Quasi-Convolution Properties of Certain Subclasses of Analytic Functions

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Abstract

Two subclasses $\mathcal{A}(p,n,\alpha)$ and $\mathcal{B}(p,n,\alpha)$ of analytic functions in the open unit disk are introduced. The object of the present paper is to give a number of quasi-convolution properties of functions belonging to each of the classes $\mathcal{A}(p,n,\alpha)$ and $\mathcal{B}(p,n,\alpha)$.

1 Introduction and Definitions

Let $\mathcal{T}(p,n)$ be the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k\, z^k \qquad (a_k \geq 0;\ \ p,n \in \mathbb{N} := \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}\,.$$

Let $\mathcal{T}(p,n,\alpha)$ denote the subclass of $\mathcal{T}(p,n)$ consisting of functions $f(z)$ which also satisfy the inequality:

$$\Re\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \qquad \left(z \in \mathcal{U};\ \ 0 \leq \alpha < \frac{1}{p}\right) \tag{1.2}$$

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for some α $(0 \leq \alpha < 1/p)$. We note that a function $f(z)$ belonging to the class $\mathcal{T}(p, n, \alpha)$ is p-valently starlike in $\mathcal{U}$.

For the class $\mathcal{T}(p, n, \alpha)$, Yamakawa [6] has shown that, if $f(z) \in \mathcal{T}(p, n)$ satisfies the inequality:

$$\sum_{k=p+n}^{\infty} (2k - \alpha pk - p)\, a_k \leq p(1 - \alpha p) \qquad \left(0 \leq \alpha < \frac{1}{p}\right), \tag{1.3}$$

then $f(z) \in \mathcal{T}(p, n, \alpha)$. Applying this fact, Yamakawa [6] gave the following equivalence relations:

$$f(z) \in \mathcal{A}(p, n, \alpha) \Leftrightarrow f(z) \in \mathcal{T}(p, n) \quad \text{and} \quad \sum_{k=p+n}^{\infty} (2k - \alpha pk - p)\ a_k \leq p(1 - \alpha p) \qquad \left(0 \leq \alpha < \frac{1}{p}\right); \tag{1.4}$$

$$f(z) \in \mathcal{B}(p, n, \alpha) \Leftrightarrow f(z) \in \mathcal{T}(p, n) \quad \text{and} \quad \sum_{k=p+n}^{\infty} k(2k - \alpha pk - p)\ a_k \leq p^2(1 - \alpha p) \qquad \left(0 \leq \alpha < \frac{1}{p}\right). \tag{1.5}$$

Let the functions $f_j(z)$ given by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j}\, z^k \qquad (j = 1, 2) \tag{1.6}$$

be in the class $\mathcal{T}(p, n)$. Then the quasi-convolution (or *modified* Hadamard product) $(f_1 * f_2)(z)$ of the functions $f_1(z)$ and $f_2(z)$ is defined here (and in what follows) by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{1.7}$$

The quasi-convolution (1.7) was introduced and studied earlier by Owa ([1] and [2]), and by Schild and Silverman [3] for $p = 1$. [See also Srivastava *et al.* ([4] and [5]).] In the present paper we aim at giving several quasi-convolution properties of functions in the subclasses $\mathcal{A}(p, n, \alpha)$ and $\mathcal{B}(p, n, \alpha)$ characterized by (1.4) and (1.5), respectively.

2 A Set of Lemmas

We begin by recalling the following lemmas due to Yamakawa [6], which will be needed in proving our main results (Theorem 1 and Theorem 2 below).

Lemma 1 (Yamakawa [6]). *If $f_j(z) \in \mathcal{A}(p, n, \alpha_j)$ $(j = 1, 2)$, then $(f_1 * f_2)(z) \in \mathcal{A}(p, n, \beta)$, where*

$$\beta = \frac{[p + 2n - \alpha_1 p(p + n)]\,[p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + 2n)}{p\,\{[p + 2n - \alpha_1 p(p + n)]\,[p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + n)\}}. \tag{2.1}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z^p - \frac{p(1-\alpha_j p)}{p+2n-\alpha_j p(p+n)}\, z^{p+n} \qquad (j=1,2). \tag{2.2}$$

Lemma 2 (Yamakawa [6]). *If $f_j(z) \in \mathcal{B}(p,n,\alpha_j)$ $(j=1,2)$, then $(f_1 * f_2)(z) \in \mathcal{B}(p,n,\beta)$, where*

$$\beta = \frac{[p+2n-\alpha_1 p(p+n)][p+2n-\alpha_2 p(p+n)](p+n)-p^2(1-\alpha_1 p)(1-\alpha_2 p)(p+2n)}{p\{[p+2n-\alpha_1 p(p+n)][p+2n-\alpha_2 p(p+n)](p+n)-p^2(1-\alpha_1 p)(1-\alpha_2 p)(p+n)\}}. \tag{2.3}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z^p - \frac{p^2(1-\alpha_j p)}{(p+n)\,[p+2n-\alpha_j p(p+n)]}\, z^{p+n} \qquad (j=1,2). \tag{2.4}$$

3 Main Results and Their Consequences

One of our main results is contained in

Theorem 1. *If $f_j(z) \in \mathcal{A}(p,n,\alpha_j)$ for each $j = 1,\cdots,m$, then*

$$(f_1 * f_2 * \cdots * f_m)(z) \in \mathcal{A}(p,n,\beta),$$

where

$$\beta = \frac{1}{p} - \frac{n\,p^{m-2}\prod_{j=1}^{m}(1-\alpha_j p)}{\prod_{j=1}^{m}[p+2n-\alpha_j p(p+n)] - p^{m-1}(p+n)\prod_{j=1}^{m}(1-\alpha_j p)}. \tag{3.1}$$

The result is sharp for the functions $f_j(z)$ $(j=1,\cdots,m)$ given by

$$f_j(z) = z^p - \frac{p(1-\alpha_j p)}{p+2n-\alpha_j p(p+n)}\, z^{p+n} \qquad (j=1,\cdots,m). \tag{3.2}$$

Proof. Our proof of Theorem 1 is by induction on m. Indeed the assertion of Theorem 1 holds true when $m=1$. For $m=2$, we find from Lemma 1 that $(f_1 * f_2)(z) \in \mathcal{A}(p,n,\beta)$ with

$$\begin{aligned}\beta &= \frac{[p+2n-\alpha_1 p(p+n)]\,[p+2n-\alpha_2 p(p+n)] - p(1-\alpha_1 p)(1-\alpha_2 p)(p+2n)}{p\,\{[p+2n-\alpha_1 p(p+n)]\,[p+2n-\alpha_2 p(p+n)] - p(1-\alpha_1 p)(1-\alpha_2 p)(p+n)\}}\\ &= \frac{1}{p} - \frac{n(1-\alpha_1 p)(1-\alpha_2 p)}{[p+2n-\alpha_1 p(p+n)]\,[p+2n-\alpha_2 p(p+n)] - p(1-\alpha_1 p)(1-\alpha_2 p)(p+n)}.\end{aligned} \tag{3.3}$$

Therefore, Theorem 1 is true also for $m=2$.

Next we suppose that Theorem 1 is true for a fixed natural number m. Then, applying Lemma 1 once again, we see that

$$(f_1 * f_2 * \cdots * f_{m+1})(z) = (f_1 * f_2 * \cdots * f_m)(z) * f_{m+1}(z) \in \mathcal{A}(p,n,\gamma), \tag{3.4}$$

where

$$\begin{aligned}\gamma &= \frac{1}{p} - \frac{n(1-\alpha_{m+1}p)(1-\beta p)}{[p+2n-\alpha_{m+1}p(p+n)][p+2n-\beta p(p+n)]-p(1-\alpha_{m+1}p)(1-\beta p)(p+n)} \\ &= \frac{1}{p} - \frac{n\,p^{m-1}\prod_{j=1}^{m+1}(1-\alpha_j p)}{\prod_{j=1}^{m+1}[p+2n-\alpha_j p(p+n)]-p^m(p+n)\prod_{j=1}^{m+1}(1-\alpha_j p)}.\end{aligned} \tag{3.5}$$

Therefore, Theorem 1 is true also for $m+1$. Thus, by mathematical induction, we conclude that Theorem 1 is true for any natural number m.

Finally, if we take the functions $f_j(z)$ $(j = 1, \cdots, m)$ given by (3.2), then we have

$$\begin{aligned}(f_1 * f_2 * \cdots * f_m)(z) &= z^p - p^m \frac{\prod_{j=1}^{m}(1-\alpha_j p)}{\prod_{j=1}^{m}[p+2n-\alpha_j p(p+n)]} z^{p+n} \\ &= z^p - \Omega z^{p+n}\end{aligned} \tag{3.6}$$

where, for convenience,

$$\Omega = p^m \frac{\prod_{j=1}^{m}(1-\alpha_j p)}{\prod_{j=1}^{m}[p+2n-\alpha_j p(p+n)]}. \tag{3.7}$$

Therefore, in view of (1.4), we obtain

$$\begin{aligned}&\sum_{k=p+n}^{\infty} \frac{2k-\beta pk-p}{p(1-\beta p)}\Omega \\ &\quad = \frac{2(p+n)-\beta p(p+n)-p}{p(1-\beta p)} \cdot \frac{p^m \prod_{j=1}^{m}(1-\alpha_j p)}{\prod_{j=1}^{m}[p+2n-\alpha_j p(p+n)]} \\ &\quad = 1,\end{aligned}$$

which shows that the assertion of Theorem 1 is sharp for the functions $f_j(z)$ $(j = 1, \cdots, m)$ given by (3.2).

Setting $\alpha_j = \alpha$ $(j = 1, \cdots, m)$ in Theorem 1, we readily obtain

Corollary 1. *If $f_j(z) \in \mathcal{A}(p, n, \alpha)$ for all $j = 1, \cdots, m$, then*

$$(f_1 * f_2 * \cdots * f_m)(z) \in \mathcal{A}(p, n, \beta),$$

where

$$\beta = \frac{1}{p} - \frac{n\,p^{m-2}(1-\alpha p)^m}{[p+2n-\alpha p(p+n)]^m - p^{m-1}(p+n)(1-\alpha p)^m}. \tag{3.8}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \cdots, m)$ given by

$$f_j(z) = z^p - \frac{p(1-\alpha p)}{p+2n-\alpha p(p+n)} z^{p+n} \qquad (j = 1, \cdots, m). \tag{3.9}$$

The proof of Theorem 1 can be applied *mutatis mutandis* in order to derive

Theorem 2. *If $f_j(z) \in \mathcal{B}(p,n,\alpha_j)$ for each $j = 1, \cdots, m$, then*

$$(f_1 * f_2 * \cdots * f_m)(z) \in \mathcal{B}(p,n,\beta),$$

where

$$\beta = \frac{1}{p} - \frac{n\,p^{m-1}\prod_{j=1}^{m}(1-\alpha_j p)}{(p+n)\left\{(p+n)^{m-2}\prod_{j=1}^{m}[p+2n-p(p+n)\alpha_j] - p^m\prod_{j=1}^{m}(1-\alpha_j p)\right\}}. \quad (3.10)$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z^p - \frac{p^2(1-\alpha_j p)}{(p+n)\,[p+2n-p(p+n)\alpha_j]}\,z^{p+n} \qquad (j = 1, \cdots, m). \quad (3.11)$$

For $\alpha_j = \alpha$ $(j = 1, \cdots, m)$, Theorem 2 immediately yields **Corollary 2**. *If $f_j(z) \in \mathcal{B}(p,n,\alpha)$ for all $j = 1, \cdots, m$, then*

$$(f_1 * f_2 * \cdots * f_m)(z) \in \mathcal{B}(p,n,\beta),$$

where

$$\beta = \frac{1}{p} - \frac{n\,p^{m-1}(1-\alpha p)^m}{(p+n)\left\{(p+n)^{m-2}[p+2n-p(p+n)\alpha]^m - p^m(1-\alpha p)^m\right\}}. \quad (3.12)$$

The result is sharp for the functions $f_j(z)$ $(j = 1, \cdots, m)$ given by

$$f_j(z) = z^p - \frac{p^2(1-\alpha p)}{(p+n)\,[p+2n-p(p+n)\alpha]}\,z^{p+n} \qquad (j = 1, \cdots, m). \quad (3.13)$$

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