^{-1}(S_i^{-1}M).$$

It is possible to deform the usual notion of a categorical topology (the "intersection" of two open sets must depend on the ordering in which one intersects) such that the desired equivalence between $Proj\, R$ and the category of quasi-coherent sheaves on this topological space holds, cf. [75]. This setting generalizes well if one replaces Ore-sets by arbitrary idempotent kernel functors, cf. [81]. The definition of a covering in [75] has been adapted in [20] in order to fit better the commutative case.

If R is the homogenization of an almost commutative ring, then it is possible to work with a genuine categorical topology, i.e., (1) holds although the localizations still do not commute ([35, 77]). The point variety (see 10 below) may be described locally by studying one-dimensional representations of the sections on a suitable covering.

5 The most important application of schematic algebras is of cohomological nature. Again we start with the cohomology groups of $Proj\, R$ for an arbitrary graded algebra R. Since $Proj\, R$ has enough injectives, we may define H^i, the i-th right derived functor of $\mathrm{Hom}_{Proj\, R}(\pi(R), -)$. In order to calculate $H^i(\pi(M))$, we should start with an injective resolution of $\pi(M)$ in $Proj\, R$, apply the functor $\mathrm{Hom}_{Proj\, R}(\pi(R), -)$ and take homology at the i-th locus. We get an injective resolution of $\pi(M)$ in $Proj\, R$ if we apply the functor π to an injective resolution $E^\cdot$ of M in $R-\mathbf{gr}$. Moreover, since

$$\mathrm{Hom}_{Proj\, R}(\pi(R), \pi(E^i)) \cong \mathrm{Hom}_{R-\mathbf{gr}}(R, \omega\pi(E^i)) \cong (\omega\pi(E^i))_0$$

we get that $H^i(\pi(M)) \cong h^i(\omega\pi(E^\cdot)_0)$, for all positive integers i. Graded cohomology-groups are obtained by the usual procedure, i.e., by putting

$$\underline{H}^i(\pi(M)) \stackrel{def}{=} \bigoplus_{n\in\mathbb{Z}} H^i(\pi(M[n])).$$

In particular, $\underline{H}^0(\pi(M)) \cong \omega\pi(M)$. These graded cohomology groups are again graded R-modules and from the reasoning above we obtain that

$$\underline{H}^i(\pi(M)) \cong h^i(\omega\pi(E^\cdot)).$$

The complex $\omega\pi(E^\cdot)$, the homology of which we want to calculate, may be described in an easier way since each graded injective R-module E may be written as a direct sum $I \oplus Q$ where I is graded torsion and Q is graded torsionfree. Moreover, both I and Q are graded injective and $\omega\pi(E) \cong Q$. We may then rewrite the injective resolution $E^\cdot$ of M as:

$$0 \longrightarrow M \longrightarrow I^0 \oplus Q^0 \xrightarrow{f_0} I^1 \oplus Q^1 \xrightarrow{f_1} I^2 \oplus Q^2 \xrightarrow{f_2} \cdots$$

Note that $f_n(I^n) \subseteq I^{n+1}$, since the image of a torsion element under a graded R-module homomorphism is again torsion. Applying $\omega \circ \pi$ yields a complex

$$0 \longrightarrow \omega\pi(M) \longrightarrow Q^0 \xrightarrow{g_0} Q^1 \xrightarrow{g_1} Q^2 \xrightarrow{g_2} \cdots$$

where $g_i = \omega\pi(f_i)$ is the composition of the maps

$$Q^i \hookrightarrow E^i \xrightarrow{f_i} E^{i+1} \longrightarrow Q^{i+1}.$$

Thus $\underline{H}^j(\pi(M))$ is the j^{th} homology-group of the complex (Q^i, g_i).

6 In algebraic geometry, it is shown that these H^i coincide with the derived functors of the global-sections functor on the category of sheaves, and the latter coincide with the more amenable Cech-cohomology groups. If R is a schematic algebra, then one can define (generalized) Cech-cohomology groups as the homology groups of the complex

$$0 \longrightarrow \bigoplus_i S_i^{-1}M \longrightarrow \bigoplus_{i,j} S_i^{-1}R \otimes_R S_j^{-1}M \longrightarrow \ldots$$

It has been shown in [76] that these Cech-cohomology groups coincide with the functors H^i. This provides a more computable approach to the H^i (see the example in [76]), and has some interesting consequences like, left and right cohomology of R coincide, or if R is a finite module over its center $Z(R)$, then its cohomology is the same as its cohomology as $Z(R)$-module. Moreover, if the schematic algebra R has finite global dimension, then the cohomology groups of any finitely generated graded R-module are finite dimensional ([12, 76]). We conclude with noting [83] that there is a dimension function for schematic algebras which separates the homogenizations of Weyl and enveloping algebras.

5 Regular algebras

1 This section is devoted to regular algebras whose *Proj* is considered to be a quantized projective space. Let A be a connected algebra again and suppose that A is generated by the finite dimensional vector space A_1.

We start by recalling the construction of the twisted homogeneous coordinate ring from [11]. Let X be an irreducible projective variety over k and σ an automorphism of X. An invertible sheaf L is called *σ-ample* if for all coherent sheaves $\mathcal{F}$ on X and all positive integers i we have

$$H^i(X, L \otimes L^{\sigma} \otimes \cdots \otimes L^{\sigma^{n-1}} \otimes \mathcal{F}) = 0,$$

for all sufficiently large n, where L^{σ} denotes the pull-back $\sigma^* L$. Fix a σ-ample invertible sheaf L and define $\mathcal{B}_0 = \mathcal{O}_X$ and $\mathcal{B}_n = L \otimes L^{\sigma} \otimes \cdots \otimes L^{\sigma^{n-1}}$ if $n > 0$. Let $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}_n$ and $B = B(X, \sigma, L) = \bigoplus_{n \geq 0} H^0(X, \mathcal{B}_n)$. The multiplication on B is defined by

$$b.c = b \otimes c^{\sigma^n} \in B_{n+m}$$

for $b \in B_n$ and $c \in B_m$. Then $Proj\, B$ is equivalent with the category of quasi-coherent $\mathcal{O}_X$-modules, the equivalence being induced by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{B}_n)$$

and $(\mathcal{B} \otimes_B -)_0$. Coherent $\mathcal{O}_X$-modules correspond to finitely generated B-modules. Moreover, B is a finitely generated noetherian algebra.

2 Let us call A a *regular* algebra of dimension d if and only if

1. $gldim(A) = d < \infty$;
2. $gkdim(A) < \infty$;
3. A is Gorenstein, i.e., $Ext^i_A(k, A) = \delta_{id}\, k$.

For example, a commutative algebra is regular if and only if it is a polynomial algebra. Therefore, the *Proj* of a regular algebra of dimension d is viewed as a noncommutative $\mathbb{P}^{d-1}$. If A is a finite module over its center and has finite global dimension, then A is regular (cf. [8]). It has been proved in [27] that a regular algebra A of dimension 2 is either of the form

$$A = k\{x, y\}/(yx - qxy),$$

with $q \in k^*$, or

$$A = k\{x, y\}/(yx - xy - x^2).$$

In particular, the quantum plane $k_q[x, y]$ is a regular algebra.

The study of regular algebras of dimension 3 was initiated in [8], with a proof of the following dichotomy:

3 Theorem. *Let A be a regular algebra of dimension* 3.

1. *All defining relations have the same degree* s *and the minimal number of defining relations equals* $r = \dim_k(A_1)$. *Moreover,* $(r, s) = (2, 3)$ *or* $(3, 2)$.

2. *With a suitable choice of the relations* $f_i = \sum_{j=1}^r m_{ij} x_j$ $(i = 1, \ldots, r)$, *there is a resolution*

$$0 \longrightarrow A \xrightarrow{x^t} A^r \xrightarrow{M} A^r \xrightarrow{x} A \longrightarrow k \longrightarrow 0$$

and the entries of $x^t M$ *are again a set of defining relations for* A.

3. *The Hilbert series* $H_A(t) = \sum_{i=0}^{\infty} \dim_k A_i t^i$ *of* A *is* $(1-t)^{-3}$ *if* $r = 3$ *and* $(1-t)^{-3}(1+t)^{-1}$ *if* $r = 2$.

Let us call an algebra A *standard* if it can be presented by r generators x_i of degree 1 and r relations f_i of degree s such that

1. $(r, s) = (3, 2)$ or $(2, 3)$;

2. if $f_i = \sum_{j=1}^r m_{ij} x_j$, then the r elements $g_j = \sum_{i=1}^r x_i m_{ij}$ are also a set of defining relations, i.e. there exists a matrix $Q = (q_{ij})$ in $Gl_r(k)$ such that $\sum_{i=1}^r x_i m_{ij} = \sum_{i=1}^r q_{ji} f_i$ for all $j \in \{1, \ldots, r\}$.

Thus regular algebras of dimension 3 are standard. This fact was then exploited to classify all regular algebras of dimension 3.

4 Another classification and a simple criterion to decide whether a given standard algebra A is regular emerged in [9].

Let T be the tensor algebra of the vector space A_1. Any element $t \in T_n$ may be viewed as a multilinear function $\tilde{t}$ on the product of n copies of the dual vector space A_1^*. Therefore we may consider its zero locus $\nu(\tilde{t})$ in the product of n copies of $\mathbb{P} = \mathbb{P}(A_1^*)$. Let Γ be the intersection of the $\nu(\tilde{f_i})$ where the f_i are the defining relations of A. Define two projections from $(\mathbb{P})^s$ onto $(\mathbb{P})^{s-1}$, the first one dropping the first component and the second one dropping the last component. If A is a standard algebra, then the images of Γ in $(\mathbb{P})^{s-1}$ under both projections coincide. They are both equal to the zero locus of $\det\widetilde{M}$ where $\widetilde{M}$ is the matrix of the $\widetilde{m_{ij}}$ if $f_i = \sum_{j=1}^r m_{ij} x_j$.

There are four possibilities for this locus, which we call E. If $\det\widetilde{M}$ is identically zero, the so-called *linear* case, then E is all of $\mathbb{P}^2$ if $r = 3$ or all of $\mathbb{P}^1 \times \mathbb{P}^1$ if $r = 2$. In the *elliptic* case we have that E is a cubic divisor if $r = 3$ or a divisor of bidegree $(2, 2)$ if $r = 2$.

5 Definition. An algebra A is *nondegenerate* if Γ is the graph of an automorphism σ of E, or equivalently, if the $r \times r$ matrix $\widetilde{M}$ has rank at least $r - 1$ at every point of $\mathbb{P}^2$ or of $\mathbb{P}^1 \times \mathbb{P}^1$.

We can now state the main result of [9]:

6 Theorem. *Let* A *be an algebra of global dimension* 3. *Then* A *is regular if and only if* A *is nondegenerate and standard.*

To prove the sufficiency, the authors associate to a nondegenerate standard algebra A a triple (E, σ, L) consisting of (i) a scheme $E \subset (\mathbb{P}(A_1^*))^{s-1}$ which is either a divisor of the type described above or is the whole ambient space, (ii) an automorphism σ of E and (iii) an invertible $\mathcal{O}_E$-module $L = \pi^*\mathcal{O}(1)$, where π is the inclusion of E in $\mathbb{P}^2$ if $r = 3$ or is the projection on the first factor $\mathbb{P}^1$ if $r = 2$. In the latter case, the projection of E on the second factor $\mathbb{P}^1$ is $\pi \circ \sigma$. Such a triple gives rise to a twisted homogeneous coordinate ring $B = B(E, \sigma, L)$ and there is a canonical epimorphism $A \to B$ which is an isomorphism in degree 1.

7 Theorem. *Suppose that A is regular. If dim $E = 2$, then A and B are isomorphic. If dim $E = 1$, then there exists a non zero-divisor g in A_{s+1} which is normalizing and such that $B \cong A/(g)$. Let λ be the class of L in the Picard group of E. Then*

1. *if $r = 3$, then $(\sigma - 1)^2\lambda = 0$;*
2. *if $r = 2$, then $(\sigma - 1)(\sigma^2 - 1)\lambda = 0$.*

Conversely, given a triple (E, σ, L) where L satisfies equation (7), one constructs a regular algebra as follows. If $r = 2$ or $r = 3$ and if $\pi : E \to \mathbb{P}^{r-1}$ is the morphism determined by the global sections of L and T is the tensor algebra on $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, then we obtain an epimorphism $T \to B$. If I is the graded ideal generated by the homogeneous part of degree $s = 5 - r$ of the kernel of this morphism, then T/I is a regular algebra of dimension 3.

These correspondences are almost each others inverse, except that if the automorphism σ of a triple with dim $E = 1$ can be extended to the whole ambient space, then the resulting regular algebra is the same as for the triple with the ambient space as scheme and the unique extension of σ as automorphism.

Recently *all* 3-dimensional regular algebras (including those *not* necessarily generated in degree 1) were classified in [63, 64].

8 Examples. Let us first consider the case $(r, s) = (2, 3)$. Let A be the enveloping algebra of the Heisenberg Lie algebra. Thus A is generated by two degree 1 elements x and y which satisfy the relations

$$x(xy - yx) - (xy - yx)x = 0 = y(xy - yx) - (xy - yx)y$$

The matrix M becomes

$$\begin{pmatrix} yx - 2xy & x^2 \\ -y^2 & 2yx - xy \end{pmatrix}$$

If we use $(x_1 : y_1; x_2 : y_2)$ as coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$, then

$$\widetilde{M} = \begin{pmatrix} y_1x_2 - 2x_1y_2 & x_1x_2 \\ -y_1y_2 & 2y_1x_2 - x_1y_2 \end{pmatrix}$$

Since $\det\widetilde{M} = -2(x_1y_2 - x_2y_1)^2$, the divisor E becomes the double diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. Using the affine coordinate $t = x/y$ in $\mathbb{P}^1$, E is given by the points $(t, t + \epsilon)$

such that $\epsilon^2 = 0$ and the automorphism σ maps $(t, t+\epsilon)$ to $(t+\epsilon, t+2\epsilon)$. Thus σ is the identity on the reduced curve but is non-trivial on the double diagonal.

Next, let us consider the case $(r, s) = (3, 2)$. Let A be the 3-dimensional Sklyanin algebra, i.e., the algebra depending on 3 parameters $a, b, c \in k$ defined by:

$$\begin{aligned} f_1 &= cx^2 + bzy + ayz \\ f_2 &= azx + cy^2 + bxz \\ f_3 &= byx + axy + cz^2 \end{aligned}$$

The corresponding multilinearizations are:

$$\begin{aligned} \tilde{f}_1 &= cx_1x_2 + bz_1y_2 + ay_1z_2 \\ \tilde{f}_2 &= az_1x_2 + cy_1y_2 + bx_1z_2 \\ \tilde{f}_3 &= by_1x_2 + ax_1y_2 + cz_1z_2 \end{aligned}$$

where the coordinates in $\mathbb{P}^2 \times \mathbb{P}^2$ are labeled as $(x_1 : y_1 : z_1; x_2 : y_2 : z_2)$. The divisor E is given by the zero locus of $\det \widetilde{M}$ where

$$\widetilde{M} = \begin{pmatrix} cx_1 & bz_1 & ay_1 \\ az_1 & cy_1 & bx_1 \\ by_1 & ax_1 & cz_1 \end{pmatrix}$$

Thus E is the cubic curve in $\mathbb{P}^2$ with equation

$$(a^3 + b^3 + c^3)x_1y_1z_1 = abc(x_1^3 + y_1^3 + z_1^3)$$

It is easy to check that A is regular if not $a^3 = b^3 = c^3$, or if at most one element in $\{a, b, c\}$ is zero. In that case,

$$\Gamma = \bigcap_{i=1}^{3} \nu(\tilde{f}_i) \subset \mathbb{P}^2 \times \mathbb{P}^2$$

is the graph of an automorphism σ of E. To compute $\sigma(x, y, z)$ one must solve the equations

$$\begin{aligned} cxx' + bzy' + ayz' &= 0 \\ azx' + cyy' + bxz' &= 0 \\ byx' + axy' + czz' &= 0 \end{aligned}$$

for (x', y', z'). One then obtains that

$$\sigma(x, y, z) = (acy^2 - b^2xz, bcx^2 - a^2yz, abz^2 - c^2xy)$$

Choosing $(1, -1, 0)$ as origin for the group law on E, one finds that σ is just translation by the point (a, b, c).

9 The authors of [9] also show that every regular algebra of dimension 3 is left and right noetherian. They prove that B is noetherian by a "reduction modulo a prime" argument (or by the result of [11]) and then lift this property to A. In a second paper [10], it is shown that a 3-dimensional regular algebra A is a finite module over its center if and only if the automorphism σ has finite order. In order to describe the other results of [10], we need some more definitions:

1. The *grade number* of an A-module M is the infimum of all positive integers n such that $Ext^n_A(M, A) \neq 0$. It is denoted by $j_A(M)$.

2. An algebra A is *Auslander-Gorenstein* resp. *Auslander-regular* of dimension d if and only if

   (a) $injdim(A) = d < \infty$ resp. $gldim(A) = d < \infty$;

   (b) for every finitely generated A-module M, for any positive integer n and any submodule N of $Ext^n_A(M, A)$, we have $j_A(N) \geq n$.

3. If an algebra A is noetherian and has finite *gkdim*, then A is said to satisfy the *Cohen-Macaulay property* if and only if for every finitely generated A-module M, we have that
$$gkdim(M) + j_A(M) = gkdim(A).$$

For example, if X is a smooth elliptic curve, $\sigma \in Aut_k(X)$ and L is a very ample invertible sheaf on X, then $B = B(X, \sigma, L)$ is Auslander-Gorenstein of dimension 2 and satisfies the Cohen-Macaulay property [37, 84]. Moreover, as shown in [10], regular algebras of dimension 3 are Auslander-regular and satisfy the Cohen-Macaulay property. Auslander-regular algebras are known to be domains [37] and so are regular algebras with global dimension and Gelfand-Kirillov dimension less than or equal to 4, by [10].

10 Let us restrict to quadratic algebras from now on, i.e., $r = 3$. Before summarizing the main results about the modules over A, we need some definitions. Let M be a finitely generated (left) module over an arbitrary algebra. We say that M is *Cohen-Macaulay* if $pdim(M) = j(M)$. On the other hand, we call M a *linear* module of dimension d, if M is cyclic and $H_M(t) = (1-t)^{-d}$.

For example, the d-dimensional linear modules over the polynomial ring $R = k[x_0, \ldots, x_n]$ are of the form $R/(f_1, \ldots, f_{n-d})$, where $f_1, \ldots, f_{n-d}$ are linearly independent elements of R_1. Linear modules of dimension 1, 2 and 3 are called *point*, *line* and *plane* modules. Over a 3-dimensional quadratic regular algebra A, these modules may be defined by their homological properties. Indeed, an A-module M is isomorphic to the shift of a point, resp. line module if and only if M is Cohen-Macaulay, $e(M) = 1$ and $gkdim(M) = 1$, resp. 2. The point modules of A correspond to the points of E, for if $p \in E$, then we obtain a point module $M(p) = \sum_i ke_i$ by $x_i.e_j = x_i(\sigma^{-j}(p))e_{j+1}$. In general, if one defines a module as the quotient of A by the submodule generated by all linear forms vanishing on a point p, then one gets a module of finite length if $p \notin E$ and $M(p)$ if $p \in E$. Therefore, E is called the *point variety* of A. Similarly, line modules correspond to lines in $\mathbb{P}(A_1^*)$, for if $a \in A_1$, then the line l with equation $a = 0$ corresponds to the line module $M_l = A/Aa$. A line module M_l maps onto a point module N_p if and only if the point p lies on the line l in the ordinary projective space! However, the automorphism σ complicates the picture. Indeed, shifting a point module by 1 and chopping off at degree 0, one gets the point module $(N_p[1])_{\geq 0} = N_{\sigma^{-1}(p)}$ corresponding to the image of the original point under σ^{-1}. If the line l intersects E in three distinct points $\{p, p', p''\}$, then one obtains the exact sequence
$$0 \longrightarrow M_{l_1}[-1] \longrightarrow M_l \longrightarrow N_p \longrightarrow 0$$

where l_1 is the line through $\sigma^{-1}(p')$ and $\sigma^{-1}(p'')$.
We refer to [1] for supplementary information about the above concepts.

11 Fix scalars $\alpha_1, \alpha_2, \alpha_3$ in k such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$. The four-dimensional Sklyanin algebra $S = S(\alpha_1, \alpha_2, \alpha_3)$ is the quotient of the tensor algebra on $V = kx_0 + kx_1 + kx_2 + kx_3$ by the ideal I generated by

$$x_0x_i - x_ix_0 = \alpha_i(x_jx_k + x_kx_j)$$
$$x_0x_i + x_ix_0 = x_jx_k - x_kx_j$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Thus S has 6 quadratic relations $f_1, \ldots, f_6$. The ring-theoretical properties of this algebra were studied in [57] using the same methods as in [9]. Suppose that

$$\{\alpha_1, \alpha_2, \alpha_3\} \cap \{0, 1, -1\} = \emptyset,$$

or that $\alpha_i = 1, \alpha_j = -1$ and $\alpha_k \notin \{0, 1, -1\}$. Then one can again define $\Gamma \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ the common locus of zeros of the multilinearizations of the relations f_i.
Let $E_i \subseteq \mathbb{P}^3$ be the image of Γ under the i^{th} projection map. Then $E_1 = E_2$ is the union of an irreducible non-singular elliptic curve E and 4 more points labeled $e_0, \ldots, e_3$. These special points are the only ones lying on infinitely many secant lines of E, or alternatively, the only singular points on the pencil of quadrics containing E. Since I_2 consists of all $f \in V \otimes V$ which vanish on Γ, we get that S is completely determined by E and σ. Moreover, π_i induces an isomorphism between Γ and E_i, and $\sigma = \pi_1 \circ \pi_2^{-1}$ is an automorphism of E_1 which restricts to the identity on $\{e_0, \ldots, e_3\}$ and to the addition with some point $\tau \in E$ on the elliptic curve.
Again point modules correspond to points of E_1 and if $M(p) = \bigoplus_i ke_i$ is the point module corresponding to $p \in E_1$ then $x_i.e_j = x_i(\sigma^{-j}(p))e_{j+1}$. If i denotes the embedding of E in $\mathbb{P}^3$ and L the pull-back $i^*\mathcal{O}_{\mathbb{P}^3}(1)$, then $B = B(E, \sigma, L)$ is a quadratic algebra generated by B_1.
There is a natural homomorphism from S to B which is an isomorphism in degree 1. The kernel is generated by two central elements Ω_1, Ω_2 of degree 2.
The main theorem of [57] states that S is Koszul with Hilbert series $(1-t)^{-4}$ and that $\{\Omega_1, \Omega_2\}$ is a regular sequence, i.e. Ω_1 is a non-zero divisor in S and Ω_2 is a non-zero divisor in $S/(\Omega_1)$. Since S is also Frobenius, one gets that S is Gorenstein and hence regular. Furthermore, S is Auslander-regular and satisfies the Cohen-Macaulay property, because B is Auslander-Gorenstein of dimension 2 and satisfies the Cohen-Macaulay property, cf. [37].
Note that there exists another way [65] to prove Auslander-regularity without using the geometric ring B.
Let us mention what happens in the so-called degenerate cases: $S(\alpha_1, -1, 1)$ is not regular, $S(0, \alpha_2 \notin \{0, 1\}, \alpha_3)$ and $S(0, 0, 0)$ are iterated Ore-extensions and have a PBW-basis in the sense of [50], hence they are regular noetherian domains of global dimension 4.
We finish this paragraph with some results of Stafford's [60], who analyses all algebras on four generators with six quadratic relations that map onto the geometric ring B and have the same good properties as S. By carefully translating these

properties to the Koszul dual, one may exhibit a 1-parameter family of isomorphism classes and show, along the way, B to be Koszul.

12 The module theory of the 4-dimensional Sklyanin algebra may be found in [38]. Point, line and plane modules may again be characterized homologically: a module M is the shift of a plane, resp. line, resp. point module if and only if M is Cohen-Macaulay, $e(M) = 1$ and $gkdim(M) = 3$, resp. 2, resp. 1. Plane modules correspond to hyperplanes in $\mathbb{P}^3$ and line modules to secant lines of E. A point p of E_1 lies on a secant line l of E if and only if the corresponding line module $M(l)$ maps onto the corresponding point module $M(p)$.
The kernel of this map is again the shift of a line module. Indeed, let $l \cap E = \{p, q\}$, then

1. if $l \cap \{e_0, e_1, e_2, e_3\} = \emptyset$, then

$$0 \longrightarrow M(p+\tau, q-\tau)[-1] \longrightarrow M(p,q) \longrightarrow M(p) \longrightarrow 0$$

2. if $l \cap \{e_0, e_1, e_2, e_3\} = e_i$, then

$$0 \longrightarrow M(p-\tau, q-\tau)[-1] \longrightarrow M(p,q) \longrightarrow M(e_i) \longrightarrow 0$$

If the order of τ is infinite, then for each positive integer k there are $k+1$-dimensional simple S-modules and all such modules are quotients of line modules, cf. [58]. On the other hand, if τ is a point of (finite) order n, then every simple S-module is of dimension at most n and S is a finite module over its center (cf. [56]). The generators and relations of the center of S (and also of the 3-dimensional Sklyanin algebra) were determined in [59]. The generators are certain liftings of generators of the center of the geometric ring B, together with the members of the regular sequence.

13 A new class of 4-dimensional regular algebras has been described in [66]. Suppose that a non-singular quadric Q and a line L in $\mathbb{P}^3$ meet in two distinct points. Let σ be an automorphism of $Q \cup L$ such that

1. $\sigma(Q) = Q$ and $\sigma(L) = L$;

2. the restriction of σ to $Q \cup L$ is the identity;

3. the restrictions of σ to both Q and L are restrictions of a linear automorphism of $\mathbb{P}^3$;

4. σ is *not* the restriction of a linear automorphism of $\mathbb{P}^3$.

One defines an algebra A in terms of these geometric data as follows. If V is the vector space of linear forms on $\mathbb{P}^3$, then A is the quotient of the tensor algebra on V by the ideal of all bilinear forms vanishing on the graph of σ. A suitable choice of coordinates allows to assume that $L = \nu(x_1, x_4)$ and $Q = \nu(x_1x_4 + x_2x_3)$. There are two cases to be considered:

1. if σ preserves the two rulings of the quadric, then A is generated by $x_1, \ldots, x_4$ with defining relations

$$\begin{array}{lll} x_2x_1 = \alpha x_1x_2 & x_3x_1 = \lambda x_1x_3 & x_4x_1 = \alpha\lambda x_1x_4 \\ x_4x_3 = \alpha x_3x_4 & x_4x_2 = \lambda x_2x_4 & x_3x_2 - \beta x_2x_3 = (\alpha\beta - \lambda)x_1x_4 \end{array}$$

   for some non-zero $\alpha, \beta, \lambda \in k$ with the property that $\lambda \neq \alpha\beta$.

2. if σ interchanges the rulings on the quadric, then A is generated by $x_1, \ldots, x_4$ with defining relations

$$\begin{array}{lll} x_3x_4 = \alpha x_1x_3 & x_2x_4 = \lambda x_1x_2 & x_4^2 = \alpha\lambda x_1^2 \\ x_4x_2 = \alpha x_2x_1 & x_4x_3 = \lambda x_3x_1 & \beta x_3x_2 - x_2x_3 = (\lambda - \alpha\beta)x_1^2 \end{array}$$

   for some non-zero $\alpha, \beta, \lambda \in k$ with the property that $\lambda \neq \alpha\beta$.

The algebra A is determined by these geometric data. Conversely, the defining relations determine the geometric data. Indeed, the graph of σ is precisely the zero locus of the multilinearizations of the relations. The family of algebras which may be defined in this way contains $M_q(2)$, the quantum 2×2 matrices. Any algebra A of this family is an iterated Ore-extension and hence a noetherian domain of global dimension 4. Its Hilbert series is $(1-t)^{-4}$ and it is a Koszul algebra. Up to scalar multiples, there is a unique element Ω in A_2 which vanishes on the $\{(q, \sigma(q))$ with $q \in Q\}$, but not on those with $x \in L$. This element Ω is normal and $A/(\Omega)$ is isomorphic to $B(Q, \sigma, \mathcal{L})$ where $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^3}(1)$ if i denotes the embedding $Q \hookrightarrow \mathbb{P}^3$. Consequently A is Auslander-regular and satisfies the Cohen-Macaulay property.

Again the properties of the point, line and plane modules are very similar to the previous cases. Plane modules correspond to planes in $\mathbb{P}^3$, point modules to points of $Q \cup L$ and line modules to lines on Q or lines intersecting L. Every point module is the quotient of a line module and the kernel is again the shift of a line module. For instance, if a line l contains at least 3 distinct points p, q, r of $Q \cup L$, then:

$$0 \longrightarrow M(l')[-1] \longrightarrow M(l) \longrightarrow M(p) \longrightarrow 0$$

where l' is the line through $\sigma^{-1}(q)$ and $\sigma^{-1}(r)$.

14 Another class of 4-dimensional regular algebras consists of central extensions of 3-dimensional regular algebras, i.e., regular algebras D of dimension 4 which have a central regular element z of degree 1 such that $D/(z)$ is isomorphic to a 3-dimensional regular algebra A. Note that any normal regular element may be turned into a central one via a twist. Given a 3-dimensional regular algebra A, all 4-dimensional regular algebras D and surjective homomorphisms $\theta : D \to A$ such that $\operatorname{Ker}\theta$ is generated by one central regular element of degree 1 are classified in [33]. This is accomplished by an analysis of when the property of A being Koszul may be lifted to D. Other properties like being noetherian, being a domain, Auslander-regularity [40] and the Cohen-Macaulay property [37] may also be lifted.

Moreover, it is shown that the first and the second projection of Γ_D coincide and that Γ_D is the graph of an automorphism σ_D of the point variety $\mathcal{P}_D$ of D. It turns out that $\mathcal{P}_A = \mathcal{P}_D \cap \nu(z)$, that $\sigma_D = \sigma_A$ on $\mathcal{P}_A$ and that σ_D is the identity on

$\mathcal{P}_D \cap \nu(z)^c$. If A is elliptic, then D is determined by its geometric data. It is possible to determine $\mathcal{P}_D$ for each generic member of each family of 3-dimensional regular algebras. If M is a line module over D, then z annihilates M or z acts as a non-zero divisor. In the first case M is a line module over A, in the second case we get that M/zM is a point module over A. For each point $p \in \mathcal{P}_A$, there is a pair of lines (not lying in $\nu(z)$) through p which varies continuously with p.

15 Again let Q be a non-singular quadric in $\mathbb{P}^3$ and $\tau \in Aut(Q)$. In [67], the authors classify all 4-dimensional regular algebras R with Hilbert series $(1-t)^{-4}$ which map onto $B = B(Q, \tau, \mathcal{O}_Q(1))$. It follows that R must have a normal element Ω of degree 2 and hence that R is a noetherian domain, Auslander-regular and satisfying the Cohen-Macaulay property. If $\mathcal{P}$ denotes the point scheme of R, then $Q \subseteq \mathcal{P}$ and $\sigma|_Q = \tau$.
If $\mathcal{P} \neq Q$, then R is determined by the geometric data $(\mathcal{P}, \sigma)$. These algebras are classified by twisting them to an algebra R' mapping onto the homogeneous coordinate ring of Q and classifying the algebras R' and their possible twists. One finds that either $\mathcal{P} = \mathbb{P}^3$, or $\mathcal{P}$ is the union of the quadric Q and a line L such that $L \cap Q$ is 2 points counted with multiplicity, or $\mathcal{P}_c = Q$ and $\mathcal{P}$ contains a double line L of multiple points on Q. In the first case, the line modules correspond to lines in $\mathbb{P}^3$, in the latter two cases the line modules are parametrized by lines on Q or lines which intersect L. Moreover, there is a regular normalizing sequence $\{v_1, v_2\} \subset R_1$ such that $L = \nu(v_1, v_2)$. All these algebras are twists of the algebras studied in [33].
If $\mathcal{P} = Q$, then R cannot be studied in the previous way. Actually, R is the first example of an algebra *not* determined by its geometric data since these data determine B. However, R may be twisted (by a twisting system) to a member of a 1-parameter family. This family is the subject of [78]. It turns out that it consists of finite free modules over their center, which is a polynomial ring in 4 variables. They are also closely related to Clifford algebras. For the first time also, there are left line modules over R which do not correspond to right line modules, although the isomorphism classes of left and right line modules still correspond.

The authors of [67] also prove the next general theorem:

16 Theorem. *Let $A = T(A_1)/(W)$ be a quadratic noetherian algebra, with $W \subset A_1 \otimes A_1$. Suppose that A is Auslander-regular of global dimension 4 and satisfies the Cohen-Macaulay property. Let*

$$\pi_i \; : \; \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*) \longrightarrow \mathbb{P}(A_1^*)$$

be the i^{th} projection map. Then

1. *the automorphism classes of left (and right) point modules over A correspond to the graph Γ of an automorphism σ of some subvariety $\mathcal{P}$ of $\mathbb{P}(A_1^*)$;*

2. *σ corresponds to shifts, i.e., $(M(p)[1])_{\geq 0} = M(\sigma^{-1}(p))$;*

3. *if $\pi_i(\nu(W))$ contains 2 distinct points for both $i = 1$ and $i = 2$, then Γ is precisely the set of closed points of the scheme $\nu(W)$.*

17 The examples of 4-dimensional regular algebras we have met so far have very nice properties, although some of these properties were lost in the previous example.

So, how does a generic 4-dimensional regular algebra behave? In [69], M. Van den Bergh showed that such an algebra has at most 20 point modules and a 1-dimensional family of line modules and introduced a class of examples, which have precisely 20 point modules.

In fact, it suffices to consider a symmetric $n \times n$-matrix Y with entries in $k[y_1, \ldots, y_n]_1$, and to construct the Clifford algebra $A = A(Y)$ with generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and relations

$$\begin{aligned} x_i x_j + x_j x_i &= Y_{ij} \\ x_i y_j - y_j x_i &= 0 \\ y_i y_j - y_j y_i &= 0 \end{aligned}$$

If one writes $Y = Y_1 y_1 + \ldots Y_n y_n$ where the Y_i are symmetric $n \times n$ matrices over k, then A is regular of global dimension n if and only if the quadrics corresponding to the Y_i have no common intersection point, cf. [31]. In that case, A is generated by the x_i only, is a finite module over its center and hence noetherian.

Such graded Clifford algebras have a two-dimensional family of line modules. Moreover, they are determined by their geometric data, cf. [68].

On the other hand, this is the first example we encounter of a regular algebra with line modules which do not map onto any point module. In [68], there is also an example of a deformed graded Clifford algebra which has precisely one point module and a 1-dimensional family of line modules.

The geometry of points and lines is thus clearly insufficient in the general case. A classification of all 4-dimensional regular algebras seems hopeless at the moment.

18 What is known about regular algebras of higher dimensions? Given an elliptic curve E and some suitable point τ on it, there is definition [48] of a n-dimensional Sklyanin algebras for any $n \geq 3$. It is interesting to note that the point modules over this algebra correspond to the points of E when $n \geq 5$.

If $\mathfrak{g}$ is a finite-dimensional Lie-algebra, then the homogenization of its enveloping algebra $U(\mathfrak{g})$ is a graded algebra $H(\mathfrak{g})$ which has a central regular element t of degree 1 such that

$$H(\mathfrak{g})/(t-1) \cong U(\mathfrak{g}) \text{ and } H(\mathfrak{g})/(t) \cong S(\mathfrak{g}),$$

where $S(\mathfrak{g})$ is the symmetric algebra on $\mathfrak{g}$. More details about homogenizations may be found in [39]. The latter isomorphism yields that $H(\mathfrak{g})$ is a regular algebra of dimension $\dim_k \mathfrak{g} + 1$. If f is a one-dimensional representation of a codimension d Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then the homogenization of

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_f$$

is a d-dimensional linear module over $H(\mathfrak{g})$ and all d-dimensional linear modules have this form ([32, 34]). The point variety has a subvariety such that every line (not annihilated by t) intersects this variety.

19 Several authors have investigated the homological properties of regular algebras. Let us mention that results are known about dualizing complexes ([84]), about the residue complex ([2, 3, 85]) and about injective resolutions ([4]).

Suppose now that R has finite global dimension d and satisfies the Gorenstein condition. From results in [12], it follows that the cohomology groups of R are then completely similar to those of projective $d-1$ space:

- $\underline{H}^0(\pi(R)) \cong R$;
- $\underline{H}^j(\pi(R)) = 0$ for all $j \notin \{0, d-1\}$;
- $\underline{H}^{d-1}(\pi(R)) \cong R^*[l]$, where $R^* = \bigoplus_n \mathrm{Hom}_k(R_{-n}, k)$ is the graded dual of R.

The converse also holds:

20 Theorem. [76] *Let R be a noetherian connected k-algebra with finite global dimension. Suppose there exists a natural number d and an integer l such that $\underline{H}^0(\pi(R)) \cong R$, $\underline{H}^j(\pi(R)) = 0$ for all $j \notin \{0, d-1\}$ and $\underline{H}^{d-1}(\pi(R)) \cong R^*[l]$. Then $d = gldim(R)$ and R is Gorenstein.*

We conclude with a result proved independently in [86] and [82]. If the connected noetherian k-algebra R is Gorenstein, then Serre-duality holds for R.

21 Theorem. *Let R be a Gorenstein-algebra of finite global dimension d (with $Ext^d(k, R) = k[l]$). Then:*

1. *for any finitely generated graded R-module M, the natural pairing*

$$Hom(\pi(M), \pi(R)[-l]) \times H^{d-1}(\pi(M)) \longrightarrow H^{d-1}(\pi(R)[-l]) \cong k$$

   is a perfect pairing of finite-dimensional vector spaces for any finitely generated graded R-module M;

2. *for any positive integer i, the vector space $Ext^i(\pi(M), \pi(R)[-l])$ is isomorphic to the dual vector space $H^{d-1-i}(\pi(M))'$ of $H^{d-1-i}(\pi(M))$.*

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