Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 50 (2009), No. 1, 119-142.

Homomorphic Images of Polynomial Near-rings

Stefan Veldsman

Department Mathematics, Sultan Qaboos University
Sultanate of Oman
e-mail: veldsman@squ.edu.om

Abstract. A polynomial near-ring can mean one of several things. Here a polynomial near-ring is a near-ring of polynomials with the coefficients from a near-ring in the sense of van der Walt and Bagley. We describe quotients of such polynomial near-rings by principal ideals leading to generalizations of some well-known ring constructions.

MSC 2000: 16Y30
Keywords: near-ring, polynomial near-ring, polynomial near-ring quotients

1. Introduction

Near-rings of polynomials should not be confused with polynomial near-rings. Typically, the former is a set of polynomials over a (commutative) ring (with identity) which is a near-ring with respect to the usual addition and composition of polynomials. These near-rings have been studied extensively and their theory and applications can be found in the books by Pilz [12] and Clay [3].

A polynomial near-ring, on the other hand, is a near-ring of polynomials in the universal algebraic sense, see for example Lausch and Nöbauer [6]. These polynomials with their coefficients from a near-ring are much more awkward to deal with, and apart from the more general universal algebraic considerations, not much has been done. This situation was partly addressed when Andries van der Walt proposed a model for polynomial near-rings using mappings as a special case of the more general group near-rings introduced by le Riche, Meldrum and van der Walt [9]. This approach led to a number of papers by Bagley [1], [2], Farag [4], [5], Lee [7] and Lee and Groenewald [8], but covers very little of all that one would like to know about polynomials.

There are thus three different kinds of polynomials studied in near-ring theory: the near-rings of polynomials over rings with respect to the usual addition and composition of polynomials, the polynomial near-rings in the universal algebraic sense and thirdly, the polynomial near-rings in the sense of van der Walt and Bagley. Here we will look at certain homomorphic images of the polynomial near-rings in this latter sense. In particular, we want to describe the quotients of these near-rings determined by the ideals generated by a single polynomial. The objective is to find a near-ring analogue of the well-known ring constructions like that of the algebraic integers or complex numbers. Partial success has been achieved. It will be shown, for example, that if $N[x]$ is the polynomial near-ring over a near-field N and the ideal generated by the polynomial x^2+1 in $N[x]$ is denoted by $\langle x^2+1\rangle$, then $N/\langle x^2+1\rangle$ is a near-ring which can be written as a union $\bigcup_{n=2}^{\infty}\mathcal{A}_n$ where

$$\mathcal{A}_2 = \{a+by \mid a,b\in N\},$$

$$\mathcal{A}_{n+1} = \{\sum_{i=1}^{k} a_i w_i \mid k\geq 1, a_i\in N, w\in\mathcal{A}_n\} \text{ for } n\geq 2 \text{ and } y^2+1=0.$$

Before we discuss polynomial near-rings, we start with a more general construction. This general construction, which has the polynomial near-ring as a special case, will have two further applications: It will be used to describe certain homomorphic images of the polynomial near-rings and it can be used to define polynomial near-rings in two or more commuting indeterminates, avoiding the usual iterative construction.

2. General construction

All near-rings will be right distributive, 0-symmetric and with identity 1. As usual, $A\lhd N$ means A is an ideal of the near-ring N. Let $(G,+)$ be a group. G is called an $N-N$-*bigroup* if there are mappings $N\times G\to G$ and $G\times N\to G$ such that, if we write the images by juxtaposition, then $(n+m)g=ng+mg$, $(g+h)n = gn+hn$, $(nm)g=n(mg)$, $g(nm)=(gn)m$ and $(ng)m=n(gm)$ for all $g,h\in G$ and $n,m\in N$. We will suppose that all actions are unital and that G is left-faithful, i.e. $(0:G)_N := \{n\in N\mid nG=0\}=0$.

As is well-known, $M_N(G) := \{f\in M_0(G)\mid f(gn)=f(g)n$ for all $g\in G, n\in N\}$ is a subnear-ring of $M_0(G)$. By the left-faithfulness, N can be embedded in $M_N(G)$ via $\eta : N\to M_N(G)$ defined by $\eta(a) := \eta_a, \eta_a(g) := ag$ for all $g\in G$. We identify $a\in N$ with η_a in $M_N(G)$ and note that the identity map on G is then the identity of N. We work mostly with the following bigroup: Let N be a 0-symmetric near-ring with identity and let $G := N^k$ be the direct sum of k copies of $(N,+)$ where $k\in\mathbb{N}$, $\mathbb{N}$ is the set of positive integers, or $k=\omega$, the first limit ordinal. With respect to the usual left and right scalar multiplication, N^k is a unital left-faithful $N-N$-bigroup. If N does not have an identity, then we can embed N as a subnear-ring in a near-ring $\overline{N}$ which has an identity. Then $\overline{N}^k$ will be a left-faithful $N-N$-bigroup with respect to the canonical actions.

Any $u \in M_N(G) - N$ will be called an *indeterminate*. A *commuting indeterminate* is an indeterminate which is an $N-N$-homomorphism, i.e. $u(ng) = nu(g)$, $u(gn) = u(g)n$ and $u(g+h) = u(g) + u(h)$ for all $g, h \in G$ and $n \in N$. For an indeterminate u, let $[N, G, u]$ be the subnear-ring of $M_N(G)$ generated by $N \cup \{u\}$. More generally, if U is a set of indeterminates, then $[N, G, U]$ denotes the subnear-ring of $M_N(G)$ generated by $N \cup U$.

We will also need: Let G be a left N-group where, as usual, N is 0-symmetric with identity and G is unital. A non-empty subset $\mathcal{B}$ of G is called a *basis for G over N* if:

(i) $\mathcal{B}$ is a *linearly independent set over* N, i.e. for any finite subset $\{b_1, b_2, b_3, \ldots, b_n\}$ of distinct elements of $\mathcal{B}$, if $\sum_{i=1}^{n} a_i b_i = 0, a_i \in N$, then $a_i = 0$ for all $i = 1, 2, 3, \ldots, n$.

(ii) $\mathcal{B}$ *generates G over N*, i.e. $G = \bigcup_{n=0}^{+\infty} \mathcal{B}_n$ where $\mathcal{B}_0 = \mathcal{B}, \mathcal{B}_{n+1} = \{\sum_{i=1}^{k} a_i w_i \mid k \geq 1, a_i \in N, w_i \in \mathcal{B}_n\}$ for $n \geq 0$.

Note that for any $\mathcal{B}$, $\emptyset \neq \mathcal{B} \subseteq G$, $\bigcup_{n=0}^{+\infty} \mathcal{B}_n$ is always an N-subgroup of G. Trivial, but necessary to mention for later use is:

Proposition 2.1. *Let N be a 0-symmetric near-ring. For any indeterminate $u \in M_N(G)$ and $a \in N$:*

(i) $-u = (-1)u$,

(ii) $(-u)u = -u^2 = (-1)u^2$,

(iii) $a(-u) = a'u$ *for some* $a' \in N$,

(iv) $(-a)u = a'u$ *for some* $a' \in N$,

(v) $(-a)(-u) = a'u$ *for some* $a' \in N$.

If u is a commuting indeterminate, then:

(vi) $u(f+g) = uf + ug$ *for all* $f, g \in M_N(G)$,

(vii) $uf = fu$ *for all* $f \in [N, G, u]$,

(viii) $u(-u) = -u^2 = (-u)u$,

(ix) $(-u)(-u) = u^2$,

(x) $(-u)a = u(-a) = (-a)u$, *but in general* $a(-u) \neq (-u)a$,

(xi) *for any* $n \geq 1$, $(-u^n)a = (-a)u^n = u^n(-a)$.

Proof. (i) and (ii) are both trivial, following from basic properties valid in any near-ring.

For (iii), note that $a(-u) = a((-1)u) = (a(-1))u = a'u$ follows from the definition of an $N-N$-bigroup.

Next we show the validity of (vii). It follows immediately that $uf = fu$ for all $f \in N \cup \{u\}$.
Let $f, g \in [N, G, u]$ and suppose $uf = fu$ and $ug = gu$. Then $(fg)u = f(gu) = f(ug) = (fu)g = (uf)g = u(fg)$ and, by (vi) above, $u(f + g) = uf + ug = fu + gu = (f + g)u$. Since each element of $[N, G, u]$ is a finite combination of sums and products of elements from $N \cup \{u\}$, (vii) follows.

For the second part of (x) let $N := M_0(\mathbb{Z}_k)$ where $\mathbb{Z}_k = (\mathbb{Z}_k, +)$ is the group of integers $\mod k$, $k > 3$. Let $G = N^2$ and define $u : N^2 \to N^2$ by $u(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$. Then u is a commuting indeterminate. If $a : \mathbb{Z}_k \to \mathbb{Z}_k$ is defined by $a(0) = 0$ and $a(t) = (t + 1) \mod k$ for $0 \neq t \in \mathbb{Z}_k$, then $a(-1) \neq -a$. Indeed, choose $1 < t < k - 1$. Then $a(-1)(t) = a(-t) = a(k - t) = k - t + 1$ and $-a(t) = -(t + 1) = k - t - 1$. Thus $a(-u)(0, 1) \neq (-u)a(0, 1)$ and so $a(-u) \neq (-u)a$ which justifies the second claim in (x). $\square$

Next we describe the elements in $[N, G, u]$. We take $u^0 = 1$.

Proposition 2.2. *For any indeterminate $u \in M_N(G)$, $[N, G, u] = \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ where $\mathcal{A}_0 = \{1, u, u^2, u^3, \ldots\}, \mathcal{A}_1 = \{\prod_{i=1}^{m} a_i u^{t_i} \mid m \geq 1, a_i \in N, t_i \geq 0\}$ and $\mathcal{A}_{n+1} = \{\sum_{i=1}^{m} s_i w_i \mid m \geq 1, s_i \in \mathcal{A}_1, w_i \in \mathcal{A}_n\}$ for $n \geq 1$.*

Proof. Firstly note that $\mathcal{A}_0 \cup N \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \cdots$. By definition $N \cup \{u\} \subseteq [N, G, u]$ and thus $\mathcal{A}_1 \subseteq [N, G, u]$. If $\mathcal{A}_n \subseteq [N, G, u]$ for some $n \geq 1$, then so is $\mathcal{A}_{n+1}$ since $[N, G, u]$ is a subnear-ring of $M_N(G)$. Hence $\bigcup_{n=0}^{+\infty} \mathcal{A}_n \subseteq [N, G, u]$. Next we show $\bigcup_{n=0}^{+\infty} \mathcal{A}_n$ is a subnear-ring of $M_N(G)$. If $h = \prod_{i=1}^{m} a_i u^{t_i} \in \mathcal{A}_1$, $t_i \geq 0$, then $-h = (-a_1)u^{t_1} a_2 u^{t_2} \ldots a_m u^{t_m} \in \mathcal{A}_1$. Hence, if $f, g \in \bigcup_{n=0}^{+\infty} \mathcal{A}_n$, then $f - g \in \bigcup_{n=0}^{+\infty} \mathcal{A}_n$. Indeed, since the $\mathcal{A}_n$'s form an ascending chain, there is an $m \geq 2$ with $f, g \in \mathcal{A}_m$. Then clearly $f - g \in \mathcal{A}_m \subseteq \bigcup_{n=0}^{+\infty} \mathcal{A}_n$. To see that $fg \in \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ for $f, g \in \bigcup_{n=0}^{+\infty} \mathcal{A}_n$, it will suffice to show that $\mathcal{A}_n \mathcal{A}_m \subseteq \mathcal{A}_{n+m+1}$ for all $n, m \geq 0$. Note that $\mathcal{A}_0 \mathcal{A}_0 \subseteq \mathcal{A}_0, \mathcal{A}_1 \mathcal{A}_1 \subseteq \mathcal{A}_1$ and thus $\mathcal{A}_1 \mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{A}_0 \mathcal{A}_1 \subseteq \mathcal{A}_1$. Let $m \geq 0$ be fixed and we proceed by induction on n. For $n = 1$, we have $\mathcal{A}_1 \mathcal{A}_m \subseteq \mathcal{A}_{m+1}$ by definition and for $n = 0$ we have $\mathcal{A}_0 \mathcal{A}_m \subseteq \mathcal{A}_1 \mathcal{A}_m \subseteq \mathcal{A}_{m+1}$. Suppose $n \geq 1$ and $\mathcal{A}_n \mathcal{A}_m \subseteq \mathcal{A}_{n+m+1}$. Let $h = \sum_{i=1}^{m} s_i w_i \in \mathcal{A}_{n+1}$, $m \geq 1$, $s_i \in \mathcal{A}_1$, $w_i \in \mathcal{A}_n$ and choose $e \in \mathcal{A}_m$. Then $w_i e \in \mathcal{A}_n \mathcal{A}_m \subseteq \mathcal{A}_{n+m+1}$ and so $he = \sum_{i=1}^{m} s_i w_i e \in \mathcal{A}_{(n+1)+m+1}$. Thus $\bigcup_{n=0}^{+\infty} \mathcal{A}_n$ is a subnear-ring of $M_N(G)$ which contains $N \cup \{u\}$ and $[N, G, u] \subseteq \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ follows. $\square$

Corollary 2.3. *Let $u \in M_N(G)$ be a commuting indeterminate. Then* $[N,G,u] = \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ *where* $\mathcal{A}_0 = \{1, u, u^2, u^3, \ldots\}$, $\mathcal{A}_1 = \{au^t \mid t \geq 0,\ a \in N\}$ *and* $\mathcal{A}_{n+1} = \{\sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n\}$ *for* $n \geq 1$.

Proof. Clearly $\mathcal{A}_1 = \{au^t \mid t \geq 0,\ a \in N\}$. For $n \geq 1$ and $\sum_{i=1}^{m} s_i w_i \in \mathcal{A}_{n+1},\ m \geq 1,\ s_i \in \mathcal{A}_1,\ w_i \in \mathcal{A}_n$, we have $s_i = a_i u^{t_i}$ for some $a_i \in N$ and $t_i \geq 0$. Then $\sum_{i=1}^{m} s_i w_i = \sum_{i=1}^{m} (a_i u^{t_i}) w_i = \sum_{i=1}^{m} a_i (u^{t_i} w_i)$ with $u^{t_i} w_i = w_i u^{t_i} \in \mathcal{A}_n \mathcal{A}_0 \subseteq \mathcal{A}_n$ for all i. Thus $\mathcal{A}_{n+1} = \{\sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n\}$ as required. □

Since N is a subnear-ring of $[N,G,u]$, $([N,G,u],+)$ is a left N-group with respect to the canonical action. If u is a commuting indeterminate, the above corollary shows that $\{1, u, u^2, \ldots\}$ generates $[N,G,u]$ as a left N-group. If, in addition, $\{1, u, u^2, \ldots\}$ is a linearly independent set over N, then $\{1, u, u^2, \ldots\}$ will be a basis for $[N,G,u]$ over N (but in general this need not be the case).

Any element of $[N,G,u]$ is in one of the $\mathcal{A}_n$'s and we will always write an element from $[N,G,u]$ in the form specified for $\mathcal{A}_n$'s elements as above. This will be our canonical representation of the elements of $[N,G,u]$. Note that this representation of an element in $[N,G,u]$ need not be unique. For example, for the near-ring N define $y : N^2 \to N^2$ by $y(\alpha_1, \alpha_2) := (-\alpha_2 + \alpha_1, \alpha_1)$ for all $\alpha_1, \alpha_2 \in N$. Then $y^2 = -1 + y$ and in $[N, N^2, y]$ we have $y(-1+y) = y^3 = -y + y^2 = -y - 1 + y$ where $y^3 \in \mathcal{A}_0, y(-1+y) \in \mathcal{A}_3$ and $-y + y^2 = -y - 1 + y \in \mathcal{A}_2$. Note that $\{1, y\}$ is linearly independent over N, but $\{1, y\}$ need not generate $[N, N^2, y]$ over N.

In general, this lack of unique representation will make it troublesome at times to deal with the elements of these near-rings. For $f \in [N,G,u] = \bigcup_{n=0}^{+\infty} \mathcal{A}_n$, we say the *level* of f is n if $f \in \mathcal{A}_n$ and $f \notin \mathcal{A}_k$ for all $k < n$. Note that the level of f is independent of the particular representation chosen for f. For the example above, $f = y(-1+y) = y^3 = -y + y^2 = -y - 1 + y$ has level 0.

As is the case for polynomials over rings, substitutions can be problematic. For $f \in [N,G,u]$, we may replace all u's in f with any other fixed element from $[N,G,u]$ or even from $M_N(G)$ (which could take values outside of $[N,G,u]$). This, in itself, is not a problem – the problem arises when functions are defined by substitution since elements of $[N,G,u]$ may not have unique representations. For example, let $y : N^2 \to N^2$ be defined by $y(\alpha_1, \alpha_2) := (-\alpha_2 + \alpha_1, \alpha_1)$ for all $\alpha_1, \alpha_2 \in N$ and define $u : N^2 \to N^2$ by $u(\alpha_1, \alpha_2) := (\alpha_1, \alpha_1)$ for all $\alpha_1, \alpha_2 \in N$. Then $y^2 = -1 + y$ and $u^2 = u$. Define $\theta_u : [N, N^2, y] \to [N, N^2, u]$ by $\theta_u(f) := \overline{f}$ where $\overline{f}$ is obtained from f by replacing all occurrences of y with u. This is not a well-defined mapping. Indeed, $y^2 = -1 + y$ but $u = u^2 \neq -1 + u$. Even $\theta_1 : [N, N^2, y] \to N$ defined by $\theta_1(f) := \overline{f}$ where $\overline{f}$ is obtained from f by replacing each y with 1 is not well-defined. To have such substitutions well-defined, one should require that the substitution is independent of the representation chosen for the element from $[N,G,u]$.

Proposition 2.4. *Let S be a near-ring and $\gamma : N \to S$ a near-ring homomorphism. Choose $w \in S$ fixed. For $f \in [N, G, u]$, let $\overline{f}$ be the element in S obtained from f by replacing each u in f with w in $\overline{f}$ and each $a \in N$ in f with $\gamma(a) \in S$ in $\overline{f}$. Suppose that for all $f \in [N, G, u]$, if $f = 0$ in $[N, G, u]$, then $\overline{f} = 0$ in S. Then $\gamma_w : [N, G, u] \to S$ defined by $\gamma_w(f) := \overline{f}$ is a well-defined homomorphism with* $\ker \gamma_w = \{f \in [N, G, u] \mid \overline{f} = 0\}$. *If γ is surjective, then so is γ_w.*

Proof. γ_w is well-defined, for if $f, g \in [N, G, u]$ with $f = g$, then $h := f - g = 0$ and so, by assumption, $\overline{f} - \overline{g} = \overline{f - g} = \overline{h} = 0$. Clearly γ_w is a homomorphism which is surjective if γ is. □

We have a number of corollaries:

Corollary 2.5. *Let S be a near-ring and $\gamma : N \to S$ a homomorphism. Let G be a unital left-faithful $N - N$-bigroup and let H be a unital left-faithful $S - S$-bigroup with indeterminates $u \in M_N(G) - N$ and $w \in M_S(H) - S$. Define $\gamma_w : [N, G, u] \to [S, H, w]$ by $\gamma_w(f) := \overline{f}$ where $\overline{f}$ is obtained from f by replacing each u in f with w and each $a \in N$ with $\gamma(a) \in S$. If $f = 0$ in $[N, G, u]$ implies $\overline{f} = 0$ in $[S, H, w]$, then γ_w is a well-defined homomorphism which is surjective if γ is.*

Corollary 2.6. *Let $k, l \in \mathbb{N} \cup \{\omega\}$ and choose indeterminates $u \in M_N(N^k) - N$ and $w \in M_N(N^l) - N$. Define $\gamma_w : [N, N^k, u] \to [N, N^l, w]$ by $\gamma_w(f) := \overline{f}$ where $\overline{f}$ is obtained from f by replacing each u with w. If $f = 0$ in $[N, N^k, u]$ implies $\overline{f} = 0$ in $[N, N^l, w]$, then γ_w is a well-defined surjective homomorphism.*

Proposition 2.7. *Let S be a near-ring and $\theta : [N, G, u] \to S$ a surjective homomorphism with $w := \theta(u) \notin \theta(N)$ and $N \cap \ker \theta = 0$. Then $S \cong [N, S, w]$, i.e. $[N, G, u]/\ker \theta \cong [N, S, w]$.*

Proof. Since θ is surjective, $[N, G, u]/\ker \theta \cong S$. For $a \in N$, we have $\theta(a) = 0$ if and only if $a = 0$; hence $\theta|_N : N \to \theta(N)$ is an isomorphism and we identify N with $\theta(N)$ in S. Then $(S, +)$ is a unital left-faithful $N - N$-bigroup and we may embed S in $M_N(S)$ via $a \mapsto \tau_a$, $\tau_a : S \to S$ is defined by $\tau_a(s) = as$ for all $s \in S$. Hence $w = \theta(u) \in S - N \subseteq M_N(S) - N$ and from $N \cup \{w\} \subseteq S \subseteq M_N(S)$, we get $[N, S, w] \subseteq S$. Since θ is surjective, for $s \in S$, there is an $f \in [N, G, u]$ with $\theta(f) = s$. Then $s = \theta(f) \in [N, S, w]$ and we conclude that $S = [N, S, w]$. □

This immediately gives

Corollary 2.8. *Let $0 \neq h \in [N, G, u]$ and let $I := \langle h \rangle$, the ideal in $[N, G, u]$ generated by h. Suppose that $I \cap N = 0$ and $u + I \notin \frac{N+I}{I}$. Then there is a near-ring S, which contains N as a subnear-ring, and an element $w \in S - N$ with $\overline{h} = 0$ where $\overline{h}$ is obtained from h by replacing u with w.*

This section is closed with an example.

Example 2.9. Let $u : N^\omega \to N^\omega$ and $w : N^2 \to N^2$ be defined by $u(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) = (\alpha_2, \alpha_1, \alpha_4, \alpha_3, \ldots)$ and $w(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$. Then both $u \in M_N(N^\omega)$ and $w \in M_N(N^2)$ are commuting indeterminates with $u^2 = 1$ and $w^2 = 1$. We will show $[N, N^\omega, u] \cong [N, N^2, w]$ by using Corollary 2.6. Here $\gamma_w : [N, N^\omega, u] \to [N, N^2, w]$ is defined by $\gamma_w(f) := \overline{f}$ where $\overline{f}$ is obtained from f by replacing each u in f with w in $\overline{f}$. In order to show that γ_w is an isomorphism, we only need to show that for $f \in [N, N^\omega, u], \overline{f} = 0$ if and only if $f = 0$. This will follow from (i) and (ii) below.

(i) For every $f \in [N, N^\omega, u]$, there is a function $F : N^2 \to N$ such that for all $k \in \mathbb{N}$ and $\alpha_i \in N$,

$$\pi_k(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) = \begin{cases} F(\alpha_k, \alpha_{k+1}) \text{ if } k \text{ is odd} \\ F(\alpha_k, \alpha_{k-1}) \text{ if } k \text{ is even} \end{cases}$$

where $\pi_k : N^\omega \to N$ is the k-th projection.

Proof. Let $f \in [N, N^\omega, u]$. Then $f \in \mathcal{A}_n$ for some $n \geq 1$ (cf. Corollary 2.3). We proceed by induction on n. For $n = 1$ and $f = au^p$ with $a \in N$ and $p \in \{0, 1\}$, define F by

$$F(s,t) := \begin{cases} as \text{ if } p = 0 \\ at \text{ if } p = 1 \end{cases}.$$

For any $k \geq 1$, if $p = 0$, then $\pi_k(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) = a\alpha_k = F(\alpha_k, \alpha_{k+1})$ and if $p = 1$, then

$$\begin{aligned} \pi_k(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) &= \begin{cases} a\alpha_{k+1} \text{ if } k \text{ is odd} \\ a\alpha_{k-1} \text{ if } k \text{ is even} \end{cases} \\ &= \begin{cases} F(\alpha_k, \alpha_{k+1}) \text{ if } k \text{ is odd} \\ F(\alpha_k, \alpha_{k-1}) \text{ if } k \text{ is even.} \end{cases} \end{aligned}$$

Suppose now $n \geq 1$ is fixed and (i) holds for any element of $\mathcal{A}_n$. Let $f \in \mathcal{A}_{n+1}$, say $f = \sum_{i=1}^{m} a_i f_i, m \geq 1, a_i \in N$ and $f_i \in \mathcal{A}_n$. By the induction assumption, for each $i = 1, 2, 3, \ldots, m$, there is a function $F_i : N^2 \to N$ such that for all $k \geq 1$,

$$\pi_k(f_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) = \begin{cases} F_i(\alpha_k, \alpha_{k+1}) \text{ if } k \text{ is odd} \\ F_i(\alpha_k, \alpha_{k-1}) \text{ if } k \text{ is even.} \end{cases}.$$

Define $F : N^2 \to N$ by $F(s,t) := \sum_{i=1}^{m} a_i F_i(s,t)$ for all $s, t \in N$. Then for any $k \geq 1$,

$$\begin{aligned} \pi_k(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) &= \sum_{i=1}^{m} a_i \pi_k(f_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) \\ &= \begin{cases} \sum_{i=1}^{m} a_i F_i(\alpha_k, \alpha_{k+1}) \text{ if } k \text{ is odd} \\ \sum_{i=1}^{m} a_i F_i(\alpha_k, \alpha_{k-1}) \text{ if } k \text{ is even} \end{cases} \\ &= \begin{cases} F(\alpha_k, \alpha_{k+1}) \text{ if } k \text{ is odd} \\ F(\alpha_k, \alpha_{k-1}) \text{ if } k \text{ is even} \end{cases} \end{aligned}$$

as required.

(ii) For any $f \in [N, N^\omega, u]$, if $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) = (\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \ldots)$, then $\overline{f}(\alpha_1, \alpha_2) = (\alpha'_1, \alpha'_2)$.

Proof. By induction on $n, f \in \mathcal{A}_n$. For $n = 1$, say $f = au^p$ for $a \in N$ and $p \in \{0, 1\}$,

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) = \begin{cases} (a\alpha_1, a\alpha_2, a\alpha_3, a\alpha_4, \ldots) \text{ if } p = 0 \\ (a\alpha_2, a\alpha_1, a\alpha_4, a\alpha_3, \ldots) \text{ if } p = 1 \end{cases}.$$

On the other hand,

$$\overline{f}(\alpha_1, \alpha_2) = \begin{cases} (a\alpha_1, a\alpha_2) \text{ if } p = 0 \\ (a\alpha_2, a\alpha_1) \text{ if } p = 1 \end{cases}$$

which completes the proof for $n = 1$.

Suppose (ii) holds for all elements from $\mathcal{A}_n$ for some fixed $n \geq 1$. Let $f \in \mathcal{A}_{n+1}$, say $f = \sum_{i=1}^{m} a_i f_i, m \geq 1, a_i \in N$ and $f_i \in \mathcal{A}_n$. If $f_i(\alpha_1, \alpha_2, \alpha_3, \ldots) = (\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \ldots)$, then the induction assumption gives $f_i(\alpha_1, \alpha_2) = (\alpha_1^{(i)}, \alpha_2^{(i)})$. Thus

$$\begin{aligned} f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) &= \sum_{i=1}^{m} a_i f_i(\alpha_1, \alpha_2, \alpha_3, \ldots) \\ &= \sum_{i=1}^{m} a_i(\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \ldots) \\ &= (\sum_{i=1}^{m} a_i\alpha_1^{(i)}, \sum_{i=1}^{m} a_i\alpha_2^{(i)}, \sum_{i=1}^{m} a_i\alpha_3^{(i)}, \ldots) \text{ and} \\ \overline{f}(\alpha_1, \alpha_2) &= \sum_{i=1}^{m} a_i\overline{f_i}(\alpha_1, \alpha_2) \\ &= \sum_{i=1}^{m} a_i(\alpha_1^{(i)}, \alpha_2^{(i)}) \\ &= (\sum_{i=1}^{m} a_i\alpha_1^{(i)}, \sum_{i=1}^{m} a_i\alpha_2^{(i)}) \end{aligned}$$

which completes the proof for (ii).

It is now shown that for $f \in [N, N^\omega, u], \overline{f} = 0$ if and only if $f = 0$. Indeed, if $f = 0$ and $\alpha_1, \alpha_2 \in N$, then $\overline{f}(\alpha_1, \alpha_2) = (\alpha'_1, \alpha'_2)$ where $0 = f(\alpha_1, \alpha_2, 0, 0, \ldots) = (\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \ldots)$. Thus $\overline{f}(\alpha_1, \alpha_2) = 0$, i.e. $\overline{f} = 0$. Conversely, suppose $\overline{f} = 0$. Let $\alpha_1, \alpha_2, \alpha_3, \ldots \in N$ and suppose $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) = (\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \ldots)$. Then $(\alpha'_1, \alpha'_2) = \overline{f}(\alpha_1, \alpha_2) = 0$. Let $F : N^2 \to N$ be the function given by (i) above. Then $F(\alpha_1, \alpha_2) = \pi_1(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) = \pi_1(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \ldots) = \alpha'_1 = 0$. This means for any $k \geq 1$, we have

$$\begin{aligned} \pi_k(f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots)) &= \begin{cases} F(\alpha_k, \alpha_{k+1}) \text{ if } k \text{ is odd} \\ F(\alpha_k, \alpha_{k-1}) \text{ if } k \text{ is even} \end{cases} \\ &= 0. \end{aligned}$$

Thus $f = 0$ and we are done.

3. Polynomial near-rings and representations of polynomials

Define $x : N^\omega \to N^\omega$ by $x(\alpha_1, \alpha_2, \alpha_3, \ldots) = (0, \alpha_1, \alpha_2, \alpha_3, \ldots)$. Then $x \in M_N(N^\omega) - N$ is a commuting indeterminate and the near-ring $[N, N^\omega, x]$ is called the *polynomial near-ring over* N. This near-ring will be denoted by $N[x]$. If N is a ring with identity, then $N[x]$ coincides with the usual polynomial ring. In general it need not be the case. For example, let N be a ring with zero multiplication and let $N_R[x]$ denote the usual polynomial ring over the ring N. Then x is not in the polynomial ring $N_R[x]$ and $0 \neq ax$ is in $N_R[x]$ for all $0 \neq a \in N$. On the other hand, for the polynomial near-ring $N[x]$, we have $0 \neq x \in N[x]$ and $ax = 0$ for all $a \in N$. The polynomial near-ring $N[x]$ has been studied by Bagley [1], [2], Farag [4], [5], Lee [7] and Lee and Groenewald [8]; in most cases dealing with their ideal theory. We investigate quotients of polynomial near-rings, but we need to develop some tools. Firstly, note that x has a left inverse $\overline{x}$ in $M_N(N^\omega)$ defined by $\overline{x}(\alpha_1, \alpha_2, \alpha_3, \ldots) = (\alpha_2, \alpha_3, \alpha_4, \ldots)$. Hence, if $x^k f = 0$ (or $f x^k = 0$) for any $k \geq 1$ and $f \in M_N(N^\omega)$, then $f = 0$.

Because x is a commuting indeterminate, we have from Corollary 2.3, $N[x] = \bigcup\limits_{n=0}^{+\infty} \mathcal{A}_n$ where

$$\mathcal{A}_0 = \{1, x, x^2, x^3, \ldots\},$$
$$\mathcal{A}_1 = \{ax^t \mid t \geq 0,\ a \in N\},$$
$$\mathcal{A}_2 = \{\sum_{i=1}^{m} a_i x^{k_i} \mid m \geq 1,\ a_i \in N,\ k_i \geq 0\}$$

and

$$\mathcal{A}_{n+1} = \{\sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n\} \text{ for } n \geq 2.$$

$\mathcal{A}_0$ is a basis for $N[x]$ over N. Indeed, that $\mathcal{A}_0$ generates $N[x]$ is clear from the previous lines and we only show $\mathcal{A}_0$ is linearly independent over N. Let $x^{k_1}, x^{k_2}, \ldots, x^{k_m}$ be distinct and $a_1, a_2, \ldots, a_m \in N$ with $\sum\limits_{i=1}^{m} a_i x^{k_i} = 0$. Suppose $k_{i_1} < k_{i_2} < \cdots < k_{i_m}$ where each $k_{i_j} \in \{k_1, k_2, \ldots, k_m\}, j = 1, 2, \ldots, m$. Then

$$0 = (\sum_{i=1}^{m} a_i x^{k_i})(1, 0, 0, \ldots) = (0, \ldots, a_{k_{i_1}}, 0, \ldots, a_{k_{i_2}}, 0, \ldots, 0, a_{k_{i_m}}, 0, \ldots)$$

and so $a_1 = a_2 = \cdots = a_m = 0$.

An element in $N[x]$ may still have many different representations, and we follow our earlier convention to use the form of the elements of the sets $\mathcal{A}_n$ as our canonical representation of the near-ring polynomials (as was also done in [7] and [8]). Amongst others, this means we always write x as far to the right as possible but not as a common factor on the right of a bracket. For example, we write

ax and not xa and we write $a(bx+cx^2)$ and not $ax(b+cx)$ nor $a(b+cx)x$. We will need an analogue of the concept of the degree of the usual polynomials. But with different representations of the same polynomial, this is not a straightforward matter. For example, one could have $f = a(bx^3+cx) = ax = a(bx^2+cx)$. We will define the height of a polynomial as an analogue of the notion of degree. Note firstly that we will distinguish between the "height of a particular representation of a polynomial" and the "height of the polynomial" which is independent of the chosen representation. For $0 \neq f \in N[x]$, choose any representation of f. The height of this representation is the highest power of x in the expression. The height of the polynomial f is the minimum amongst all the heights of all the possible representations for f. When $f = 0$, the height of f is by definition $-\infty$. For the polynomial f as above, the representation $a(bx^2+cx)$ has height 2, but f has height 1 (if we know $ax \neq 0$).

Proposition 3.1. *Let $f \in N[x]$ with given representation of height k, $k \geq 0$. Then there is a function $F : N^{k+1} \to N$ such that for all $i \geq 1$ and for all $\alpha_1, \alpha_2, \alpha_3, \ldots \in N$, $\pi_i(f(\alpha_1, \alpha_2, \alpha_3, \ldots)) = F(\alpha_{i-k}, \alpha_{i-k+1}, \ldots, \alpha_{i-1}, \alpha_i)$ where $\pi_i : N^\omega \to N$ is the i-th projection and where we take $\alpha_j = 0$ for $j \leq 0$.*

Proof. For $f \in N[x]$ with given representation of height k, $k \geq 0$, define $F : N^{k+1} \to N$ by $F(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) := \pi_{k+1}(f(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}, 0, 0, \ldots))$ for all $\alpha_1, \alpha_2, \ldots, \alpha_{k+1} \in N$. Note that in the definition of F it is immaterial what we put in positions $k+2, k+3, k+4, \ldots$ of the vector on the right hand side, since these components will not appear in the $(k+1)^{th}$ component of $f(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}, 0, 0, \ldots)$. We show that F has the desired property by induction on the level n of f.

If $f = ax^k$ for some $a \in N$ and $k \geq 0$, then $F(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = a\alpha_1$. For any $i \geq 1$,

$$\begin{aligned} \pi_i(f(\alpha_1, \alpha_2, \alpha_3, \ldots)) &= \begin{cases} 0 \text{ if } i \leq k \\ a\alpha_{i-k} \text{ if } i > k \end{cases} \\ &= a\alpha_{i-k} \text{ where } \alpha_j = 0 \text{ for } j \leq 0 \\ &= F(\alpha_{i-k}, \alpha_{i-k+1}, \ldots, \alpha_{i-1}, \alpha_i). \end{aligned}$$

Let $n \geq 1$ and suppose that for any polynomial in $\mathcal{A}_n$, its associated function has the desired property. Let $f = \sum_{j=1}^{m} a_j w_j \in \mathcal{A}_{n+1}$ with given representation of height k, $k \geq 0$, where $m \geq 1$, $a_i \in N$ and $w_i \in \mathcal{A}_n$. We suppose w_i has height k_i and then, by definition, $k = \max\{k_1, k_2, \ldots, k_m\}$. By the induction assumption, for each $j = 1, 2, \ldots, m$ there is a function $W_j : N^{k_j+1} \to N$ with $\pi_i(w_j(\alpha_1, \alpha_2, \alpha_3, \ldots)) = W_j(\alpha_{i-k_j}, \alpha_{i-k_j+1}, \ldots, \alpha_{i-1}, \alpha_i)$ for all $i \geq 1$ and $\alpha_1, \alpha_2, \alpha_3, \ldots \in N$. Using the definition of F,

$$\begin{aligned}
F(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) &= \pi_{k+1}(f(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}, 0, 0, \ldots)) \\
&= \pi_{k+1}(\sum_{j=1}^{m} a_j w_j((\alpha_1, \alpha_2, \ldots, \alpha_{k+1}, 0, 0, \ldots))) \\
&= \sum_{j=1}^{m} a_j \pi_{k+1}(w_j(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}, 0, 0, \ldots)) \\
&= \sum_{j=1}^{m} a_j W_j(\alpha_{k+1-k_j}, \alpha_{k+2-k_j}, \ldots, \alpha_k, \alpha_{k+1}).
\end{aligned}$$

But

$$\begin{aligned}
\pi_i(f(\alpha_1, \alpha_2, \alpha_3, \ldots)) &= \pi_i(\sum_{j=1}^{m} a_j w_j(\alpha_1, \alpha_2, \alpha_3, \ldots)) \\
&= \sum_{j=1}^{m} a_j \pi_i(w_j(\alpha_1, \alpha_2, \alpha_3, \ldots)) \\
&= \sum_{j=1}^{m} a_j W_j(\alpha_{i-k_j}, \alpha_{i+1-k_j}, \ldots, \alpha_{i-1}, \alpha_i).
\end{aligned}$$

Since $k_j \leq k$ for all $j = 1, 2, \ldots, m$, it follows from the above

$$\begin{aligned}
F(\alpha_{i-k}, \alpha_{i-k+1}, \ldots, \alpha_{i-1}, \alpha_i) &= \sum_{j=1}^{m} a_j W_j(\alpha_{i-k_j}, \alpha_{i+1-k_j}, \ldots, \alpha_{i-1}, \alpha_i) \\
&= \pi_i(f(\alpha_1, \alpha_2, \alpha_3, \ldots))
\end{aligned}$$

for all $\alpha_1, \alpha_2, \alpha_3, \ldots \in N$ as required. □

For later use, we record a number of properties of a polynomial and its associated function:

Remark 3.2. (1) Recall, for given $f \in N[x]$ with height k, the representation of f may not be unique. Suppose $f = g$ where $g \in N[x]$ is another representation for f with height l, say $k \leq l$. Let $F : N^{k+1} \to N$ and $G : N^{l+1} \to N$ be the associated functions. We now describe the relationship between F and G. Since $f = g$, we have for all $i \geq 1$ and for all $\alpha_1, \alpha_2, \alpha_3, \ldots \in N$a,

$$\begin{aligned}
F(\alpha_{i-k}, \alpha_{i-k+1}, \ldots, \alpha_{i-1}, \alpha_i) &= \pi_i(f(\alpha_1, \alpha_2, \alpha_3, \ldots)) \\
&= \pi_i(g(\alpha_1, \alpha_2, \alpha_3, \ldots)) \\
&= G(\alpha_{i-l}, \alpha_{i-l+1}, \ldots, \alpha_{i-1}, \alpha_i).
\end{aligned}$$

In particular, for $i = k + 1$ and since $k + 1 - l \leq l + 1 - l = 1$,

$$\begin{aligned}
&F(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) \\
&= G(\alpha_{k+1-l}, \alpha_{k+2-l}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_k, \alpha_{k+1}) \\
&= G(0, 0, \ldots, 0, \alpha_1, \alpha_2, \ldots, \alpha_{k+1}) \text{ with zeros in the first } l-k \text{ positions.}
\end{aligned}$$

Likewise, with $i = l + 1$, we get

$$G(\alpha_1, \alpha_2, \ldots, \alpha_{l+1}) = F(\alpha_{l-k+1}, \alpha_{l-k+2}, \ldots, \alpha_l, \alpha_{l+1}).$$

(2) For $f \in N[x]$ with $ht(f) = k$, the associated function $F : N^{k+1} \to N$ completely describes the polynomial f. For some special cases, it may be possible to use a function with fewer than $k + 1$ arguments (e.g. $f = ax^k$), but in general it may not be possible nor desirable. However, it is possible, and we will have a need to do so later, to increase the number of arguments of the associated function to l with $l \geq k + 1$. This is achieved by defining $G : N^l \to N$ by $G(\alpha_1, \alpha_2, \ldots, \alpha_l) = \pi_l(f(\alpha_1, \alpha_2, \ldots, \alpha_l, 0, 0, \ldots))$. Then $G(\alpha_1, \alpha_2, \ldots, \alpha_l) =$

$F(\alpha_{l-k}, \alpha_{l-k+1}, \ldots, \alpha_{l-1}, \alpha_l)$ which can also be written as $F(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = G(0, 0, \ldots, 0, \alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ for all α_i where there are $l-k-1$ zeros at the start of G's argument. In fact, it does not matter what we put in these $l-k-1$ positions. Indeed, $G(\alpha_1, \ldots, \alpha_{l-k-1}, \alpha_{l-k}, \ldots, \alpha_{l-1}, \alpha_l) = \pi_l(f(\alpha_1, \alpha_2, \ldots, \alpha_l, 0, 0, \ldots))$. Since f has height k, we will have at most $x^0, x, x^2, \ldots, x^k$ somewhere in the representation of f which means that the only α_i's we will find in the l-th component will be from $\alpha_{l-k}, \alpha_{l-k+1}, \ldots, \alpha_{l-1}$ and α_l.

(3) Let $F : N^{k+1} \to N$ be the function associated with the polynomial $f \in N[x]$ with a given representation of height k. Then F and f determine each other according to the following rule: Given f, then F can be obtained from f by replacing each x^j in f by α_{k+1-j} for $j = 1, 2, \ldots, k$. Conversely, given F, then the given representation for f can be obtained from F by replacing each α_j with x^{k+1-j}. Care must be taken with x^0 – the only $a \in N$ which is actually regarded as ax^0 will be those a in f which do not have a left bracket immediate to their right. For example, $a(b + cx) + d = a(bx^0 + cx) + dx^0$. For this association it is important to remember that we always write the polynomials in the agreed canonical form. If this is overlooked, this association may not be valid. For example, if $g = axb$, then the canonical representation is $f = abx$ (we take $ab \neq 0$). The associated function $F : N^2 \to N$ is given by $F(\alpha_1, \alpha_2) = ab\alpha_1$ which may not be the same as $a\alpha_1 b$. In fact, the associated function can be used to get the canonical representation of any polynomial: For a polynomial, whether in canonical form or not, determine the associated function and use the rule above (replace α_j with x^{k+1-j}) to get the canonical representation of the polynomial. We will adopt the following notation: If we write $f = f[x^0, x, x^2, \ldots, x^k]$, then $F(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = f[x^0 = \alpha_{k+1}, x = \alpha_k, x^2 = \alpha_{k-1}, \ldots, x^k = \alpha_1]$ where the latter means x^j in f is replaced by α_{k+1-j} for all $j = 0, 1, \ldots, k$.

(4) For $f \in N[x]$ with $ht(f) = k$, define the coefficient of x^j in f by $f_j := \operatorname{coeff}(x^j; f) := \pi_{j+1}(f(1, 0, 0, \ldots))$. Note that $\operatorname{coeff}(x^j; f) = F(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is in position $k + 1 - j$ for $0 \leq j \leq k$. As usual, the coefficient of x^0 is called the constant term of f. The coefficients are uniquely determined by the polynomial and do not depend on the chosen representation. The converse is not true: For example, if $f = a(b + cx) + d$ and $g = ab + d + acx$ then $\operatorname{coeff}(x^0; f) = ab + d = \operatorname{coeff}(x^0; g)$ and $\operatorname{coeff}(x; f) = ac = \operatorname{coeff}(x; g)$ but f and g need not be the same. Another unexpected consequence of this definition is given by the polynomial $f = a(b + x) - ax - ab$. Here f need not be 0, but $\operatorname{coeff}(x^0; f) = 0 = \operatorname{coeff}(x; f)$.

(5) Bagley [1] defined the degree of a polynomial by $\deg(f) := \max\{|f(\alpha)| - |\alpha| \mid \alpha \in N^{(\omega)}\}$ where $N^{(\omega)} := \{\alpha \in N^\omega \mid \alpha \text{ has finite support}\}$ and for $\beta \in N^\omega$, $|\beta| = \min\{k \in \mathbb{N} \mid \alpha_i = 0 \text{ for all } i > k\}$. It was shown that any polynomial has some degree. The unusual behaviour of near-ring polynomials as demonstrated by their coefficients above is also seen with respect to this notion of degree. If $a(b + c) - ac - ab \neq 0$ in the near-ring N, then $f = a(b + cx) - acx - ab$ has $\deg(f) = 0$. However, $ht(f) = 1$. This is one of the main reasons why we found it necessary to work with the height rather than the degree.

(6) For ring polynomials, we know that a polynomial is 0 if and only if all the coefficients are 0. For a zero near-ring polynomial, also all the coefficients must be 0, but the converse need not be true (see (4) above).

(7) For $f \in N[x]$ with $ht(f) = k$ and any $\alpha_1, \alpha_2, \ldots, \alpha_m \in N, m \geq 1$, $f(\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0, \ldots) \in (N, N, \ldots, N, 0, 0, 0, \ldots)$ where the non-zero components could go up to the $(k+m)$-th position. Indeed, let $F : N^{k+1} \to N$ be the associated function. Since all the functions in $N[x]$ are 0-symmetric, $F(0, 0, \ldots, 0) = 0$. If we take $\alpha_{m+j} = 0$ for all $j \geq 1$, then by Proposition 3.1 we get

$$\begin{aligned}
&\pi_{k+m+j}(f(\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0, \ldots)) \\
&= F(\alpha_{k+m+j-k}, \alpha_{k+m+j-k+1}, \ldots, \alpha_{k+m+j-1}, \alpha_{k+m+j}) \\
&= F(\alpha_{m+j}, \alpha_{m+j+1}, \ldots, \alpha_{m+k}, \alpha_{m+k+1}) \\
&= F(0, 0, \ldots, 0) = 0 \text{ for all } j \geq 1.
\end{aligned}$$

We close this section with some remarks on the construction of polynomial near-rings with multiple indeterminates. For a ring R, the usual definition of $R[x, y]$ is as an iterated polynomial ring construction $R[x, y] = (R[x])[y]$. This means the constants of $R[x, y]$ are the elements from $R[x]$ which is not really desirable and a more formal approach is better. For polynomial near-rings, Lee and Groenewald [8] have shown that $(N[x])[y] \cong (N[y])[x]$ by showing that both are isomorphic to a certain near-ring of self-mappings of a group of infinite matrices. We give a different construction of $N[x, y]$ which avoids the iterated approach and also the use of infinite matrices. Moreover, this construction is easy to generalize to polynomial near-rings of three or more indeterminates.

If $(N^\omega)^2$ denotes the direct sum of two copies of $(N^\omega, +)$, then $(N^\omega)^2$ is an $N-N$-bigroup with respect to the canonical actions. Here we will use $s : N^\omega \to N^\omega$ to denote the shift function $s(\alpha) := (0, \alpha_1, \alpha_2, \alpha_3, \ldots)$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) \in N^\omega$. Define $x : (N^\omega)^2 \to (N^\omega)^2$ and $y : (N^\omega)^2 \to (N^\omega)^2$ by $x(\alpha, \beta) := (s(\alpha), \beta)$ and $y(\alpha, \beta) = (\alpha, s(\beta))$ for $\alpha, \beta \in N^\omega$. Then both x and y are commuting indeterminates in $M_N((N^\omega)^2)$ and also $xy = yx$. We then define $N[x, y]$, the *polynomial near-ring in the two commuting indeterminates* x and y over N, as the subnear-ring of $M_N((N^\omega)^2)$ generated by $N \cup \{x, y\}$. If N is a ring with identity, then $N[x, y]$ is just the ring of polynomials in the two commuting indeterminates x and y.

It can be shown that $N[x, y] = \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ where

$$\mathcal{A}_0 = \{x^n y^m \mid n, m \geq 0\},$$
$$\mathcal{A}_1 = \{a x^n y^m \mid n, m \geq 0, a \in N\},$$
$$\mathcal{A}_2 = \{\sum_{i=1}^{m} a_i x^{n_i} y^{m_i} \mid m \geq 1,\ a_i \in N,\ n_i, m_i \geq 0\}$$

and

$$\mathcal{A}_{n+1} = \{\sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n\} \text{ for } n \geq 2.$$

4. Divisibility

We will need a division algorithm analogous to that of ring polynomials. Let $0 \neq h \in N[x]$ with height $k \geq 1$ and let $H := \langle h \rangle$ be the ideal in $N[x]$ generated by h. We suppose that we can write x^k as $x^k = h_1 + p$ where $h_1 \in H$ and $p \in N[x]$ has height $\leq k-1$. In a near-field N, the expression for x^k can readily be obtained for some choices of h. For example, if $h = a(b + cx^3) + dx$ with $a \neq 0$, $c \neq 0$, we have

$$\begin{aligned} x^3 &= c^{-1}(a^{-1}(h - dx) - b) \\ &= c^{-1}(a^{-1}(h - dx) - a^{-1}(-dx) + a^{-1}(-dx) - b) \\ &= c^{-1}(h_1 + a^{-1}(-dx) - b) - c^{-1}(a^{-1}(-dx) - b) + c^{-1}(a^{-1}(-dx) - b) \\ &= h_2 + p \end{aligned}$$

for some $h_1, h_2 \in H$ and $p := c^{-1}(a^{-1}(-dx) - b) \in N[x]$ has height at most 1. However, when for example $h = a(b + x^2) - ax^2 - ab$, then it is not clear (if indeed possible) how to isolate x^2.

Proposition 4.1. *Let $0 \neq h \in N[x]$ with height $k \geq 1$ and let $H := \langle h \rangle$ be the ideal in $N[x]$ generated by h. Suppose x^k can be expressed as $x^k = h_1 + p$ where $h_1 \in H$ and $p \in N[x]$ has height $\leq k - 1$. Then any $f \in N[x]$ can be written as $f = h' + r$ where $h' \in H$ and $r \in N[x]$ with $r = 0$ or if $r \neq 0$, then it has height $\leq k - 1$.*

Proof. If $0 \neq f \in N[x]$ has height $< k$, let $f = 0 + r$ where $r = f$ and we are done. Suppose thus f has height $k + n$, $n \geq 0$ and we proceed by induction on n.

Let $n = 0$. We show that f has the required form by induction on the level of f. If $f = ax^k$ for some $a \in N$, then

$$\begin{aligned} f &= ax^k \\ &= a(h_1 + p) \text{ where } h_1 \in H \text{ and } p \in N[x] \text{ has height } \leq k - 1 \\ &= a(h_1 + p) - ap + ap \\ &= h_2 + ap \text{ with } h_2 \in H \text{ and } ap = 0 \text{ or } ap \text{ has height } \leq k - 1. \end{aligned}$$

Let $f = \sum_{j=1}^{m} a_j x^{k_j} \in \mathcal{A}_2$ with $k_j \leq k$ for all $j = 1, 2, \ldots, m$. Suppose $k = k_{j_1} = k_{j_2} = \cdots = k_{j_t}$ (there must be at least one such k_{j_i}) and $k_j < k$ for all $k_j \in \{k_1, k_2, \ldots, k_m\} - \{k_{j_1}, k_{j_2}, \ldots, k_{j_t}\}$. By the first step of this induction, we know $a_{j_i} x^{k_{j_i}} = h_{j_i} + r_{j_i}$ for some $h_{j_i} \in H$ and $r_{j_i} = 0$ or height $r_{j_i} \leq k - 1$ for all $i = 1, 2, \ldots, t$. Then

$$\begin{aligned} f &= \sum_{j=1}^{m} a_j x^{k_j} \\ &= a_1 x^{k_1} + \cdots + (h_{j_1} + r_{j_1}) + \cdots + a_j x^{k_j} + \cdots + (h_{j_t} + r_{j_t}) + \cdots + a_m x^{k_m} \\ &= h' + a_1 x^{k_1} + \cdots + r_{j_1} + \cdots + a_j x^{k_j} + \cdots + r_{j_t} + \cdots + a_m x^{k_m} \\ &= h' + r \end{aligned}$$

for some $h' \in H$ and $r := a_1 x^{k_1} + \cdots + r_{j_1} + \cdots + a_j x^{k_j} + \cdots + r_{j_t} + \cdots + a_m x^{k_m}$ is zero or has height $\leq k - 1$. Let $l \geq 2$ and suppose the statement is true for all polynomials $g \in \mathcal{A}_l$ with height $\leq k$. Let $f = \sum_{j=1}^{m} a_j w_j \in \mathcal{A}_{l+1}$ with height k

where $m \geq 1$, $a_j \in N$ and $w_j \in \mathcal{A}_l$. We suppose aw_j has height k_j and then, by definition, $k = \max\{k_1, k_2, \ldots, k_m\}$. By the induction assumption, $w_j = h_j + r_j$ for some $h_j \in H$ and $r_j = 0$ or height $r_j \leq k-1$ for all $j = 1, 2, \ldots, m$. Then

$$\begin{aligned} f &= \sum_{j=1}^{m} a_j w_j \\ &= \sum_{j=1}^{m} (a_j(h_j + r_j) - a_j r_j + a_j r_j) \\ &= \sum_{j=1}^{m} (h'_j + a_j r_j) \text{ for some } h'_j \in H \\ &= h' + \sum_{j=1}^{m} a_j r_j \text{ for some } h' \in H \end{aligned}$$

where either $\sum_{j=1}^{m} a_j r_j = 0$ or it has height $\leq k-1$. We thus conclude that for $n = 0$ the result holds, i.e. for any $f \in N[x]$ with $ht(f) \leq k$, f has the required form.

Let $n \geq 1$ be fixed and suppose for any $g \in N[x]$ with height $\leq k + (n-1)$ we have $g = h' + r$ where $h' \in H$ and $r = 0$ or r has height $\leq k-1$. Let $f \in N[x]$ with $ht(f) = k + n$. We show f has the required form by induction on the level of f.

If $f = ax^{k+n}$ for some $a \in N$, then

$$\begin{aligned} f &= ax^{k+n} \\ &= (ax^{k+n-1})x \\ &= (h_1 + r_1)x \text{ for some } h_1 \in H, r_1 = 0 \text{ or } ht(r_1) \leq k-1 \\ &= h_1 x + (h_2 + r_2) \text{ for some } h_2 \in H, r_2 = 0 \text{ or } ht(r_2) \leq k-1 \\ &= (h_1 x + h_2) + r_2 \text{ as required.} \end{aligned}$$

Let $f = \sum_{j=1}^{m} a_j x^{k_j} \in \mathcal{A}_2$ with $ht(f) = k + n = \max\{k_j \mid j = 1, 2, \ldots, m\}$. Then for each j, $a_j x^{k_j} = h_j + r_j$ for some $h_j \in H$ and $r_j = 0$ or $ht(r_j) \leq k-1$. Hence $f = \sum_{j=1}^{m} (h_j + r_j) = h' + \sum_{j=1}^{m} r_j$ where $h' \in H$ and $\sum_{j=1}^{m} r_j = 0$ or has height $\leq k-1$.

Finally, let $l \geq 2$ be fixed and suppose for each $g \in \mathcal{A}_l$ with $ht(g) \leq k + n$ we have g of the required form. Let $f = \sum_{j=1}^{m} a_j w_j \in \mathcal{A}_{l+1}$ with $ht(f) = k + n = \max\{ht(a_j w_j) \mid j = 1, 2, \ldots, m\}$ where $m \geq 1$, $a_j \in N$ and $w_j \in \mathcal{A}_l$. By the induction assumption, $w_j = h_j + r_j$ for some $h_j \in H$ and $r_j = 0$ or $ht(r_j) \leq k-1$ for all $j = 1, 2, \ldots, m$. Then

$$\begin{aligned} f &= \sum_{j=1}^{m} a_j w_j \\ &= \sum_{j=1}^{m} (a_j(h_j + r_j) - a_j r_j + a_j r_j) \\ &= \sum_{j=1}^{m} (h'_j + a_j r_j) \text{ for some } h'_j \in H \\ &= h' + \sum_{j=1}^{m} a_j r_j \end{aligned}$$

for some $h' \in H$ where either $\sum_{j=1}^{m} a_j r_j = 0$ or it has height $\leq k-1$. $\square$

5. Quotients of polynomial near-rings

In this section we develop the theory and techniques to describe quotients of polynomial near-rings of the type $\frac{N[x]}{\langle h \rangle}$ for certain $h \in N[x]$ where $\langle h \rangle$ is the ideal in $N[x]$ generated by h. But firstly, we recall: For rings there is a well-developed theory with many applications. For example, if $h = x^2 + 1 \in \mathbb{R}[x]$, the ring of polynomials over the reals, then $\frac{\mathbb{R}[x]}{\langle x^2+1 \rangle} \cong \{a + by \mid a, b \in \mathbb{R}, y^2 + 1 = 0\}$. In this set y is, like x in $\mathbb{R}[x]$, a placeholder and a reminder of the multiplication rule, but not an indeterminate in the ring polynomial sense of the word. Also note that $f := f_0 + f_1 x + \cdots + f_n x^n = 0$ means the coefficients $f_0 = f_1 = \cdots = f_n = 0$. But for y as above if, for example, $a_0 + a_1 y + a_2 y^2 = 0$, then it does not necessarily mean $a_0 = a_1 = a_2 = 0$ while $a_0 + a_1 y = 0$ does imply $a_0 = a_1 = 0$. In the model we are using for polynomial near-rings, x is not an indeterminate in the ring theory sense – here it is a very specific function with a clearly described definition. Also, concepts like coefficient and degree are no longer as obvious as in the ring case (if at all).

Let $0 \neq h := x^k - p$ where $p \in N[x]$ with $ht(p) \leq k-1$ and $k \geq 2$. Let $P : N^k \to N$ be the function associated with p. Even if $ht(p) < k-1$, we will always take the domain of the associated function P as N^k (cf. Remark 3.2(2)). Let $H := \langle h \rangle$. From Proposition 2.7, if $N \cap H = 0$ and $x + H \notin (N+H)/H$, then $\frac{N[x]}{H} \cong [N, S, y]$ where $S = \frac{N[x]}{H}$ and $y := x + H$. Here $y : \frac{N[x]}{H} \to \frac{N[x]}{H}$ is defined by $y(f+H) := xf + H = r_f + H$ where xf is reduced to $r_f \mod H$ with $r_f = 0$ or r_f has height $\leq k-1$. Note that y, being the homomorphic image of the commuting indeterminate x, is a commuting indeterminate and $y^k = x^k + H = p + H = \overline{p}$ where $\overline{p}$ is obtained from p by replacing each x in p with y in $\overline{p}$ (and for $a \in N$, since $N \cap H = 0$, we identify $a + H$ with a).

Whereas any $f \in N[x]$ is a function $f : N^\omega \to N^\omega$, $f + H$ could be thought of as having maximum height $k-1$ (after reduction $\mod H$). This means it may be possible to describe the elements of $\frac{N[x]}{H}$ as functions $f + H : N^k \to N^k$ rather than as functions $\frac{N[x]}{H} \to \frac{N[x]}{H}$. We explore this possibility further.

Let $f \in N[x]$ with $ht(f) = k-1$ and let $F : N^k \to N$ be the associated function. From $y(f+H) = xf + H$ and remembering that the action of x is just a shift to the right, we may think that the "associated function" of $y(f+H)$ is $\overline{F} : N^k \to N$ defined by $\overline{F}(\alpha_1, \alpha_2, \ldots, \alpha_k) = F(\overline{P(\alpha)}, \alpha_1, \ldots, \alpha_{k-1})$ where $\overline{P(\alpha)}$ has the following meaning: Every $a\alpha_1$ in F comes from ax^{k-1} in f. This means, in $y(f+H)$, ax^{k-1} becomes ax^k which $\mod H$ is ap and thus $P(a\alpha)$ in $\overline{F}$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in N^k$. We cannot just replace every $a\alpha_1$ in F with $aP(\alpha)$ in $\overline{F}$ since in general $aP(\alpha)$ and $P(a\alpha)$ need not coincide. In particular, we may think of $y = x + H : N^k \to N^k$ as a shift function which is, unlike x, not open-ended at the right. Rather, x^k is reincarnated and appears as $P(\alpha)$ in the original position of α_1. In particular, $y(\beta_1, \beta_2, \ldots, \beta_k) := (P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-1})$ for all $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$. But still there is a problem. In $N[x]$, x is a commuting indeterminate and if y is to be its homomorphic image, it better also be commuting. However, this need not be the case and to ensure that y will be commuting, we have to assume that p is such that $P(a\alpha) = aP(\alpha)$ and $P(\alpha+\beta) = P(\alpha) + P(\beta)$ for all $a \in N$ and $\alpha, \beta \in N^k$. Note that, since $k \geq 2$, we have $y \in M_N(N^k) - N$. Indeed, if $y = a$ for some $a \in N$, then for all $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$,

$$\begin{aligned}(a\beta_1, a\beta_2, \ldots, a\beta_k) &= a(\beta_1, \beta_2, \ldots, \beta_k) \\ &= y(\beta_1, \beta_2, \ldots, \beta_k) \\ &= (P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-1}).\end{aligned}$$

Thus $P(\beta) = a\beta_1$ and $\beta_i = a\beta_{i+1}$ for all $i \geq 1$. Since $k \geq 2$, this gives $y = a = 0$ and so $(P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-1}) = y(\beta) = 0$ for all β_i. But then the contradiction $\beta_1 = 0$ for all $\beta_1 \in N$ follows. Hence $y \in M_N(N^k) - N$ and it can be verified that y is a commuting indeterminate. We will show that $\frac{N[x]}{H} \cong [N, N^k, y]$, but for this we need to establish more tools.

Lemma 5.1. *For $k \geq 2$, let $P : N^k \to N$ be a function which satisfies $P(\beta a) = P(\beta)a$ for all $\beta \in N^k, a \in N$. Define $y : N^k \to N^k$ by $y(\beta_1, \beta_2, \ldots, \beta_k) := (P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-1})$ for all $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$. For $f \in N[x]$ let $\overline{f}$ be the element of $[N, N^k, y]$ obtained from f by replacing every x if f with y in $\overline{f}$. Then:*

1. *For any $j \geq 1$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$, $y^j(\beta) = (P(y^{j-1}(\beta)), P(y^{j-2}(\beta)), \ldots, P(y^{j-k}(\beta)))$ where, by definition, we take $P(y^{-i}(\beta)) := \beta_i$ for $i = 1, 2, \ldots, k$. In particular, the i-th component of $y^j(\beta)$ is $\pi_i(y^j(\beta)) = P(y^{j-i}(\beta))$.*
2. *If $f \in N[x]$ has $ht(f) = m$ and associated function $F : N^{m+1} \to N$, then for all $i = 1, 2, \ldots, k$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$, $\pi_i(\overline{f}(\beta_1, \beta_2, \ldots, \beta_k)) = F(P(y^{m-i}(\beta)), P(y^{m-i-1}(\beta)), \ldots, P(y^{-i+1}(\beta)), P(y^{-i}(\beta)))$.*
3. *If $r \in N[x]$ with $ht(r) = l \leq k-1$ and $\overline{r} = 0$, then $r = 0$.*

Proof. (1) By definition of y, for $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, we have

$$\begin{aligned}y(\beta) &= (P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-1}), \\ y^2(\beta) &= (P(y(\beta)), P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-2}), \\ y^3(\beta) &= (P(y^2(\beta)), P(y(\beta)), P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-3}) \text{ etc.}\end{aligned}$$

If we let $P(y^{-i}(\beta)) := \beta_i$ for $i = 1, 2, \ldots, k$, then

$$y^j(\beta) = (P(y^{j-1}(\beta)), P(y^{j-2}(\beta)), \ldots, P(y^{j-k}(\beta))) \text{ for all } j = 0, 1, 2, 3, \ldots.$$

In particular, the i-th component of $y^j(\beta), 1 \leq i \leq k$, is $\pi_i(y^j(\beta)) = P(y^{j-i}(\beta))$.

(2) For $f \in N[x]$ with $ht(f) = m$, we have $\overline{f} : N^k \to N^k$. Let $F : N^{m+1} \to N$ be the function associated with f. By Remark 3.2 (3), x^j in f becomes α_{m+1-j} in F (and conversely) and x^j becomes y^j in $\overline{f}$. The i-th component of $y^j(\beta)$ is $P(y^{j-i}(\beta))$ as was shown in (1) above. This means we have the correspondences

$$\alpha_j \text{ in } F \leftrightarrow x^{m+1-j} \text{ in } f \leftrightarrow y^{m+1-j} \text{ in } \overline{f}$$

and $y^{m+1-j}(\beta)$ has i-th component $P(y^{m+1-j-i}(\beta))$ for all $j = 1, 2, \ldots, m+1$ which justifies the claim.

(3) Let $R : N^{l+1} \to N$ be the function associated with r. For all $i = 1, 2, 3, \ldots$, $\pi_i(r(\alpha_1, \alpha_2, \alpha_3, \ldots)) = R(\alpha_{i-l}, \alpha_{i-l+1}, \ldots, \alpha_i)$ for all $\alpha_i \in N$. By the assumption and (2) above, we have for all $1 \le i \le k$ and all β_j, $0 = \pi_i(\overline{r}(\beta_1, \beta_2, \ldots, \beta_k)) = R(P(y^{l-i}(\beta)), P(y^{l-i-1}(\beta)), \ldots, P(y^{-i+1}(\beta)), P(y^{-i}(\beta)))$. Recall that $P(y^{-t}(\beta)) = \beta_t$ for $t = 1, 2, \ldots, k$. Since $l < k$,

$$\begin{aligned} 0 &= \pi_k(\overline{r}(\beta_1, \beta_2, \ldots, \beta_k)) \\ &= R(P(y^{l-k}(\beta)), P(y^{l-k-1}(\beta)), \ldots, P(y^{-k+1}(\beta)), P(y^{-k}(\beta))) \\ &= R(\beta_{k-l}, \beta_{k-l+1}, \ldots, \beta_{k-1}, \beta_k) \text{ for all } \beta_j. \end{aligned}$$

Thus, for any $\alpha_1, \alpha_2, \ldots, \alpha_{l+1} \in N$,
$R(\alpha_1, \alpha_2, \ldots, \alpha_{l+1}) = \pi_k(\overline{r}(\ldots, \alpha_1, \alpha_2, \ldots, \alpha_{l+1})) = 0$ where the first $k - (l + 1)$ places in the argument of r can be filled arbitrarily. This means for any $\alpha_1, \alpha_2, \ldots \in N$ and $i = 1, 2, 3, \ldots$ it then follows that $\pi_i(r(\alpha_1, \alpha_2, \alpha_3, \ldots)) = R(\alpha_{i-l}, \alpha_{i-l+1}, \ldots, \alpha_i) = 0$. Thus $r = 0$. □

This brings us to our main result:

Proposition 5.2. *Let $h = x^k - p \in N[x]$ with $p \in N[x]$, $ht(p) \le k-1$ and $k \ge 2$. Define $y : N^k \to N^k$ by $y(\beta) = (P(\beta), \beta_1, \beta_2, \ldots, \beta_{k-1})$ where $P : N^k \to N$ is the function associated with p and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$. Suppose the function P satisfies: $P(a\alpha) = aP(\alpha)$ and $P(\alpha + \beta) = P(\alpha) + P(\beta)$ for all $\alpha, \beta \in N^k, a \in N$. Then $\frac{N[x]}{\langle h \rangle} \cong [N, N^k, y]$ and $\overline{h} = 0$ where $\overline{h}$ is obtained from h by replacing all occurrences of x in h with y in $\overline{h}$.*

Proof. Define $\theta : N[x] \to [N, N^k, y]$ by $\theta(f) := \overline{f}$ where $\overline{f}$ is obtained from f by replacing all occurences of x in f with y in $\overline{f}$. We firstly show θ is well-defined. Let $f, g \in N[x]$ with $f = g, ht(f) = m, ht(g) = n$, say $m \le n$, and associated functions $F : N^{m+1} \to N, G : N^{n+1} \to N$. We show $\overline{f} = \overline{g}$.

For all $i = 1, 2, \ldots, k$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$ we have from Lemma 5.1 and Remark 3.2(1),

$$\begin{aligned} &\pi_i(\overline{g}(\beta_1, \beta_2, \ldots, \beta_k) \\ &= G(P(y^{n-i}(\beta)), P(y^{n-i-1}(\beta)), \ldots, P(y^{-i+1}(\beta)), P(y^{-i}(\beta))) \\ &= F(P(y^{-i+m}(\beta)), P(y^{-i+m-1}(\beta)), \ldots, P(y^{-i+1}(\beta)), P(y^{-i}(\beta))) \\ &= \pi_i(\overline{f}(\beta_1, \beta_2, \ldots, \beta_k). \end{aligned}$$

Hence $\overline{f} = \overline{g}$ and θ is well-defined. Clearly θ is a homomorphism and surjective with $\ker\theta = \{f \in N[x] \mid \overline{f} = 0\}$. To complete the proof, we show $\ker\theta = \langle h \rangle$. For $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in N^k$ and $i = 1, 2, \ldots, k$, by Lemma 5.1 $\pi_i(\overline{h}(\beta)) = \pi_i(y^k(\beta) - \overline{p}(\beta)) = P(y^{k-i}(\beta)) - P(y^{k-i}(\beta)) = 0$. Thus $\overline{h} = 0$ and $h \in \ker\theta$. Conversely, let $f \in \ker\theta$. Then $\overline{f} = 0$ and by Proposition 4.1 $f = h_1 + r$ where $h_1 \in \langle h \rangle$ and $r = 0$ or $ht(r) \le k - 1$. We show $r = 0$. Now $0 = \overline{f} = \overline{h_1} + \overline{r} = \overline{r}$ since $h_1 \in \langle h \rangle \subseteq \ker\theta$. From Lemma 5.1 (3) we have $r = 0$ and we conclude that $\ker\theta = \langle h \rangle$. □

Proposition 5.3. *With the notation as in the previous proposition, $\mathcal{B} := \{1, y, y^2, \ldots, y^{k-1}\}$ is a basis for $[N, N^k, y]$ over N.*

Proof. Let $t \in \{1,2,\dots,k\}$ be fixed; let $a_1,a_2,\dots,a_t \in N$ and $k_1,k_2,\dots,k_t$ distinct elements from $\{0,1,2,\dots,k-1\}$ such that $\sum_{i=1}^{t} a_i y^{k_i} = 0$. Let $r := \sum_{i=1}^{t} a_i x^{k_i}$. Then $r \in N[x]$ with $ht(r) \le k-1$ and $\overline{r} = \sum_{i=1}^{t} a_i y^{k_i} = 0$. By Lemma 5.1(3) $\sum_{i=1}^{t} a_i x^{k_i} = r = 0$. Since $\{1,x,x^2,\dots\}$ is a basis for $N[x]$ over N, we get $a_1 = a_2 = \cdots = a_t = 0$. Lastly, it is clear that $\mathcal{B}$ generates $[N,N^k,y]$ over N since $y^k = \overline{p} \in \mathcal{B}_n$ for some $n \ge 1$. $\square$

6. Examples

6.1. Let $H \neq 0$ be a proper subgroup of the group G. Let N be the near-ring $N := \{a \in M_0(G) \mid a(H) \subseteq H\}$. N is 0-symmetric and has an identity.

(a) Let a,b and c be non-zero elements of N with $a(H) = 0, b(G) \subseteq H$ and $c(G) \subseteq H$. For any $u,v,w \in N$ and positive integers k,l,m and n, we have $0 = a(bx^k + c(ux^n + v(x^m + wx^l))) \in N[x]$ of level 4 and height $\max\{k,n,m,l\}$.

(b) Let $a,b,c \in N$ with $a(H) = 0, a(G-H) = a_0$ where $0 \neq a_0 \in G, 0 \neq b(G) \subseteq H$ and $0 \neq c(H) \subseteq H, c(G-H) \subseteq G-H$. For $k \ge 1$, $a(bx^k + cx) = ax \neq 0$. Here $a(bx^k + cx)$ has level 2 and height k and ax has level 1 and height 1 (actually a distributes over $bx^k + cx$, $ab = 0$ and $ac = a$).

6.2. Let N be the following 0-symmetric near-ring with identity on $N = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which for ease of reference, we write as $N = \{0,a,b,1\}$ with $a+b = 1$, $a+1 = b$, $b+1 = a$, $at = 0$ for all $t, ba = a$ and $b^2 = b$. Then $-t = t$ for all $t \in N$, $(N,+)$ is commutative and $x(-1) = x = -x$. From Proposition 5.2 it follows $\frac{N[x]}{\langle x^2+1\rangle} \cong [N,N^2,y]$ where $y : N^2 \to N^2$ is defined by $y(\alpha_1,\alpha_2) := (-\alpha_2,\alpha_1) = (\alpha_2,\alpha_1)$. This near-ring has a unit y since $y^2 = -1 = 1$. Note that in $N[x]$ we have $a(a+ax) = 0$ and $b(a+ax) = a+ax$.

6.3. For any $h = x^k - p$ where $p \in N[x]$ with $ht(p) \le k-1, k \ge 2$, we can define $y : N^k \to N^k$ by $y(\beta_1,\beta_2,\dots,\beta_k) := (P(\beta_1,\beta_2,\dots,\beta_k),\beta_1,\beta_2,\dots,\beta_{k-1})$. Here $P : N^k \to N$ is the function associated with p. Then we may form the near-ring $[N,N^k,y]$ in which $y^k = \overline{p}$ where $\overline{p}$ is obtained from p by replacing each x with y. In general the indeterminate y need not be commuting. On the other hand, we have $\frac{N[x]}{\langle x^k-p\rangle} \cong [N,S,w]$ where $S := \frac{N[x]}{\langle x^k-p\rangle}$ and $w : S \to S$ is the commutative indeterminate defined by $w(f+H) := xf+H$ (provided $x+H \notin (N+H)/H$ and $N \cap H = 0$). Here $H = \langle x^k - p\rangle$. In general we need not have $\frac{N[x]}{\langle x^k-p\rangle} \cong [N,N^k,y]$.

6.4. Let N, as usual, be a 0-symmetric near-ring with identity and let $e \in N$ be a central element (i.e. $ea = ae$ for all $a \in N$). Let $k,l \in \mathbb{N}$ with $k \ge 2$ and $0 \le l \le k-1$. Let $h = x^k - ex^l \in N[x]$ and let $p := ex^l$ with associated function $P : N^k \to N$ defined by $P(\alpha_1,\alpha_2,\dots,\alpha_k) = e\alpha_{k-l}$. Then $y : N^k \to N^k$ is defined by $y(\beta_1,\beta_2,\dots,\beta_k) := (e\beta_{k-l},\beta_1,\dots,\beta_{k-1})$. It is clearly a commuting indeterminate. From Proposition 5.2 we know $\frac{N[x]}{\langle x^k - ex^l\rangle} \cong [N, N^k, y] = \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ where $y^k = ey^l$ and

$$\begin{aligned} \mathcal{A}_0 &= \{1, y, y^2, \ldots, y^{k-1}\} \cup \{e^n y^{l+j} \mid n \geq 1,\ j = 0, 1, 2, \ldots, k-l-1\}, \\ \mathcal{A}_1 &= \{ay^t \mid t \geq 1,\ a \in N\} \text{ and} \\ \mathcal{A}_{n+1} &= \left\{ \sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n \right\} \text{ for } n \geq 1. \end{aligned}$$

Some special cases are:

(a) Let $h = x^2 + 1$, i.e. $k = 2$, $l = 0$, $e = -1$. Suppose N is a near-ring with $a(-1) = -a$ for all $a \in N$ (and then $(N, +)$ is necessarily commutative, e.g. if N is a near-field). Then $\frac{N[x]}{\langle x^2+1\rangle} \cong [N, N^2, y] = \bigcup_{n=0}^{+\infty} \mathcal{A}_n$ where $y^2 + 1 = 0$ and

$$\begin{aligned} \mathcal{A}_0 &= \{1, -1, y, -y\}, \\ \mathcal{A}_1 &= \{ay^t \mid 0 \leq t \leq 1,\ a \in N\} \text{ and} \\ \mathcal{A}_{n+1} &= \left\{ \sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n \right\} \text{ for } n \geq 1. \end{aligned}$$

(b) $h = x^k - 1 \in N[x]$ meets the above requirements. This example will be discussed in more detail later (see 6.5 below).

(c) If $e = 1$, then $h = x^k - x^l$ is another example that meets the above requirements.

(d) For $e = 0$, we get $h = x^k$ and $y(\beta_1, \beta_2, \ldots, \beta_k) := (0, \beta_2, \ldots, \beta_k)$ with $y^k = 0$. Then $\frac{N[x]}{\langle x^k\rangle} \cong [N, N^k, y]$. Recall from Lee [7] that $\langle x^k\rangle = \{f \in N[x] \mid f(\alpha) \in (0, 0, \ldots, 0, N, N, \ldots)$ for all $\alpha \in N^\omega\}$ where the first k positions of $(0, 0, \ldots, 0, N, N, \ldots)$ are zeros.

6.5. We firstly recall the construction of the well-known circulant matrix rings. Let R be a ring, $k \geq 2$ and $\mathbb{M}_k(R)$ the ring of $k \times k$ matrices over R. If $R[x]$ is the polynomial ring over R, then

$$\begin{aligned} \frac{R[x]}{\langle x^k - 1\rangle} &\cong \{a_0 + a_1 y + a_2 y^2 + \cdots + a_{k-1} y^{k-1} \mid y^k = 1, a_i \in R\} \\ &\cong \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{k-1} \\ a_{k-1} & a_0 & a_1 & \ldots & a_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{bmatrix} \mid a_i \in R \right\} \end{aligned}$$

which is a subring of $\mathbb{M}_k(R)$.

It will be shown that a similar result holds for near-rings. Let N be a near-ring with $N[x]$ the polynomial near-ring. For $k \geq 2$ and $h = x^k - 1$, $\frac{N[x]}{\langle x^k-1\rangle} \cong [N, N^k, y]$

where $y\colon N^k \to N^k$ is the commuting indeterminate defined by $y(\beta_1, \beta_2, \ldots, \beta_k) := (\beta_k, \beta_1, \ldots, \beta_{k-1})$. Then $y^k = 1$ and y is a unit with inverse $y^{-1} = y^{k-1} \in [N, N^k, y]$. The elements of $[N, N^k, y]$ are functions $f : N^k \to N^k$, but so are the elements of the $k \times k$ matrix near-ring $\mathbb{M}_k(N)$ over N. For the basics on matrix near-rings, see Meldrum and van der Walt [10] or Clay [3]. We recall just the following: $\mathbb{M}_k(N)$ is the subnear-ring of $M_0(N^k)$ generated by $\{f_{ij}^a \mid 1 \leq i, j \leq k, a \in N\}$ where $f_{ij}^a : N^k \to N^k$ is defined by $f_{ij}^a(\beta_1, \beta_2, \ldots, \beta_k) := (0, \ldots, 0, a\beta_j, 0, \ldots, 0)$ with $a\beta_j$ in the i-th position. (Contrary to what we do here, it is customary in matrix near-ring theory to write the vectors in the arguments as column vectors.)

We will show that $\frac{N[x]}{\langle x^k - 1\rangle} \cong [N, N^k, y] \subseteq \mathbb{M}_k(N)$. The isomorphism is by Proposition 5.2 and the inclusion will follow if we can show the inclusions $N \cup \{y\} \subseteq \mathbb{M}_k(N) \subseteq M_N(N^k) \subseteq M_0(N^k)$ because $[N, N^k, y]$ is the subnear-ring of $M_N(N^k)$ generated by $N \cup \{y\}$.

For the first inclusion, it can be shown that for $a \in N$, regarded (as usual) as the function $a : N^k \to N^k$ defined by $a(\beta_1, \beta_2, \ldots, \beta_k) = (a\beta_1, a\beta_2, \ldots, a\beta_k)$, we have $a = \sum\limits_{j=1}^{k} f_{jj}^a \in \mathbb{M}_k(N)$ and $y = f_{1k}^1 + \sum\limits_{j=2}^{k} f_{j,j-1}^1$. The other two inclusions are obvious and we conclude that $\frac{N[x]}{\langle x^k - 1\rangle} \cong [N, N^k, y] \subseteq \mathbb{M}_k(N)$. The near-ring $[N, N^k, y]$ can be regarded as the near-ring analogue of the circulant matrix rings. When $(N, +)$ is commutative, then the defining classes simplify somewhat and take on a more ring-like appearance: $[N, N^k, y] = \bigcup\limits_{n=0}^{+\infty} \mathcal{A}_n$ where $y^k = 1$,

$$\begin{aligned}
\mathcal{A}_0 &= \{1, y, y^2, \ldots, y^{k-1}\}, \\
\mathcal{A}_1 &= \{ay^t \mid 0 \leq t \leq k-1,\ a \in N\}, \\
\mathcal{A}_2 &= \{a_1 + a_2 y + \cdots + a_k y^{k-1} \mid a_i \in N\} \text{ and} \\
\mathcal{A}_{n+1} &= \left\{ \sum_{i=1}^{m} a_i w_i \mid m \geq 1,\ a_i \in N,\ w_i \in \mathcal{A}_n \right\} \text{ for } n \geq 2.
\end{aligned}$$

6.6. We conclude with another ring analogy. Let $\mathbb{M}_\omega(R)$ denote the ring of $\omega \times \omega$ column finite matrices over the ring R. Then the polynomial ring $R[x]$ can be embedded in $\mathbb{M}_\omega(R)$ by

$$f_0 + f_1 x + f_2 x^2 + \cdots + f_n x^n \mapsto \begin{bmatrix} f_0 & f_1 & f_2 & \ldots & f_n & 0 & 0 & \ldots \\ 0 & f_0 & f_1 & f_2 & \ldots & f_n & 0 & \ldots \\ 0 & 0 & f_0 & f_1 & f_2 & \ldots & f_n & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

(One may still think of this image of a polynomial as an infinite circulant matrix.)

It will be shown that a similar result holds for near-rings, but firstly we need some preparations. For the near-ring N, let $N^{(\omega)} := \{\alpha \mid \alpha : \mathbb{N} \to N$ a function with finite support$\}$. As is well known (see for example [9]), if the polynomial near-ring $N[x]$ is defined using $N^{(\omega)}$ instead of N^ω, then a near-ring isomorphic to $N[x]$ is obtained. Indeed, for $N[x] = [N, N^\omega, x]$ and $N_f[x] = [N, N^{(\omega)}, x]$ where in both cases x is just the shift function, the function $\theta : N[x] \to M_N(N^{(\omega)})$ defined by restricting $f \in N[x]$ to a function $f : N^{(\omega)} \to N^{(\omega)}$, is a well-defined monomorphism (see 3.2(7)) with $\theta(N[x]) = N_f[x]$. We will thus identify $N[x]$ with $N_f[x]$.

For every $s, t \in \mathbb{N}$ and $a \in N$ define the function $f^a_{st} : N^{(\omega)} \to N^{(\omega)}$ by

$$(f^a_{st}(\alpha))(r) := \begin{cases} a\alpha(t) \text{ if } r = s \\ 0 \text{ otherwise} \end{cases}$$

for all $\alpha \in N^{(\omega)}$, $r \in \mathbb{N}$. Then $f^a_{st} \in M_N(N^{(\omega)})$ and one may think of this function as the elementary matrix with a in 'row s, column t'. For notational reasons, f^a_{st} is often written as $[s, t; a]$.

Next we want to give meaning to infinite sums $\sum\limits_{i=1}^{+7\infty}[r_i, s_i; a_i]$ as elements of $M_N(N^{(\omega)})$. The infinite sum $\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]$ is said to be *defined* if for every $\alpha \in N^{(\omega)}$, $[r_i, s_i; a_i](\alpha) = 0$ for almost all i (i.e., it is 0 for all but possibly a finite number of i's). In such a case, the sum $\left(\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]\right)(\alpha) := \sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i](\alpha)$ where the right-hand side is a finite sum of non-zero terms. Any $[r, s; a]$ can be written as such an infinite sum $\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]$ with $r_1 = r, s_1 = s, a_1 = a, a_i = 0$ for all $i \geq 2$ and r_i, s_i arbitrary for $i \geq 2$. Not all infinite sums need to be defined. For example, $\sum\limits_{i=1}^{+\infty}[i, 1; 1]$ is not defined. In fact, if $a_i = 0$ for almost all i, then $\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]$ is defined and when most of the a_i's are not zero, then it can be verified that the infinite sum $\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]$ is defined if and only if almost all of the s_i's are different.

Let $\mathbb{M}_\omega(N)$ be the subnear-ring of $M_N(N^{(\omega)})$ generated by the set of all defined infinite sums $\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]$. For our purposes, we will call this near-ring the *infinite matrix near-ring* over N. When N is a ring, then $\mathbb{M}_\omega(N)$ is just the ring of all $\omega \times \omega$ column finite matrices over N. For example, the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} & a_{26} & \cdots \\ 0 & a_{32} & a_{33} & 0 & a_{35} & 0 & \cdots \\ 0 & a_{42} & 0 & 0 & a_{45} & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_{55} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

is the defined infinite sum $\sum\limits_{i=1}^{+\infty}[r_i, s_i; a_i]$ with $(r_1, s_1) = (1, 1)$, $a_1 = a_{11}$, $(r_2, s_2) = (2, 1)$, $a_2 = a_{21}$, $(r_3, s_3) = (1, 2)$, $a_3 = a_{12}$, $(r_4, s_4) = (2, 2)$, $a_4 = a_{22}$, $(r_5, s_5) = (3, 2)$, $a_5 = a_{32}$, etc.

We now show $N[x] \subseteq \mathbb{M}_\omega(N)$. This will follow if we can establish $N \cup \{x\} \subseteq \mathbb{M}_\omega(N) \subseteq M_N(N^{(\omega)})$. Firstly, for $a \in N$, the mapping $a : N^{(\omega)} \to N^{(\omega)}$ coincides with $\sum_{i=1}^{+\infty} f_{ii}^a$ and this mapping $\sum_{i=1}^{+\infty} f_{ii}^a$ is defined. Indeed, if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k, 0, 0, \dots) \in N^{(\omega)}$ with $\alpha_k \neq 0$, then $f_{ii}^a(\alpha_1, \alpha_2, \dots, \alpha_k, 0, 0, \dots) = 0$ for all $i \geq k + 1$. Secondly, the shift function $x : N^{(\omega)} \to N^{(\omega)}$ defined by $x(\alpha_1, \alpha_2, \dots, \alpha_k, 0, 0, \dots) := (0, \alpha_1, \alpha_2, \dots, \alpha_k, 0, 0, \dots)$ can be written as $x = \sum_{i=1}^{+\infty} f_{i+1,i}^1$ and this sum is defined. We conclude that $N[x] \subseteq \mathbb{M}_\omega(N)$.

An alternative way to see $N[x]$ as a subnear-ring of an infinite matrix near-ring is as follows. In [11] Meyer defines an $\omega \times \omega$ row-finite matrix near-ring $RF_\omega(N)$ over the near-ring N as the subnear-ring of $M_0(N^\omega)$ generated by the set $C := \{f \mid f : N^\omega \to N^\omega$ *a* function such that for each $i \geq 1$ there is an $r \in N$ and a $j \geq 1$ such that $\pi_i f = f^r \pi_j\}$. Here $\pi_i : N^\omega \to N$ is the i-th projection and $f^r : N \to N$ is the function $f^r(n) := rn$ for all $n \in N$. It can be shown that $C \subseteq M_N(N^\omega)$ and hence $RF_\omega(N) \subseteq M_N(N^\omega)$. Moreover, $N \cup \{x\} \subseteq RF_\omega(N)$ and so $N[x] \subseteq RF_\omega(N)$.

References

[1] Bagley, S.: *Polynomial near-rings, distributor ideals and J_2 ideals of generalized centralizer near-rings.* Doctoral dissertation, Texas A&M University, 1993.

[2] Bagley, S.: *Polynomial near-rings: Polynomials with coefficients from a near-ring.* In: Saad, G. et al. (eds.), Nearrings, Nearfields and K-Loops. Kluwer Academic Publishers, Math. Appl. **426** (1997), 179–190. Zbl 0891.16027

[3] Clay, J. R.: *Nearrings: Geneses and Applications.* Oxford Science Publications, New York 1992. Zbl 0790.16034

[4] Farag, M.: *On the structure of polynomial near-rings.* Doctoral dissertation, Texas A&M University, 1999.

[5] Farag, M.: *A new generalization of the center of a near-ring with applications to polynomial near-rings.* Commun. Algebra **29** (2001), 2377–2387. Zbl 0986.16018

[6] Lausch, H.; Nöbauer, W.: *Algebra of Polynomials.* North Holland, Amsterdam 1973. Zbl 0283.12101

[7] Lee, E. K. S.: *Theory of polynomial near-rings.* Commun. Algebra **32** (2004), 1619–1634. Zbl 1086.16028

[8] Lee, E. K. S.; Groenewald, N. J.: *Polynomial near-rings in k indeterminates.* Bull. Aust. Math. Soc. **70** (2004), 441–449. Zbl 1080.16049

[9] le Riche, L. R.; Meldrum, J. D. P.; van der Walt, A. P. J.: *On group near-rings.* Arch. Math. **52** (1989), 132–139. Zbl 0671.16018

[10] Meldrum, J. D. P.; van der Walt, A. P. J.: *Matrix near-rings.* Arch. Math. **47** (1986), 312–319. Zbl 0611.16025

[11] Meyer, J. H.: *Matrix near-rings*. Ph.D. thesis, University of Stellenbosch, 1986.

[12] Pilz, G.: *Near-rings*. North Holland, Amsterdam 1983. Zbl 0521.16028

Received August 22, 2007