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Characterizing Certain Staircase Convex Sets in $\mathbb{R}^d$

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Abstract. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Let $T \subseteq S$. The set T lies in a staircase convex subset of S if and only if for every a, b in T there is an $a - b$ staircase path in S. This result, in turn, yields necessary and sufficient conditions for S to be a union of k staircase convex sets, $k \geq 1$. Analogous results characterize S as a union of k staircase starshaped sets.

Further, when $d \geq 3$, the set S above will be staircase convex if and only if for every chain A of boxes in $\mathcal{C}$, each projection of A into a coordinate hyperplane is staircase convex.

Finally, if S is any orthogonal polytope in $\mathbb{R}^d$, $d \geq 2$, S is staircase convex if and only if, for every j-flat F parallel to a coordinate flat, $F \cap S$ is connected, $1 \leq j \leq d - 1$.

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1. Introduction

We begin with some definitions from [2] and [3]. A set B in $\mathbb{R}^d$ is called a *box* if and only if B is a convex polytope (possibly degenerate) whose edges are parallel to the coordinate axes. A set S in $\mathbb{R}^d$ is an *orthogonal polytope* if and only if S is a connected union of finitely many boxes. Let λ be a simple polygonal path in

$\mathbb{R}^d$ whose edges are parallel to the coordinate axes. For x, y in S, the path λ is called an $x - y$ *path* in S if and only if λ lies in S and contains the points x and y; λ is an $x - y$ *geodesic* in S if and only if λ is an $x - y$ path of minimal length in S. (Clearly an $x - y$ geodesic need not be unique.) The path λ is a *staircase path* if and only if no two of its edges have opposite directions. That is, for each standard basis vector $e_i, 1 \le i \le d$, all the edges of λ parallel to e_i have the same direction. For convenience of notation, we use e_i or $-e_i$ to indicate the associated direction. Clearly if λ is a staircase path in S with endpoints x and y, then λ is an $x-y$ geodesic in S. Moreover, if S contains an $x-y$ staircase path, then every $x - y$ geodesic in S is an $x - y$ staircase.

For points x and y in a set S, we say x *sees* y (x is *visible* from y) *via staircase paths* if and only if there is a staircase path in S that contains both x and y. A set S is *staircase convex (orthogonally convex)* if and only if for every pair x, y in S, x sees y via staircase paths. Similarly, a set S is *staircase starshaped (orthogonally starshaped)* if and only if for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S. For a set S in the plane, S is called *horizontally convex* if and only if for each x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Using a result by Motwani et al. [13, Lemma 1], an orthogonal polygon S in the plane is staircase convex if and only if S is both horizontally convex and vertically convex.

We will use a few standard terms from graph theory. For $F = \{C_1, \dots, C_n\}$ a finite collection of distinct sets, the *intersection graph* G of F has vertex set $c_1, \dots, c_n$. Further, for $1 \le i < j \le n$, the points c_i, c_j determine an edge in G if and only if the corresponding sets C_i, C_j in F have a nonempty intersection. A graph G is a *tree* if and only if G is connected and acyclic. A sequence $v_1, \dots, v_k$ of vertices in G is a *walk* if and only if each consecutive pair v_i, v_{i+1} determines an edge of G, $1 \le i \le k - 1$. A walk $v_1, \dots, v_n$ is a *path* if and only if its points are distinct.

Finally, for $B_1, \dots, B_n$ a collection of distinct boxes in $\mathbb{R}^d$, we say that their union is a *chain* of boxes (relative to our ordering) if and only if the intersection graph of $\{B_1, \dots, B_n\}$ is the path $b_1, \dots b_n$ (where b_i represents the set B_i in the intersection graph, $1 \le i \le n$). That is, relative to our labeling, for $1 \le i < j \le k, B_i \cap B_j \neq \emptyset$ if and only if $j = i + 1$.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that instead use the idea of visibility via staircase paths. For example, the familiar Krasnosel'skii theorem [8] says that, for a nonempty compact set S in the plane, S is starshaped via segments if and only if every three points of S see via segments in S a common point. In the staircase analogue [1], for a nonempty simply connected orthogonal polygon S in $\mathbb{R}^2, S$ is staircase starshaped if and only if every two points of S see via staircase paths in S a common point. Moreover, in an interesting study involving rectilinear spaces, Chepoi [4] has generalized the planar result to any finite union S of boxes in $\mathbb{R}^d$ whose corresponding intersection graph is a tree. In this paper, we examine such a union S of boxes in $\mathbb{R}^d$ and extend other planar results to S, obtaining necessary and sufficient conditions for S to be a union of k staircase convex (or k staircase starshaped) sets. Finally, we generalize a planar result [13, Lemma 1] to give necessary and sufficient conditions for an arbitrary union of boxes in $\mathbb{R}^2$ to be staircase convex.

We will use the following terminology. For convenience, we call each of the hyperplanes $\{(x_1, \dots, x_d) : x_i = 0\}, 1 \leq i \leq d$, a *coordinate hyperplane*. Similarly, any intersection of coordinate hyperplanes will be a *coordinate flat*, while any projection of $\mathbb{R}^d$ onto a coordinate hyperplane will be a *coordinate projection.* If λ is a simple path containing points x and y, then $\lambda(x, y)$ will denote the subpath of λ from x to y (ordered from x to y). Readers may refer to Valentine [14], to Lay [10], to Danzer, Grünbaum, Klee [5], to Eckhoff [6], to Martini and Soltan [11], and to Martini and Wenzel [12] for discussions concerning visibility via straight line segments and starshaped sets. Readers may refer to Harary [7] for information on intersection graphs, trees, and other graph theoretic concepts.

2. The results

We begin with a lemma.

Lemma 1. *Let $\mathcal{C} = \{C_1, \dots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \dots \cup C_n$. Set S is staircase convex if and only if, for every chain A of boxes in $\mathcal{C}$, A is staircase convex.*

Proof. We begin with some preliminary comments. Observe that any path δ in S corresponds in an obvious way to an associated walk (not necessarily unique) in the intersection graph of $\mathcal{C}$, where the walk is determined by the C_i sets met by δ. For points a, b in S and for any $a-b$ geodesic $\lambda = \lambda(a, b)$ in S, λ corresponds to a path $w(\lambda)$ in the intersection graph of $\mathcal{C}$. The path $w(\lambda)$ need not be unique, since the point a may belong to two members of $\mathcal{C}$, as may the point b. Similarly, the geodesic λ lies in a corresponding chain $C_{1(\lambda)} \cup \dots \cup C_{k(\lambda)}$ of boxes in $\mathcal{C}$, where $a \in C_{1(\lambda)}, b \in C_{k(\lambda)}$. (Again, $C_{1(\lambda)}, \dots, C_{k(\lambda)}$ need not be unique.) Since the intersection graph of $\mathcal{C}$ is a tree, clearly $C_{1(\lambda)} \cup \dots \cup C_{k(\lambda)}$ contains every $a-b$ geodesic in S. Moreover, if $k(\lambda) \geq 3$ then intermediate boxes $C_{2(\lambda)}, \dots, C_{(k-1)(\lambda)}$ are uniquely determined by a and b. Of course, if $a \in C_{2(\lambda)}$ then $\lambda \subseteq C_{2(\lambda)} \cup \dots \cup C_{(k-1)(\lambda)}$. Similarly, if $a \in C_{2(\lambda)}$ and $b \in C_{(k-1)(\lambda)}$, then $\lambda \subseteq C_{2(\lambda)} \cup \dots \cup C_{(k-1)(\lambda)}$. This implies that if $k(\lambda)$ is as small as possible and $k(\lambda) \geq 2$ then the corresponding boxes $C_{1(\lambda)}, \dots, C_{k(\lambda)}$ are uniquely determined by a and b.

To establish the lemma, assume that S is staircase convex, with A a chain of boxes in $\mathcal{C}$. For convenience of notation, let $A = C_1 \cup \dots \cup C_k$. To see that A is staircase convex, select a, b in A to find a corresponding $a-b$ staircase path in A. If $k = 1$ or $k = 2$, the result is easy, so assume that $k \geq 3$. Without loss of generality, assume that $a \in C_1 \backslash C_2, b \in C_k \backslash C_{k-1}$ (for otherwise we could restrict our attention to a subchain). The set S contains an $a-b$ staircase path $\lambda(a, b)$, and by our preliminary comments $\lambda(a, b)$ lies in a chain of boxes $C'_1 \cup \dots \cup C'_m$ in $\mathcal{C}$, where $a \in C'_1, b \in C'_m$, and where m is as small as possible. Since the intersection graph of $\mathcal{C}$ is a tree, each set $C'_1, \dots, C'_m$ appears in $\{C_1, \dots, C_k\}$. Then C'_1 is either C_1 or C_2, and since $a \notin C_2$, $C'_1 = C_1$. Similarly, $C'_m = C_k$, and the chains are identical. Therefore $\lambda(a, b)$ lies in our original chain A. We conclude that A is staircase convex, the desired result.

Conversely, assume that each chain of boxes in $\mathcal{C}$ is staircase convex to prove that S is staircase convex. Let s, t belong to S, and let $\lambda(s, t)$ be any $s - t$ geodesic in S. By our preliminary comments, $\lambda(s, t)$ lies in a chain A of boxes in $\mathcal{C}$. Since A is staircase convex, $\lambda(s, t)$ must be a staircase path. Hence S is staircase convex, finishing the proof of the lemma.

The first theorem is a d-dimensional analogue of [3, Lemmas 1 and 2].

Theorem 1. *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Let $T \subseteq S$. If for every a, b in T, there is an $a - b$ staircase in S, then T lies in a staircase convex union of boxes $B_1 \cup \cdots \cup B_m$, where B_i is a subset of some associated box $C_i, 1 \le i \le m$ (for an appropriate labeling of members of $\mathcal{C}$).*

To establish the theorem, for each a, b in T, let $U(a, b)$ denote the union of all staircase $a - b$ paths in S, and let $U = \cup\{U(a, b) : a, b \text{ in } T\}$. For each $C_i, 1 \le i \le n$, select a smallest box B_i (possibly empty) such that $B_i \subseteq C_i$ and $B_i \cap U = C_i \cap U$. That is, B_i is the smallest subbox of C_i containing $C_i \cap U$. Remove any empty B_i sets. For convenience of notation, assume that $B_1, \ldots, B_m$ are the remaining (that is, nonempty) B_i sets, with $B_1, \ldots, B_m$ distinct. Let $\mathcal{B} = \{B_1, \ldots, B_m\}$. Clearly $B_1 \cup \cdots \cup B_m$ is connected, and the intersection graph of $\mathcal{B}$ is a tree.

We assert that the union $B_1 \cup \cdots \cup B_m$ satisfies the theorem. Certainly $T \subseteq B_1 \cup \cdots \cup B_m$, so we need only show that $B_1 \cup \cdots \cup B_m$ is staircase convex. We will prove that any chain of boxes from $\mathcal{B}$ is staircase convex. For convenience of notation, let $B_1 \cup \cdots \cup B_k$ be a chain of boxes in $\mathcal{B}$. The result is true for $k = 1$ and for $k = 2$. Inductively, let $k \ge 3$ and assume that the result holds for chains of $k-1$ or fewer boxes in $\mathcal{B}$. Without loss of generality, choose $x \in B_1 \backslash B_2, y \in B_k \backslash B_{k-1}$, to find an $x - y$ staircase in $B_1 \cup \cdots \cup B_k$. Select x' in B_2 closest to x. Observe that any $x - x'$ staircase lies in B_1 and uses at least one and at most d directions. For convenience of notation, we label these directions $e_1, \ldots, e_j$ for some j, $1 \le j \le d$, where e_i is orthogonal to a hyperplane H_i supporting B_2 at x' and where x is beyond H_i, $1 \le i \le j$. (In case B_2 is fully d-dimensional, then each H_i is determined by a facet F_i of B_2 with $x' \in F_i$ and with x beyond H_i, $1 \le i \le j$.) For future reference, observe that any staircase from x to B_2 must use at least the directions $e_1, \ldots, e_j$. Moreover, since $B_1 \cup \cdots \cup B_{k-1}$ (by our induction hypothesis) is staircase convex, it follows that all points of $B_2 \cup \cdots \cup B_{k-1}$ must lie on or beneath each H_i, $1 \le i \le j$.

Recall that $B_2 \cup \cdots \cup B_k$ is staircase convex as well and hence contains an $x' - y$ staircase. If point y is on or beneath each of the hyperplanes $H_i, 1 \le i \le j$, then no $x' - y$ staircase uses direction $-e_i$, $1 \le i \le j$. We may combine any $x - x'$ staircase in B_1 with any $x' - y$ staircase in $B_2 \cup \cdots \cup B_k$ to produce an $x - y$ staircase in $B_1 \cup \cdots \cup B_k$, the desired result.

It remains to show that y does lie on or beneath each of the hyperplanes H_i, $1 \le i \le j$. Suppose on the contrary that y lies beyond at least one of these hyperplanes, to obtain a contradiction. For convenience, assume that y lies beyond the hyperplane H_1. Then for some pair c, d in T and for some staircase $c - d$ path $\lambda(c, d)$ in S, $\lambda(c, d)$ contains a point y_1 in B_k beyond H_1. Clearly at least one of c, d must lie beyond H_1. Without loss of generality, assume that c lies beyond H_1, and consider the subpath $\lambda(y_1, c)$ of $\lambda(d, c)$ from y_1 to c. The staircase path $\lambda(y_1, c)$ lies in a chain $B_k \cup \cdots \cup B_r$ of boxes in $\mathcal{B}$, with $c \in B_r$. Observe that the boxes in $\{B_k, \ldots, B_r\}$ are distinct from those in $\{B_2, \ldots, B_{k-1}\}$, since the staircase path $\lambda(y_1, c)$ is entirely beyond H_1 while all points of $B_2 \cup \cdots \cup B_{k-1}$ are beneath (or on) H_1.

Similarly, the point x is beyond H_1, so for some c', d' in T and some staircase $c' - d'$ path $\lambda'(c', d')$ in S, $\lambda'(c', d')$ contains a point x_1 in B_1 beyond H_1. Assume that c' lies beyond H_1, and choose a chain $B'_s \cup \cdots \cup B_1$ of boxes in $\mathcal{B}$ containing $\lambda'(c', x_1)$, with $c' \in B'_s$. Again observe that the boxes in $\{B'_s, \ldots, B_1\}$ are distinct from those in $\{B_2, \ldots, B_{k-1}\}$. Also, since the intersection graph of $\mathcal{B}$ is a tree, the union $B'_s \cup \cdots \cup B_1 \cup B_2 \cup \cdots \cup B_{k-1} \cup B_k \cup \cdots \cup B_r$ defines a chain. Clearly any $c' - c$ geodesic in S must lie in this chain. Moreover, since c', $c \in T$ any $c' - c$ geodesic is a staircase path. However, since c' and c are beyond H_1, the travel from c' to B_1 to B_2 requires direction e_1, while travel from B_2 to c requires direction $-e_1$. That is, no $c' - c$ geodesic in this chain can be staircase. We have a contradiction, our supposition is false, and y lies on or beneath each hyperplane H_1, $1 \le i \le j$.

Using an earlier argument, we conclude that $B_1 \cup \cdots \cup B_k$ does contain an $x - y$ staircase, finishing the induction. That is, any chain of boxes from $\mathcal{B}$ is staircase convex. By Lemma 1, it follows that $B_1 \cup \cdots \cup B_m$ is indeed staircase convex, completing the proof of Theorem 1.

The following corollaries are d-dimensional analogues of [3, Theorems 1 and 3].

Corollary 1.1. *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Assume that, for every finite subset F of S, there is a k-partition of F into subsets $F_1, \ldots, F_k$ such that every pair in F_i can be joined by a staircase path in S, $1 \le i \le k$. Then S is a union of k staircase convex sets.*

Proof. The argument, just like the proof of [3, Theorem 1], is included for completeness. By a result of Lawrence, Hare, Kenelly [9, Theorem 1], the hypothesis for finite subsets of S implies that there is a corresponding k-partition of S, say $\{S_1, \ldots, S_k\}$, such that, for every finite subset F of S, every pair in $F \cap S_i$ can be joined by a staircase path in S, $1 \le i \le k$. Then every pair in S_i can be joined by a staircase path in S and, by Theorem 1, S_i lies in a staircase convex union P_i of boxes in S, $1 \le i \le k$. Hence $S = \cup\{S_i : 1 \le i \le k\}$ is a union of the k staircase convex sets $P_1, \ldots, P_k$.

Corollary 1.2. *Let $\mathcal{C} = \{C_1, \ldots C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Assume that, for every sequence $v_1, \ldots, v_m, v_{m+1} = v_1$, in S, m odd, at least one consecutive pair v_i, v_{i+1},* *can be joined by a staircase path in S. Then S is a union of two staircase convex sets.*

Proof. The proof replicates the argument in [3, Theorem 3]. For F any finite subset of S, define a corresponding graph G_F as follows: The vertices of G_F correspond to points in F. Further, two points of G_F are adjacent if and only if their associated points in F can be joined by a staircase path in S.

Let G^C represent the graph complement of G_F. It is not hard to show that G^C contains no odd cycles and hence G^C is a bigraph. (See [3, Theorem 3] for details.)

Finally, let $\{A_1, A_2\}$ be a partition of the vertex set of G^C satisfying the definition of a bigraph. Then $\{A_1, A_2\}$ induces a corresponding partition of F, and every two points of A_1 (of A_2) can be joined by a staircase path in S. The result follows from Corollary 1.1 above.

Remark. Clearly, the converse of Corollary 1.1 and the converse of Corollary 1.2 hold as well.

Using results of Chepoi [4], we obtain the following starshaped analogues of Theorem 1 and its corollaries. Notice that Theorem 2 is a d-dimensional variation of [3, Lemma 3].

Theorem 2. *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Let T be a nonempty subset of S. If every two points of T see a common point of S via staircase paths, then T lies in a subset of S that is starshaped via staircase paths.*

Proof. For completeness, we include some definitions from [4]. A metric space is called a *median space* if and only if, for each triple of points x, y, z, there is a unique "median" point between each pair of x, y, z. A *median graph* is a graph whose standard graph-metric generates a median space, while a *median polyhedron* in $\mathbb{R}^d$ is the geometric realization of some finite median graph. (See [4], [15] for detailed discussions.)

For each point t in T, define $V(t) = \{x : t \text{ sees } x \text{ via staircase paths in } S\}$. By Chepoi's results in [4, Corollary 2], S is a median polyhedron. Moreover, by [4, Theorem], each set $V(t)$ is compact and convex in the corresponding median space. Using Helly's theorem for median spaces [15], since every two of the $V(t)$ sets meet, they all meet. Choose $t_0 \in \cap \{V(t) : t \text{ in } T\}$. Every point of T sees t_0 via staircase paths, so T lies in the starshaped set $V(t_0) \subseteq S$.

The next two corollaries are d-dimensional analogues of [3, Theorems 4 and 5].

Corollary 2.1. *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Assume that for every finite subset F of S there is a k-partition of F into subsets $F_1, \ldots, F_k$ such that every two points in F_i see a common point of S via staircase paths, $1 \leq i \leq k$. Then S is a union of k staircase starshaped sets.*

Proof. The argument parallels the proof of Corollary 1.1, using Theorem 2 in place of Theorem 1.

Corollary 2.2. *Let $\mathcal{C} = \{C_1, \dots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \dots \cup C_n$. Assume that for every sequence $v_1, \dots, v_m, v_{m+1} = v_1$, in S, m odd, at least one consecutive pair v_i, v_{i+1} sees a common point of S via staircase paths. Then S is a union of two staircase starshaped sets.*

Proof. The argument parallels the proof of Corollary 1.2.

Remark. Clearly, the converse of each corollary holds.

Theorem 3 will use projections to determine whether a chain of boxes is staircase convex. The following easy lemma will be helpful.

Lemma 2. *For $d \geq 2$ and for each $i, 1 \leq i \leq d$, let Π_i denote the coordinate projection from $\mathbb{R}^d$ onto the coordinate hyperplane $\{(x_1, \dots, x_d) : x_i = 0\}$. Let A, C be boxes in $\mathbb{R}^d$. If $A \cap C = \emptyset$, then for at least $d-1$ of the projections $\Pi_i, 1 \leq i \leq d$, $\Pi_i(A) \cap \Pi_i(C) = \emptyset$.*

Proof. Let A be the product of intervals $[a_i, b_i]$, $1 \leq i \leq d$, and let C be the product of intervals $[c_i, d_i]$, $1 \leq i \leq d$. If $A \cap C = \emptyset$, then for at least one i, say for $i = 1$, $[a_1, b_1] \cap [c_1, d_1] = \emptyset$. Without loss of generality, assume $a_1 \leq b_1 < c_1 \leq d_1$. Let H_1 be a hyperplane orthogonal to the x_1 axis at any point of the segment (b_1, c_1). Then H_1 strictly separates A and C. For $i \neq 1$, the $(d-2)$-flat $\Pi(H_i)$ strictly separates $\Pi_i(A)$ and $\Pi_i(C)$. That is, $\Pi_i(A) \cap \Pi_i(C) = \emptyset$ for $2 \leq i \leq d$.

Theorem 3. *Let $A \equiv B_1 \cup \dots \cup B_k$ be a chain of boxes in $\mathbb{R}^d, d \geq 3$. The chain A is staircase convex if and only if, for every subchain D of A, each projection of D into a coordinate hyperplane is staircase convex.*

Proof. If A is staircase convex, so are its subchains. Let D be any subchain of A, and let $\Pi(D)$ denote the projection of D into the coordinate hyperplane defined by $x_1 = 0$. Let $a', b' \in \Pi(D)$, where $a' = \Pi(a), b' = \Pi(b)$ for a, b in D. Since D is staircase convex, D contains an $a - b$ staircase $\lambda(a, b)$. It is easy to see that $\Pi(\lambda)$ defines an $a' - b'$ staircase in $\Pi(D)$: Vectors in λ parallel to the x_1-axis map to singleton sets in $\Pi(\lambda)$. Each remaining vector $(\vec{v})$ in λ maps to a vector $\Pi(\vec{v})$, parallel to $\vec{v}$ and having the same direction as $\vec{v}$ in $\Pi(\lambda)$.

To establish the converse, assume that, for every subchain D of A, each projection of D into a coordinate hyperplane is staircase convex, to prove that A is staircase convex. We use induction on the number of boxes in the chain A. Clearly the result holds for chains of one or two boxes. Inductively, assume that the result holds for chains of $k-1$ or fewer boxes, $3 \leq k$, to prove the result for a chain of k boxes. Let A be the chain $B_1 \cup \dots \cup B_k$. To show that A is staircase convex, select a, b in A to find an appropriate $a - b$ path in A. Without loss of generality, assume that $a \in B_1 \backslash B_2, b \in B_k \backslash B_{k-1}$. As in the proof of Theorem

1, select a' in B_2 closest to a. Observe that any $a - a'$ staircase lies in B_1 and uses exactly the j directions $e_1, \ldots, e_j$ for some $1 \leq j \leq d$, where e_i is orthogonal to the hyperplane H_i supporting B_2 at x' and where x is beyond H_i, $1 \leq i \leq j$. Moreover, any staircase from a to B_2 must use at least the directions $e_1, \ldots, e_j$. Since $B_1 \cup \cdots \cup B_{k-1}$ is staircase convex, at all points of $B_2 \cup \cdots \cup B_{k-1}$ must lie on or beneath each H_i, $1 \leq i \leq j$.

We will show that the point b must lie on or beneath each $H_i, 1 \leq i \leq j$, as well. Suppose on the contrary that b lies beyond the hyperplane H_1, to obtain a contradiction. (See Figure 1.) Since $k \geq 3$, the boxes B_1 and B_k are disjoint. Hence we may use Lemma 2 to conclude that for at least $d - 1$ of the coordinate projections $\Pi_i, 1 \leq i \leq d$, $\Pi_i(B_1) \cap \Pi_i(B_k) = \emptyset$. Since $d \geq 3$, we may select such a Π_i with $i \neq 1$. For convenience of notation, assume that $\Pi_2(B_1) \cap \Pi_2(B_k) = \emptyset$. However, then the corresponding projection of our chain, $\Pi_2(B_1 \cup \cdots \cup B_k)$, cannot be staircase convex, since any geodesic joining $\Pi_2(a)$ to $\Pi_2(b)$ will require at least one vector in direction e_1 (to travel from $\Pi(a)$ to $\Pi(B_2 \cup \cdots \cup B_{k-1})$) and at least one vector in direction $-e_1$ (to travel from $\Pi(B_2 \cup \cdots \cup B_{k-1})$ to $\Pi(b)$). This contradicts our hypothesis. Our supposition must be fals, and we conclude that point b indeed lies in or beneath each H_i, $1 \leq i \leq j$.

Finally, as in the proof of Theorem 1, we may combine any $a - a'$ staircase in B_1 with any $a' - b$ staircase in $B_2 \cup \cdots \cup B_k$ to produce an $a - b$ staircase in $B_1 \cup \cdots \cup B_k$. This finishes the induction and completes the proof of Theorem 3.

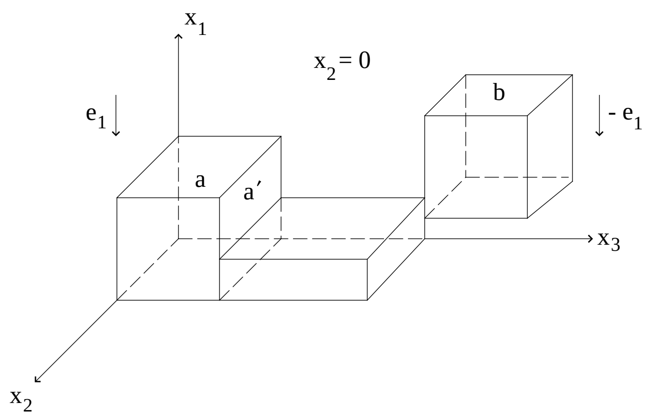

Figure 1.

Corollary 3.1. *For $d \geq 3$, let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \ldots \cup C_n$. The set S is staircase convex if and only if for every chain A of boxes in $\mathcal{C}$ each projection of A into a coordinate hyperplane is staircase convex.*

Proof. This follows immediately from Lemma 1 and Theorem 3.

Our final results concern arbitrary unions of boxes in $\mathbb{R}^d$ and provide a d-dimensional analogue of a well-known planar result obtained from [13, Lemma 1].

Theorem 4. *Let S be a connected, finite union of boxes in $\mathbb{R}^d$, $d \geq 2$. The set S is staircase convex if and only if, for every hyperplane H parallel to a coordinate hyperplane, $H \cap S$ is staircase convex.*

Proof. For the necessity, assume that S is staircase convex, and let H denote a hyperplane parallel to a coordinate hyperplane with $H \cap S \neq \emptyset$. For a, b in $H \cap S$, S contains a staircase $a - b$ path $\lambda \equiv \lambda(a, b)$. Clearly λ contains no vector orthogonal to H, since leaving H on such a vector in one direction e would require a return to H on a vector in the opposite direction $-e$. Thus $\lambda \subseteq H \cap S$, so $H \cap S$ is staircase convex.

To establish the sufficiency, assume that every hyperplane parallel to a coordinate hyperplane satisfies the condition in our hypothesis, to prove that S is staircase convex. We use a contrapositive argument. Suppose on the contrary that S fails to be staircase convex. Then for certain pairs a, b in S, every $a - b$ geodesic in S requires vectors in opposing directions. Select such a pair a_0, b_0 whose $a_0 - b_0$ geodesic $\lambda = \lambda(a_0, b_0)$ has fewest possible edges n. (That is, no such a, b has a corresponding geodesic with fewer than n edges.) Say $\lambda(a_0, b_0) = [v_0, v_1] \cup \cdots \cup [v_{n-1}, v_n]$, where $a_0 = v_0, b_0 = v_n$. Clearly $n \geq 3$. Moreover, by our choice of λ, we may assume that none of the intermediate vectors $\overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{n-2}v_{n-1}}$ are parallel to $\overrightarrow{v_0v_1}$, while $\overrightarrow{v_0v_1}$ and $\overrightarrow{v_{n-1}v_n}$ are parallel and in opposite directions. Choose a hyperplane H containing $\overrightarrow{v_1v_2}, \ldots \overrightarrow{v_{n-2}v_{n-1}}$ and parallel to a coordinate hyperplane such that v_0 and v_n are in the same open halfspace determined by H. Of course, H will be orthogonal to $\overrightarrow{v_0v_1}$. (See Figure 2.) Without loss of generality, assume that v_0 is at least as close to H as v_n is to H. The translate H_0 of H containing v_0 meets $[v_{n-1}, v_n]$, say at v'_n. Observe that there can be no $v_0 - v'_n$ staircase in $H_0 \cap S$, for such a staircase δ would use no edge parallel to $[v_0, v_1]$, so $\delta \cup [v'_n, v_n]$ would provide a $v_0 - v_n$ staircase in S. That is, there exists a hyperplane H_0 parallel to a coordinate hyperplane such that $H_0 \cap S$ is not staircase convex. The contrapositive statement yields the desired result, establishing Theorem 4.

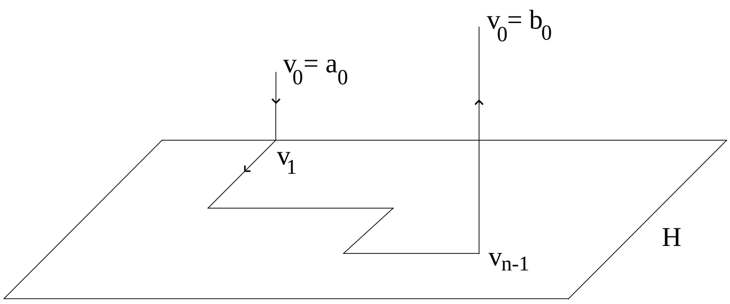

Figure 2.

Corollary 4.1. *Let S be a finite union of boxes in $\mathbb{R}^d$, $d \geq 2$. The set S is staircase convex if and only if S is connected and for every j, $1 \leq j \leq d - 1$, and for every j-flat F parallel to a coordinate flat, $F \cap S$ is connected.*

Proof. We use induction on d. Let $d = 2$. Using [13, Lemma 1], S is staircase convex in $\mathbb{R}^2$ if and only if S is connected and both horizontally and vertically convex. Thus the result is true in the plane.

Inductively, assume that the result holds in $\mathbb{R}^{k-1}$, $3 \le k$, to prove it in $\mathbb{R}^k$. Let S be a finite union of boxes in $\mathbb{R}^k$. If S is staircase convex, certainly S is connected. Moreover, by Theorem 4, for every hyperplane H parallel to a coordinate hyperplane, $H \cap S$ is staircase convex, hence connected. For any j-flat F parallel to a coordinate flat, $1 \le j \le d-1$, F lies in a hyperplane H_F parallel to a coordinate hyperplane. Since $H_F \cap S$ is staircase convex, by our induction hypothesis in $\mathbb{R}^{k-1} \equiv H_F$, $F \cap S$ is connected, too.

To establish the converse in $\mathbb{R}^k$, assume that for every j, $1 \le j \le d$, and for every j-flat F parallel to a coordinate flat, $F \cap S$ is connected. Then S is connected. Let H be any hyperplane parallel to a coordinate hyperplane. If we identify H with $\mathbb{R}^{k-1}$, then by our induction hypothesis $H \cap S$ is staircase convex. Since this is true for every such H, we may use Theorem 4 to conclude that S is staircase convex. This finishes the induction and establishes the corollary for every $d \ge 2$.

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