Measurability of linear operators in the Skorokhod topology

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Abstract

It is proved that bounded linear operators on Banach spaces of "cadlag" functions are measurable with respect to the Borel σ-algebra associated with the Skorokhod topology.

1 Introduction and notation.

Throughout this paper $\mathbb{C}^n$ is understood to be equipped with an inner product $\langle \cdot, \cdot \rangle$, defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{C}^n$. We shall write $|x| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathbb{C}^n$.
A function $f : [0, 1] \to \mathbb{C}^n$ is said to be a *cadlag function* ("continu à droite, limite à gauche") if for all $t \in [0, 1]$ one has:

$$\lim_{s \downarrow t} f(s) = f(t+) = f(t) \qquad \text{and} \qquad \lim_{s \uparrow t} f(s) = f(t-) \quad \text{exists}$$

As can be proved in an elementary way, for every cadlag function f and every $\varepsilon > 0$ the set

$$\{t \in [0, 1] : |f(t) - f(t-)| \geq \varepsilon\}$$

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is finite. It follows from this that a cadlag function can be uniformly approximated by step functions on $[0,1]$. Consequently, every cadlag function is a bounded Borel function. The linear space of all cadlag functions assuming values in $\mathbb{C}^n$ will be denoted by $\mathfrak{D}(\mathbb{C}^n)$ or, if there can be no confusion, simply by $\mathfrak{D}$.
Now $\mathfrak{D}$ is equipped with the supremum norm $\|\bullet\|$:

$$\|f\| = \sup\{|f(t)| : t \in [0,1]\}$$

In this way $\mathfrak{D}$ becomes a non-separable Banach space, we shall denote it by $\mathfrak{D}_B$.
In [8] and [9] Skorokhod introduced on $\mathfrak{D}$ a weaker topology which turns it into a Polish space. We shall refer to this topology as the Skorokhod topology. The space $\mathfrak{D}$, equipped with this topology, will be denoted by $\mathfrak{D}_S$.
It can be proved (see Billingsley [1]) that the identity map $I : \mathfrak{D}_S \to \mathfrak{D}_B$ is continuous in every f which is continuous on $[0,1]$. In particular I is continuous in the origin.
The map I is of course not continuous everywhere on $\mathfrak{D}_S$. It thus appears that the topology on $\mathfrak{D}_S$ is not translation invariant; consequently $\mathfrak{D}_S$ is not a topological vector space.
Although the Skorokhod topology is not compatible with the linear structure on $\mathfrak{D}$, the corresponding Borel σ-algebra is. In fact we shall see (theorem 3) that it presents the “cylindrical” σ-algebra on the Banach space $\mathfrak{D}_B$.
In the sequel the only thing that we shall need in connection to the Skorokhod topology is that for all $t \in [0,1]$ the map

$$f \to f(t)$$

is a Borel function on $\mathfrak{D}_S$ (see Billingsley [1]).
It follows from this that for all $t \in [0,1]$ the map

$$f \to f(t-) = \lim_{n\to\infty} f(t-\frac{1}{n}) \ ,$$

being the pointwise limit of a sequence of Borel functions, is also a Borel function on $\mathfrak{D}_S$.

2 The dual space of the Banach space $\mathfrak{D}_B$

In this section we are going to study the structure of continuous linear forms on $\mathfrak{D}_B(\mathbb{C}^n)$, that is, we are going to describe the dual space $\mathfrak{D}_B^*$ of $\mathfrak{D}_B$ (see also Corson [2]).
For any index set I and any $\varphi : I \to \mathbb{C}^n$ we define:

$$\sum_{a\in I} |\varphi(a)| = \sup\{ \sum_{a\in F} |\varphi(a)| \ : \ F \text{ a finite subset of } I\}$$

If $\sum_{a\in I} |\varphi(a)| < +\infty$, then the limit

$$\lim_F \sum_{a\in F} \varphi(a) = \sum_{a\in I} \varphi(a)$$

exists in $\mathbb{C}^n$, where the filtration on the collection of finite sets F is understood to be defined by inclusion.
The set of all $\varphi : I \to \mathbb{C}^n$ such that $\sum_{a\in I} |\varphi(a)| < +\infty$ will be denoted by $\ell^1(I, \mathbb{C}^n)$.
If $m_1, \ldots, m_n$ are complex Borel measures on $[0,1]$ then we shall write:

$$\mathbf{m} = (m_1, \ldots, m_n)$$

For all $\mathbf{m}$ and all $\varphi \in \ell^1([0,1], \mathbb{C}^n)$ we define a map $[\mathbf{m}, \varphi] : \mathfrak{D} \to \mathbb{C}$ by:

$$[\mathbf{m}, \varphi](f) = \sum_{i=1}^{n} \int f_i \, d\overline{m}_i + \sum_{a\in[0,1]} \langle f(a) - f(a-), \varphi(a) \rangle \ ,$$

where $f = (f_1, \ldots, f_n) \in \mathfrak{D}(\mathbb{C}^n)$.
The following theorem is stated in the notations introduced above:

Theorem 1. (i) For all $\mathbf{m} = (m_1, \ldots, m_n)$ and $\varphi \in \ell^1([0,1], \mathbb{C}^n)$ the map $[\mathbf{m}, \varphi] : \mathfrak{D}_B(\mathbb{C}^n) \to \mathbb{C}$ is a continuous linear form.
(ii) For every continuous linear form l on the Banach space $\mathfrak{D}_B(\mathbb{C}^n)$ there exists a unique $\mathbf{m} = (m_1, \ldots, m_n)$ and a unique $\varphi \in \ell^1([0,1], \mathbb{C}^n)$ such that $l = [\mathbf{m}, \varphi]$.

Proof. The proof of (i) is left to the reader.
We prove statement (ii) in the case where $n = 1$. The general case can easily be deduced from this, for $\mathfrak{D}_B(\mathbb{C}^n)$ is in an obvious way the direct sum of copies of $\mathfrak{D}_B(\mathbb{C})$.
Let l be an arbitrary continuous linear form on $\mathfrak{D}_B = \mathfrak{D}_B(\mathbb{C})$. By Riesz's representation theorem the restriction of l to the subspace $C([0,1])$ of continuous functions on $[0,1]$ defines a complex Borel measure on $[0,1]$. This measure will be denoted by m.
The continuous linear form $\tilde{l}$ on $\mathfrak{D}_B$ is defined by

$$\tilde{l}(f) = l(f) - \int f \, dm \qquad \text{for all } f \in \mathfrak{D}$$

Now one has $\tilde{l}(f) = 0$ for every $f \in C([0,1])$.
For every finite set $F \subset [0,1]$ we define the linear subspace $\mathfrak{M}_F$ by:

$$\mathfrak{M}_F = \{ f \in \mathfrak{D} \ : \ f(a) - f(a-) = 0 \quad \text{if } a \notin F \}$$

In other words, $\mathfrak{M}_F$ comprises those $f \in \mathfrak{D}$ which have a possible jump in the points of F only.
For every $a \in (0,1]$ and sufficiently small $\delta > 0$ we define the function $\mathbf{1}_a^\delta$ by:

$$\left.\begin{aligned} \mathbf{1}_a^\delta(t) &= \tfrac{1}{\delta}(t-a+\delta) && \text{if } t \in (a-\delta, a) \\ &= 0 && \text{elsewhere on } [0,1] \end{aligned}\right\}$$

If $f \in \mathfrak{M}_F$, then for sufficiently small $\delta > 0$ the function

$$f + \sum_{a\in F} \{ f(a) - f(a-) \} \ \mathbf{1}_a^\delta$$

is an element of $C([0,1])$. Therefore:

$$\widetilde{l}\left(f + \sum_{a\in F}\{f(a) - f(a-)\}\, \mathbf{1}_a^\delta\right) = 0$$

Consequently we have for all $f \in \mathfrak{M}_F$

$$\widetilde{l}(f) = -\sum_{a\in F}\{f(a) - f(a-)\}\, \widetilde{l}(\mathbf{1}_a^\delta)$$

Keeping a fixed, the difference of two functions of type $\mathbf{1}_a^\delta$ is in $C[0,1]$. We see in this way that the expression $\widetilde{l}(\mathbf{1}_a^\delta)$ does not depend on δ.
For every $a \in [0,1]$, define $\varphi(a) = -\,\widetilde{l}(\mathbf{1}_a^\delta)$. We then have:

$$\widetilde{l}(f) = \sum_{a\in F}\varphi(a)\{f(a) - f(a-)\} \qquad \text{for all } f \in \mathfrak{M}_F$$

Our next goal is to prove that $\varphi \in \ell^1([0,1],\mathbb{C})$. For every finite $F \subset [0,1]$ we define the "complex saw tooth function" f_F in the following way:

- $f_F(a) = \dfrac{\overline{\varphi(a)}}{|\varphi(a)|}$ if $a \in F$ and $\varphi(a) \neq 0$
- $f_F(a) = \quad 1 \qquad$ if $a \in F$ and $\varphi(a) = 0$
- f_F is a linear function on each connected component of F^c, such that for all $a \in F$ one has $f_F(a+) = f_F(a)$ and $f_F(a-) = 0$

Now $\|f_F\| \leq 1$ for all F. Therefore we have:

$$\sup_F \sum_{a\in F} |\varphi(a)| = \sup_F |\widetilde{l}(f_F)| < +\infty$$

It follows from this that $\varphi \in \ell^1([0,1],\mathbb{C})$, so the map

$$f \to \sum_{a\in[0,1]}\{f(a) - f(a-)\}\,\varphi(a)$$

is continuous on $\mathfrak{D}_B$.
For all $f \in \bigcup_F \mathfrak{M}_F$ we have

$$\widetilde{l}(f) = \sum_{a\in[0,1]}\{f(a) - f(a-)\}\,\varphi(a) \tag{$*$}$$

The linear space $\bigcup \mathfrak{M}_F$ being dense in $\mathfrak{D}_B$, this implies that $(*)$ holds for *all* $f \in \mathfrak{D}_B$. In this way we see, by definition of $\widetilde{l}$, that $l = [\overline{m}, \overline{\varphi}]$.
Unicity of m and φ can be proved easily; this is left to the reader.

Next, let Ω be an arbitrary set, $\mathcal{F}$ a σ-algebra of subsets of Ω and M a topological space. A map $X : \Omega \to M$ is said to be $\mathcal{F}$-measurable (or simply measurable if no confusion can arise) if $X^{-1}(A) \in \mathcal{F}$ for all Borel sets A in M. If M is a Banach space then a map $X : \Omega \to M$ is said to be *scalarly measurable* if for every continuous linear form l on M the composition $l \circ X : \Omega \to \mathbb{C}$ is measurable. A well-known theorem in functional analysis (due to B.J. Pettis [6]) states that in case of a *separable* Banach space, measurability is equivalent to scalar measurability. If M is non-separable then this statement is in general not true. In fact, it is easy to construct a counterexample in case $M = \mathfrak{D}_B(\mathbb{C})$:

Example. Let $\Omega = [0,1]$ and let $\mathcal{F}$ be the σ-algebra consisting of all Borel sets in $[0,1]$. Define $X : \Omega \to \mathfrak{D}_B(\mathbb{C})$ by:

$$X(s) = \mathbf{1}_{[0,s)} \qquad \text{for all } s \in [0,1]$$

For any continuous linear form $l = [m, \varphi]$ we have:

$$l(X(s)) = m\{[0,s)\} + \varphi(s) \qquad \text{for all } s \in [0,1]$$

The condition that $\sum_a |\varphi(a)| < +\infty$ implies that the set of points s for which $\varphi(s) \neq 0$ is at most countably infinite. Keeping this in mind, measurability of the map $s \to l(X(s))$ can be proved by easy verification. It thus appears that X is scalarly measurable.

Next we are going to prove that $X : \Omega \to \mathfrak{D}_B$ is not measurable.

Let $A \subset [0,1]$ be a set which is not Borel. Define

$$\mathfrak{A} = \{\mathbf{1}_{[0,s)} \ : \ s \in A\} \subset \mathfrak{D}_B$$

Denote the convex hull of $\mathfrak{A}$ by $\mathfrak{C}$. It is not hard to prove that for all $t \notin A$

$$\|\mathbf{1}_{[0,t)} - f\| \geq \frac{1}{2}$$

for every $f \in \mathfrak{C}$, and consequently also for every f in the closure $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ in $\mathfrak{D}_B$. In this way it turns out that $X^{-1}(\overline{\mathfrak{C}}) = A$. This shows that X is neither measurable in the norm, nor in the weak topology associated with the Banach space $\mathfrak{D}_B$. (To the author it is not known whether the Borel σ-algebras corresponding to the norm and the weak topology on $\mathfrak{D}_B$ really differ (see also Edgar [3]). Talagrand proved in [10] and [11] the existence of Banach spaces where both σ-algebras are different).

3 Measurability in the Skorokhod topology.

As announced earlier, the linear space $\mathfrak{D}$ equipped with the Skorokhod topology will be denoted by $\mathfrak{D}_S$. A map $X : \mathfrak{D}_S \to M$, where M is a topological space, is said to be measurable if it is measurable with respect to the Borel σ-algebra of $\mathfrak{D}_S$.

Theorem 2. Let l be a continuous linear form on the Banach space $\mathfrak{D}_B(\mathbb{C}^n)$. Then $l : \mathfrak{D}_S(\mathbb{C}^n) \to \mathbb{C}$ is measurable.

Proof. The proof is split up into three steps.
If $f = (f_1, \dots, f_n) \in \mathfrak{D}(\mathbb{C}^n)$ and $\mathbf{m} = (m_1, \dots, m_n)$ where $m_1, \dots, m_n$ are complex Borel measures on $[0,1]$, then we shall write

$$\int \langle f, d\mathbf{m} \rangle = \sum_{j=1}^{n} \int f_j \, d\overline{m}_j$$

step 1: If δ_a is the Dirac measure in the point a and if $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, then we denote

$$\mathbf{m} = c\delta_a = (c_1\delta_a, \dots, c_n\delta_a)$$

It is known that the map $f \to f(a)$ is measurable on $\mathfrak{D}_S$ (see Billingsley [1]), so it follows that, in case $\mathbf{m} = c\delta_a$, the map

$$f \to \int \langle f, d\mathbf{m} \rangle = \langle f(a), c \rangle$$

is also measurable on $\mathfrak{D}_S$.

step 2: Next we are going to prove that for arbitrary complex measures $m_1, \dots, m_n$ on $[0,1]$ the map

$$f \to \int \langle f, d\mathbf{m} \rangle$$

is measurable on $\mathfrak{D}_S$.
For every $k \in \mathbb{N}$ we define the 2^k intervals I_i^k by

$$I_i^k = [(i-1)/2^k, \ i/2^k) \qquad\qquad i = 1, 2, \dots, 2^k$$

Moreover, for every $f \in \mathfrak{D}$ a sequence $f_k \in \mathfrak{D}$ is defined by:

$$f_k = \left(\sum_{i=1}^{2^k} f(i/2^k) \mathbf{1}_{I_i^k} \right) + f(1) \ \mathbf{1}_{\{1\}}$$

Now if $k \to \infty$ one has (because $f(t+) = f(t)$) that $f_k(t) \to f(t)$ for every $t \in [0,1]$.
For all Borel sets $A \subset [0,1]$ we write

$$\mathbf{m}(A) = (m_1(A), \dots, m_n(A))$$

and we define

$$\mathbf{m}_k = \left(\sum_{i=1}^{2^k} \mathbf{m}(I_i^k) \ \delta_{i/2^k} \right) + \mathbf{m}(\{1\})\delta_1$$

Then

$$\int \langle f, d\mathbf{m}_k \rangle = \sum_{i=1}^{2^k} \langle f(i/2^k), \mathbf{m}(I_i^k) \rangle + \langle f(1), \mathbf{m}\{1\} \rangle = \int \langle f_k, d\mathbf{m} \rangle$$

So by Lebesgue's bounded convergence theorem, we have for all $f \in \mathfrak{D}$

$$\int \langle f, d\mathbf{m} \rangle = \lim_{k \to \infty} \int \langle f, d\mathbf{m}_k \rangle$$

By step 1 the maps

$$f \to \int \langle f, d\mathbf{m}_k \rangle$$

are measurable on $\mathfrak{D}_S$. It follows from this that the map

$$f \to \int \langle f, d\mathbf{m} \rangle \ ,$$

being the pointwise limit of a sequence of measurable maps, is measurable on $\mathfrak{D}_S$.

step 3: If $\varphi \in \ell^1([0,1], \mathbb{C}^n)$ then the map

$$f \to \sum_{a \in [0,1]} \langle f(a) - f(a-), \varphi(a) \rangle$$

is measurable on $\mathfrak{D}_S$.

To prove this, we observe that the set $\{a \mid \varphi(a) \neq 0\}$ is at most countably infinite. Measurability is now easily verified, for the maps

$$f \to f(a) \qquad \text{and} \qquad f \to f(a-)$$

are measurable on $\mathfrak{D}_S$.

Finally, by step 2, step 3, and theorem 1 we conclude that every continuous linear form on $\mathfrak{D}_B$ is measurable on $\mathfrak{D}_S$. This proves the theorem.

The following theorem gives a characterization of the Borel σ-algebra of $\mathfrak{D}_S$.

Theorem 3. The Borel σ-algebra of $\mathfrak{D}_S$ is generated by the maps $l : \mathfrak{D}_S \to \mathbb{C}$, where $l \in \mathfrak{D}_B^*$.

Proof. This is a direct consequence of theorem 2 and the fact that the maps of type $f \to f(a)$ generate the Borel σ-algebra of $\mathfrak{D}_S$ (see Billingsley [1] or apply Fernique's theorem, see Schwartz [7]).

The theorem above enables us to prove:

Theorem 4. If $T : \mathfrak{D}_B(\mathbb{C}^m) \to \mathfrak{D}_B(\mathbb{C}^n)$ is a bounded linear operator then $T : \mathfrak{D}_S(\mathbb{C}^m) \to \mathfrak{D}_S(\mathbb{C}^n)$ is measurable.

Proof. To prove that $T : \mathfrak{D}_S(\mathbb{C}^m) \to \mathfrak{D}_S(\mathbb{C}^n)$ is measurable it is, by theorem 3, sufficient to prove that for all $l \in \mathfrak{D}_B^*(\mathbb{C}^n)$ the composition $l \circ T : \mathfrak{D}_S(\mathbb{C}^m) \to \mathbb{C}$ is measurable. This is trivial, because $l \circ T \in \mathfrak{D}_B^*(\mathbb{C}^m)$.

Closing remarks

In stochastic analysis one is sometimes encountered with variables assuming values in $\mathfrak{D}_S$. By theorem 3, measurability of such variables is equivalent to scalar measurability with respect to the Banach space $\mathfrak{D}_B$. There is no loss of measurability if bounded linear transformations are applied (see for example J. Kormos e.a. [4] or T. van der Meer [5]).

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