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Cohomology of Torus Bundles over Kuga Fiber Varieties

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Abstract. A Kuga fiber variety is a family of abelian varieties parametrized by a locally symmetric space and is constructed by using an equivariant holomorphic map of Hermitian symmetric domains. We construct a complex torus bundle $\mathcal{T}$ over a Kuga fiber variety Y parametrized by X and express its cohomology $H^*(\mathcal{T}, \mathbb{C})$ in terms of the cohomology of Y as well as in terms of the cohomology of the locally symmetric space X.

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1. Introduction

A Kuga fiber variety is a family of abelian varieties parametrized by an arithmetic variety and can be constructed by using an equivariant holomorphic map of Hermitian symmetric domains. The goal of this paper is to construct a complex torus bundle over a Kuga fiber variety and study its cohomology.

Let G be a semisimple Lie group of Hermitian type defined over $\mathbb{Q}$, and let $\mathcal{D}$ be the associated Hermitian symmetric domain, which can be identified with the quotient G/K of G by a maximal compact subgroup K. We assume that there are a homomorphism $\rho : G \to Sp(V, \alpha)$ of Lie groups and a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}_n$ that is equivariant with respect to ρ, where $Sp(V, \alpha)$ is the symplectic group associated to an alternating bilinear form α on a real vector

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space V of dimension $2n$ and $\mathcal{H}_n$ is the Siegel upper half space of degree n. Let Γ be a torsion-free cocompact arithmetic subgroup of G, so that the corresponding locally symmetric space $X = \Gamma\backslash D$ is a compact complex manifold. When it is regarded as a complex projective variety, X is called an arithmetic variety or a Shimura variety. Then we can construct a fiber bundle Y, called a Kuga fiber variety, over X whose fiber is a polarized abelian variety by pulling back the standard family of abelian varieties associated to $Sp(V, \alpha)$ via the map of X into the Siegel modular variety $\Gamma_0\backslash\mathcal{H}_n$, induced by τ, for some discrete subgroup $\Gamma_0 \subset Sp(V, \alpha)$ with $\rho(\Gamma) \subset \Gamma_0$. Kuga fiber varieties can be considered as a special case of mixed Shimura varieties in modern terms (see [11], [12]), and various aspects of Kuga fiber varieties have been investigated extensively over the years in connection with number theory and algebraic geometry (see e.g. [7], [10]).

In this paper we consider a torus bundle $\mathcal{T}$ over a Kuga fiber variety Y whose fiber is isomorphic to a complex torus of the form $(\mathbb{C}^\times)^d$. Torus bundles of this kind are also mixed Shimura varieties, and they arise naturally in the study of toroidal compactifications of arithmetic varieties, or more generally, Shimura varieties (see e.g. [1], [4], [11], [12]). Such torus bundles are also related to certain generalized Jacobi forms of several variables. Although the construction of such torus bundles is essentially contained in [1], Satake introduced a systematic method of constructing such bundles in [14] using the notion of generalized Heisenberg groups. Our construction of $\mathcal{T}$ uses a further generalization of Satake's method.

The cohomology of arithmetic groups, or equivalently, the cohomology of the associated arithmetic varieties plays an important role in the theory of automorphic forms (see e.g. [2], [3]), and the cohomology of a Kuga fiber variety is closely linked to the cohomology of the associated arithmetic group. The purpose of this paper is to express the cohomology $H^*(\mathcal{T}, \mathbb{C})$ of a torus bundle $\mathcal{T}$ over a Kuga fiber variety Y parametrized by X and express its cohomology $H^*(\mathcal{T}, \mathbb{C})$ in terms of the cohomology of Y as well as in terms of the cohomology of the locally symmetric space X.

2. Group operations

Let V be a real vector space of dimension $2n$ defined over $\mathbb{Q}$, and let I_0 be a complex structure on V, that is, a linear endomorphism of V satisfying $I_0^2 = -1_V$, where 1_V denotes the identity map on V. We assume that there is an alternating bilinear form $\alpha : V \times V \to \mathbb{R}$ such that the bilinear map $(v, v') \mapsto A(v, I_0 v')$ is symmetric and positive definite. Then the map α defines the symplectic group

$$Sp(V, \alpha) = \{g \in GL(V) \mid \alpha(gv, gv') = \alpha(v, v') \text{ for all } v, v' \in V\}.$$

Let G be a semisimple Lie group of Hermitian type defined over $\mathbb{Q}$. Thus $G = \mathbb{G}(\mathbb{R})$ for some semisimple linear algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$, and, if K is a maximal compact subgroup of G, the associated Riemannian symmetric space $\mathcal{D} = G/K$ is a Hermitian symmetric domain. Note that the symplectic group $Sp(V, \alpha)$ is a semisimple Lie group defined over $\mathbb{Q}$, and the associated symmetric domain $\mathcal{H} = \mathcal{H}(V, A)$ can be identified with the Siegel upper half space of degree

n. We assume that there are a homomorphism $\rho : G \to Sp(V, A)$ of Lie groups defined over $\mathbb{Q}$ and a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}$ such that $\tau(gz) = \rho(g)\tau(z)$ for all $g \in G$ and $z \in \mathcal{D}$, so that τ is equivariant with respect to ρ.

We consider another real vector space U defined over $\mathbb{Q}$ and an alternating bilinear map $A : V \times V \to U$ defined over $\mathbb{Q}$. We denote the associated symplectic group by $Sp(V, A)$ contains the image $\rho(G)$ of the homomorphism ρ, which means that

$$A(\rho(g)v, \rho(g)v') = A(v, v') \tag{2.1}$$

for all $g \in G$ and $v, v' \in V$. The following example displays the case considered by Satake in [14].

Example 2.1. Let α be as above, and let G_2 be a semisimple algebraic subgroup of $Sp(V, \alpha)$ defined over $\mathbb{Q}$. Let $\mathcal{D}_2$ be the symmetric domain associated to G_2, so that there is a holomorphic embedding $\mathcal{D}_2 \to \mathcal{H}_m$ that is equivariant with respect to the inclusion map $G_2 \to Sp(V, \alpha)$. Let $\mathrm{Alt}(V)$ be the space of all alternating bilinear forms on $V \times V$, and set

$$U^* = \{\alpha \in \mathrm{Alt}(V) \mid G_2 \subset Sp(V, \alpha)\}.$$

If U denotes the dual space of U^*, then there is a uniquely defined alternating bilinear map $A : V \times V \to U$ such that

$$\alpha(v, v') = \langle \alpha, A(v, v') \rangle$$

for all $\alpha \in U^*$ and $v, v' \in V$. Then this bilinear map A satisfies (2.1) with ρ equal to the inclusion map $G_2 \to Sp(V, \alpha)$.

We now introduce a multiplication operation on the set $G \times V \times U$ defined by

$$(g, v, u)(g', v', u') = (gg', v + \rho(g)v', u + u' - A(v, \rho(g)v')/2) \tag{2.2}$$

for all $g, g' \in G$, $v, v' \in V$ and $u, u' \in U$.

Lemma 2.2. *The formula in* (2.2) *defines a group operation on the set* $G \times V \times U$.

Proof. Given elements (g_1, v_1, u_1), (g_2, v_2, u_2), and (g_3, v_3, u_3) of $G \times V \times U$, we have

$$\begin{aligned}\big((g_1, v_1, u_1)&(g_2, v_2, u_2)\big)(g_3, v_3, u_3)\\ &= (g_1g_2, v_1 + \rho(g_1)v_2, u_1 + u_2 - A(v_1, \rho(g_1)v_2)/2)(g_3, v_3, u_3)\\ &= (g_1g_2g_3, v_1 + \rho(g_1)v_2 + \rho(g_1g_2)v_3, \widetilde{u})\end{aligned}$$

with

$$\begin{aligned}\widetilde{u} &= u_1 + u_2 - A(v_1, \rho(g_1)v_2)/2 + u_3 - A(v_1 + \rho(g_1)v_2, \rho(g_1g_2)v_3)/2\\ &= u_1 + u_2 + u_3 - A(v_1, \rho(g_1)v_2)/2 - A(v_1, \rho(g_1g_2)v_3)/2\\ &\qquad\qquad - A(\rho(g_1)v_2, \rho(g_1)\rho(g_2)v_3)/2\\ &= u_1 + u_2 + u_3 - A(v_1, \rho(g_1)v_2)/2 - A(v_1, \rho(g_1g_2)v_3)/2 - A(v_2, \rho(g_2)v_3)/2,\end{aligned}$$

where we used (2.1). On the other hand, we have

$$\begin{aligned}(g_1, v_1, u_1)\big((g_2, v_2, u_2)(g_3, v_3, u_3)\big) \\ &= (g_1, v_1, u_1)(g_2g_3, v_2 + \rho(g_2)v_3, u_2 + u_3 - A(v_2, \rho(g_2)v_3)/2) \\ &= (g_1g_2g_3, v_1 + \rho(g_1)(v_2 + \rho(g_2)v_3), \widehat{u}) \\ &= (g_1g_2g_3, v_1 + \rho(g_1)v_2 + \rho(g_1g_2)v_3, \widehat{u})\end{aligned}$$

with

$$\begin{aligned}\widehat{u} &= u_1 + u_2 + u_3 - A(v_2, \rho(g_2)v_3)/2 - A(v_1, \rho(g_1)(v_2 + \rho(g_2)v_3))/2 \\ &= u_1 + u_2 + u_3 - A(v_2, \rho(g_2)v_3)/2 - A(v_1, \rho(g_1)v_2))/2 - A(v_1, \rho(g_1)\rho(g_2)v_3)/2.\end{aligned}$$

Thus we see that $\widetilde{u} = \widehat{u}$, and therefore the operation is associative. We see easily that $(1, 0, 0)$ is the identity element and $(g^{-1}, -\rho(g)^{-1}v, -u)$ is the inverse of $(g, v, u) \in G \times V \times U$ with respect to the operation in (2.1); hence the lemma follows. □

Note that the subgroup $\{1\} \times V \times U$ of the group $G \times V \times U$ in Lemma 2.2 can be identified with the Heisenberg group $\mathbb{H} = V \times U$ associated to A whose multiplication operation is given by

$$(v, u) \cdot (v', u') = (v + v', u + u' - A(v, v')/2) \tag{2.3}$$

for all $(v, u), (v', u') \in V \times U$. We shall denote the group in Lemma 2.2 by $G \cdot \mathbb{H}$. We also note that the subgroup $G \times V = G \times V \times \{0\}$ is the usual semidirect $G \ltimes V$ with respect to the action of G on V via ρ.

We now discuss an action of the group $G \cdot \mathbb{H}$ on the space $\mathcal{D} \times V \times U$. If $(g, v, u) \in G \times V \times U$, we set

$$(g, v, u) \cdot (z, v', u') = (gz, v + \rho(g)v', u + u' - A(v, \rho(g)v')/2) \tag{2.4}$$

for all $(z, v', u') \in \mathcal{D} \times V \times U$.

Lemma 2.3. *The formula in* (2.4) *defines an action of the group $G \cdot \mathbb{H}$ on the space $\mathcal{D} \times V \times U$.*

Proof. Given $(g, v, u) \in G \times V \times U = G \cdot \mathbb{H}$ and $(z, v', u') \in \mathcal{D} \times V \times U$, by using (2.2) and (2.4) we obtain

$$\begin{aligned}\big((g_1, v_1, u_1)(g_2, v_2, u_2)\big) \cdot (z, v', u') \\ &= (g_1g_2, v_1 + \rho(g_1)v_2, u_1 + u_2 - A(v_1, \rho(g_1)v_2)/2) \cdot (z, v', u') \\ &= (g_1g_2z, v_1 + \rho(g_1)v_2 + \rho(g_1g_2)v', \widetilde{u}')\end{aligned}$$

with

$$\begin{aligned}\widetilde{u}' &= u_1 + u_2 - A(v_1, \rho(g_1)v_2)/2 + u' - A(v_1 + \rho(g_1)v_2, \rho(g_1g_2)v')/2 \\ &= u_1 + u_2 + u' - A(v_1, \rho(g_1)v_2)/2 - A(v_1, \rho(g_1g_2)v')/2 - A(\rho(g_1)v_2, \rho(g_1)\rho(g_2)v')/2 \\ &= u_1 + u_2 + u' - A(v_1, \rho(g_1)v_2)/2 - A(v_1, \rho(g_1g_2)v')/2 - A(v_2, \rho(g_2)v')/2,\end{aligned}$$

where we used (2.1). On the other hand, we have

$$\begin{aligned}(g_1, v_1, u_1) \cdot \big((g_2, v_2, u_2) \cdot (z, v', u')\big) \\ &= (g_1, v_1, u_1) \cdot (g_2 z, v_2 + \rho(g_2)v', u_2 + u' - A(v_2, \rho(g_2)v')/2) \\ &= (g_1 g_2 z, v_1 + \rho(g_1)v_2 + \rho(g_1 g_2)v', \widehat{u}')\end{aligned}$$

with

$$\begin{aligned}\widehat{u}' &= u_1 + u_2 + u' - A(v_2, \rho(g_2)v')/2 - A(v_1, \rho(g_1)(v_2 + \rho(g_2)v')/2 \\ &= u_1 + u_2 + u' - A(v_2, \rho(g_2)v')/2 - A(v_1, \rho(g_1)v_2)/2 - A(v_1, \rho(g_1 g_2)v')/2.\end{aligned}$$

Thus we have $\widetilde{u}' = \widehat{u}'$, and therefore the lemma follows. □

3. Torus bundles over Kuga fiber varieties

Let the semisimple Lie group G and the associated symmetric domain $\mathcal{D} = G/K$ as well as the group $G \cdot \mathbb{H}$ acting on $\mathcal{D} \times V \times U$ be as in Section 2. We extend this action to an action of $G \cdot \mathbb{H}$ on $\mathcal{D} \times V \times U_{\mathbb{C}}$, where $U_{\mathbb{C}} = U \otimes \mathbb{C}$ is the complexification of the real vector space U.

We take an arithmetic subgroup $L_{\mathbb{H}}$ of the Heisenberg group $\mathbb{H} = V \times U$, and set

$$L = p_V(L_{\mathbb{H}}), \quad L_U = p_U(L_{\mathbb{H}} \cap (\{0\} \times U)),$$

where $p_V : \mathbb{H} \to V$ and $p_U : \mathbb{H} \to U$ are the natural projection maps. Then L and L_U are lattices in V and U, respectively, and we have $L = L_{\mathbb{H}}/L_U$. We consider elements $(l, 0), (l', 0) \in L_{\mathbb{H}}$ with $l, l' \in L$. Then by (2.3) their product is given by

$$(l, 0) \cdot (l', 0) = (l + l', -A(l, l')/2) \in L_{\mathbb{H}}.$$

Since $(l + l', 0)^{-1} = (-l - l', 0) \in L_{\mathbb{H}}$, we have

$$(l + l', -A(l, l')/2) \cdot (-l - l', 0) = (0, -A(l, l')/2) \in L_{\mathbb{H}}.$$

Thus it follows that $A(L, L) \subset L_U$.

Let Γ be a torsion-free cocompact arithmetic subgroup of G, so that the corresponding locally symmetric space $\Gamma\backslash\mathcal{D}$ is a compact complex manifold. The space $\Gamma\backslash\mathcal{D}$ is also called an arithmetic variety when it is considered as a complex projective variety. Using the relation $L = L_{\mathbb{H}}/L_U$ and the fact that the image of the natural projection map $G \cdot \mathbb{H} = G \times V \times U \to G \times V$ is the semidirect product $G \ltimes V$, we obtain the identification

$$\Gamma \cdot L_{\mathbb{H}}/L_U = \Gamma \ltimes L;$$

hence we see that the action of $G \cdot \mathbb{H}$ on $\mathcal{D} \times V \times U_{\mathbb{C}}$ induces actions of the discrete groups $\Gamma \cdot L_{\mathbb{H}}$ and $\Gamma \ltimes L$ on the spaces $\mathcal{D} \times V \times U_{\mathbb{C}}$ and $\mathcal{D} \times V$, respectively. We denote the associated quotient spaces by

$$\mathcal{T} = \Gamma \cdot L_{\mathbb{H}}\backslash\mathcal{D} \times V \times U_{\mathbb{C}}, \quad Y = \Gamma \ltimes L\backslash\mathcal{D} \times V.$$

The natural projection map $\mathcal{D} \times V \to \mathcal{D}$ induces a surjective map $\pi : Y \to X$, which provides Y with a structure of a fiber bundle over X whose fiber is the quotient space V/L. In fact, by introducing an appropriate complex structure I_x on V for each $x \in X$ the fiber $\pi^{-1}(x)$ of Y over x can be equipped with a structure of a polarized abelian variety. Thus Y is a family of abelian varieties parametrized by the arithmetic variety X and can be embedded into a complex projective space. The resulting projective variety is known as a *Kuga fiber variety*, which has many interesting arithmetic as well as geometric properties (see e.g. [7], [10], [13]).

On the other hand, the surjective map $\pi' : \mathcal{T} \to Y$ induced by the natural projection map $\mathcal{D} \times V \times U \to \mathcal{D} \times U$ enables us to view $\mathcal{T}$ as a fiber bundle over the Kuga fiber variety Y whose fiber is isomorphic to the quotient space $U_{\mathbb{C}}/L_U$. By identifying $U_{\mathbb{C}}$ and L_U with $\mathbb{C}^d$ and $\mathbb{Z}^d$, respectively, we obtain the isomorphism

$$U_{\mathbb{C}}/L_U \cong (\mathbb{C}/\mathbb{Z})^d. \tag{3.1}$$

If $\mathbb{C}^\times$ denotes the set of nonzero complex numbers, the map $z \mapsto e^{2\pi i z}$ determines a group homomorphism $\mathbb{C} \to \mathbb{C}^\times$ from the additive group $\mathbb{C}$ onto the multiplicative group $\mathbb{C}^\times$ whose kernel is $\mathbb{Z}$. Hence we obtain the isomorphism $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$, and therefore (3.1) can be written as

$$U_{\mathbb{C}}/L_U \cong (\mathbb{C}^\times)^d. \tag{3.2}$$

Thus $\mathcal{T}$ can be regarded as a torus bundle over the Kuga fiber variety Y whose fiber is the complex torus $(\mathbb{C}^\times)^d$.

Remark 3.1. If U is as in Example 2.1, then the associated torus bundle $\mathcal{T}$ is essentially the same as the one considered by Satake in [14]. Note that our action of $G \cdot \mathbb{H}$ on $\mathcal{D} \times V \times U_{\mathbb{C}}$ is different from the one used by Satake. For example, from (2.2) we see that the operation on the semidirect product $G \ltimes V$ in our paper is given by

$$(g, v)(g', v') = (gg', v + \rho(g)v')$$

for $g, g' \in G$ and $v, v' \in V$. On the other hand, in order for the action of $G \times V \times U$ on $\mathcal{D} \times V \times U$ used by Satake in [14] to work, the semidirect product operation should be given by

$$(g, v)(g', v') = (gg', \rho(g)^{-1}v + v')$$

instead.

4. Spectral sequences

In this section we consider the theorem of Hochschild and Serre about spectral sequences associated to group extensions. We apply this theorem to two short exact sequences involving the discrete groups that are used in the construction of the torus bundle over a Kuga fiber variety described in Section 3.

Theorem 4.1. *Let Φ, $\mathfrak{G}$ and Δ be groups, and assume that there is a short exact sequence of the form*

$$1 \to \Phi \to \mathfrak{G} \to \Delta \to 1.$$

If $\mathcal{M}$ is a $\mathfrak{G}$-module, then there is a spectral sequence $\{E_r = \bigoplus_{p,q\geq 0} E_r^{p,q}\}_{r=2}^{\infty}$ that converges to the cohomology $H^*(\mathfrak{G}, \mathcal{M})$ such that

$$E_2^{p,q} = H^p(\Delta, H^q(\Phi, \mathcal{M}))$$

for all $p, q \geq 0$.

Proof. See [5, Proposition 7] or Theorem 10.1 in [8, Chapter XI]. □

Let the discrete groups L_U, $\Gamma \cdot L_{\mathbb{H}}$ and $\Gamma \ltimes L$ be as in Section 3. By applying the Hochschild-Serre theorem to the short exact sequence

$$1 \to L_U \to \Gamma \cdot L_{\mathbb{H}} \to \Gamma \ltimes L \to 1$$

and the trivial $(\Gamma \cdot L_{\mathbb{H}})$-module $\mathbb{C}$ we see that there is a spectral sequence $\{E_r\}_{r=2}^{\infty}$ such that

$$E_2^{p,q} = H^p(\Gamma \ltimes L, H^q(L_U, \mathbb{C})) \tag{4.1}$$

for all $p, q \geq 0$ and

$$H^r(\Gamma \cdot L_{\mathbb{H}}, \mathbb{C}) \cong \bigoplus_{p+q=r} E_{\infty}^{p,q} \tag{4.2}$$

for each $r \geq 2$.

Lemma 4.2. *For each nonnegative integer q the cohomology $H^q(L_U, \mathbb{C})$ of the group L_U with coefficients in the trivial L_U-module $\mathbb{C}$ can be written as*

$$H^q(L_U, \mathbb{C}) = \wedge^q(\mathbb{C}^d), \tag{4.3}$$

where $d = \dim U$.

Proof. Since $U_{\mathbb{C}}$ is contractible, the cohomology group $H^q(L_U, \mathbb{C})$ with respect to the trivial representation of L_U on $\mathbb{C}$ is isomorphic to the cohomology group $H^q(U_{\mathbb{C}}/L_U, \mathbb{C})$ of the quotient space $U_{\mathbb{C}}/L_U$. As is indicated in (3.2), the quotient $U_{\mathbb{C}}/L_U$ can be identified with the complex torus $(\mathbb{C}^{\times})^d$. Using this and the fact that $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ is a deformation retract of the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, the cohomology of $U_{\mathbb{C}}/L_U$ is the same as the cohomology of the torus $(S^1)^d$. Therefore we have

$$H^q(U_{\mathbb{C}}/L_U, \mathbb{C}) = H^q((S^1)^d, \mathbb{C}) = \wedge^q(\mathbb{C}^d)$$

(see e.g. [6, p. 180]); hence the lemma follows. □

Using (4.3), we see that the spectral sequence $\{E_r\}_{r=2}^{\infty}$ satisfying (4.1) and (4.2) has the E_2-term

$$E_2^{p,q} = H^p(\Gamma \ltimes L, H^q(L_U, \mathbb{C})) = H^p(\Gamma \ltimes L, \wedge^q(\mathbb{C}^d)) \tag{4.4}$$

for $p, q \geq 0$. Here the action of $\Gamma \ltimes L$ on $\wedge^q(\mathbb{C}^d)$ is induced by the action of $\Gamma \ltimes L = \Gamma \cdot L_{\mathbb{H}}/L_U$ on the normal subgroup L_U of $\Gamma \cdot L_{\mathbb{H}}$, which is given by conjugation. Thus the action of an element $(\gamma, \ell) \in \Gamma \ltimes L$ on L_U is given by

$$\begin{aligned} u^{(\gamma,\ell)} &= (\gamma, \ell, 0)^{-1}(1, 0, u)(\gamma, \ell, 0) = (\gamma^{-1}, -\rho(\gamma)\ell, 0)^{-1}(1, 0, u)(\gamma, \ell, 0) \\ &= (\gamma^{-1}, -\rho(\gamma)^{-1}\ell, u)(\gamma, \ell, 0) = (1, -\rho(\gamma)^{-1}\ell + \rho(\gamma^{-1})\ell, u) = (1, 0, u) = u \end{aligned}$$

for all $u \in L_U$, where we identified u and (γ, ℓ) with the elements $(1, 0, u)$ and $(\gamma, \ell, 0)$, respectively, of $\Gamma \cdot \mathbb{H}$. This means that the action of $\Gamma \ltimes L$ on L_U is trivial, and therefore $\wedge^q(\mathbb{C}^d)$ in (4.4) is a trivial $(\Gamma \ltimes L)$-module.

We now apply the Hochschild-Serre theorem to the short exact sequence

$$1 \to L \to \Gamma \ltimes L \to \Gamma \to 1$$

and the trivial $(\Gamma \ltimes L)$-module $\mathbb{C}$ to obtain a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = H^p(\Gamma, H^q(L, \mathbb{C})), \quad H^r(\Gamma \ltimes L, \mathbb{C}) \cong \bigoplus_{p+q=r} E_\infty^{p,q}. \tag{4.5}$$

Note that there is a canonical isomorphism

$$H^q(L, \mathbb{C}) = \wedge^q(L \otimes \mathbb{C})^*. \tag{4.6}$$

Indeed, under this isomorphism the q-cocycle $c[\phi]$ in $H^q(L, \mathbb{C})$ corresponding to an element $\phi \in \wedge^q(L \otimes \mathbb{C})^*$ is given by

$$\begin{aligned} c[\phi](\ell_0, \ell_1, \dots, \ell_q) &= \sum_{i=0}^{q} (-1)^i \phi(\ell_0 \wedge \dots \wedge \widehat{\ell_i} \wedge \dots \wedge \ell_q) \\ &= \phi((\ell_1 - \ell_0) \wedge (\ell_2 - \ell_0) \wedge \dots \wedge (\ell_q - \ell_0)) \end{aligned} \tag{4.7}$$

for all $\ell_0, \ell_1, \dots, \ell_q \in L \otimes \mathbb{C}$. Hence the E_2 term in the spectral sequence in (4.5) can be written as

$$E_2^{p,q} = H^p(\Gamma, \wedge^q(L \otimes \mathbb{C})^*),$$

where the action of $\Gamma = \Gamma \ltimes L/L$ on $\wedge^q(L \otimes \mathbb{C})^*$ is induced by the Γ-action on L by conjugation. Thus, the action of an element $\gamma \in \Gamma$ on L is given by

$$\ell^\gamma = (\gamma, 0)^{-1}(1, \ell)(\gamma, 0) = (\gamma^{-1}, 0)(\gamma, \ell) = (1, \rho(\gamma)^{-1}\ell) = \rho(\gamma)^{-1}\ell$$

for all $\ell \in L$, where we identified Γ and L with the subgroups $\Gamma \times \{0\}$ and $\{1\} \times L$, respectively, of $\Gamma \ltimes L$. Therefore the action of $\gamma \in \Gamma$ on the cocycle $c[\phi]$ in (4.7) is given by

$$\begin{aligned} (\gamma \cdot c[\phi])(\ell_0, \ell_1, \dots, \ell_q) &= c[\phi](\rho(\gamma)^{-1}\ell_0, \rho(\gamma)^{-1}\ell_1, \dots, \rho(\gamma)^{-1}\ell_q) \\ &= \phi(\rho(\gamma)^{-1}(\ell_1 - \ell_0) \wedge \rho(\gamma)^{-1}(\ell_2 - \ell_0) \wedge \dots \wedge \rho(\gamma)^{-1}(\ell_q - \ell_0)) \\ &= c[\wedge^q(\rho(\gamma))^*\phi](\ell_0, \ell_1, \dots, \ell_q); \end{aligned}$$

hence the action of γ on $\wedge^q(L \otimes \mathbb{C})^*$ is simply multiplication by the matrix

$$\wedge^q(\rho(\gamma))^* = {}^t\big(\wedge^q(\rho(\gamma)^{-1})\big), \tag{4.8}$$

where ${}^t(\cdot)$ denotes the transpose of the matrix $(\cdot)$.

5. Cohomology of torus bundles

Let $\mathcal{T} = \Gamma \cdot L_{\mathbb{H}} \backslash \mathcal{D} \times V \times U_{\mathbb{C}}$ be the torus bundle over the Kuga fiber variety $Y = \Gamma \ltimes L \backslash \mathcal{D} \times V$ considered in Section 4. In this section we express the cohomology of $\mathcal{T}$ in terms of cohomology of Y as well as in terms of cohomology of the arithmetic variety $X = \Gamma \backslash \mathcal{D}$.

Given a positive integer ν we define a map $\eta_\nu : \Gamma \cdot L_{\mathbb{H}} \to \Gamma \cdot L_{\mathbb{H}}$ by

$$\eta_\nu(\gamma, \ell, k) = (\gamma, \nu\ell, \nu^2 k)$$

for all $\gamma \in \Gamma$, $\ell \in L$ and $k \in L_U$.

Lemma 5.1. *The map $\eta_\nu : \Gamma \cdot L_{\mathbb{H}} \to \Gamma \cdot L_{\mathbb{H}}$ is a group homomorphism.*

Proof. Given elements (γ, ℓ, k) and (γ', ℓ', k') of $\Gamma \cdot L_{\mathbb{H}}$, by using (2.2) we obtain

$$\begin{aligned}
\big(\eta_\nu(\gamma, \ell, k)\big)\big(\eta_\nu(\gamma', \ell', k')\big) &= (\gamma, \nu\ell, \nu^2 k)(\gamma', \nu\ell', \nu^2 k') \\
&= \big(\gamma\gamma', \nu\ell + \rho(\gamma)(\nu\ell'), a^2 k + \nu^2 k' - A(\nu\ell, \rho(\gamma)(\nu\ell'))/2\big) \\
&= \big(\gamma\gamma', \nu(\ell + \rho(\gamma)\ell'), \nu^2(k + k' - A(\ell, \rho(\gamma)\ell')/2)\big) \\
&= \eta_\nu\big((\gamma, \ell, k)(\gamma', \ell', k')\big);
\end{aligned}$$

hence the lemma follows. □

Theorem 5.2. *For each nonnegative integer r the r-th complex cohomology group of $\mathcal{T}$ has the decomposition*

$$H^r(\mathcal{T}, \mathbb{C}) = \bigoplus_{p+q=r} H^p(Y, \mathbb{C}) \otimes \wedge^q(\mathbb{C}^d) \tag{5.1}$$

in terms of cohomology of the Kuga fiber variety Y.

Proof. We shall show first that the spectral sequence $\{E_r\}$ satisfying (4.1) and (4.2) degenerates at E_2. The homomorphism $\eta_\nu : \Gamma \cdot L_{\mathbb{H}} \to \Gamma \cdot L_{\mathbb{H}}$ in Lemma 5.1 induces the map

$$\eta_{\nu,r} : H^r(\Gamma \cdot L_{\mathbb{H}}, \mathbb{C}) \to H^r(\Gamma \cdot L_{\mathbb{H}}, \mathbb{C})$$

and $\eta^{p,q}_{\nu,r} : E^{p,q}_r \to E^{p,q}_r$ for each $r \geq 0$ such that

$$d_r \circ \eta^{p,q}_{\nu,r} = \eta^{p+r,q-r+1}_{\nu,r} \circ d_r,$$

where $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ is the boundary map. Since $E^{p,q}_2 = H^p(\Gamma \ltimes L, \wedge^q(\mathbb{C}^d))$ and the restriction $\eta_\nu\big|_{L_U}$ of η_ν maps $m \in L_U$ to $\nu^2 m$, the map $\eta^{p,q}_{\nu,2}$ can be considered as the multiplication by ν^{2q}. Hence we have

$$d_2(\nu^{2q} v) = \nu^{2q-2}(d_2(v))$$

for all $v \in E^{p,q}_2$. Since ν is an arbitrary positive integer, it follows that $d_2 = 0$. Similarly, we obtain $d_s = 0$ for all $s \geq 2$. Hence $\{E_r\}$ degenerates at E_2. Thus we obtain

$$H^r(\Gamma \cdot L_{\mathbb{H}}, \mathbb{C}) = \bigoplus_{p+q=r} E^{p,q}_\infty = \bigoplus_{p+q=r} E^{p,q}_2 = \bigoplus_{p+q=r} H^p(\Gamma \ltimes L, \wedge^q(\mathbb{C}^d)).$$

Since the space $\mathcal{D} \times V \times U$ is contractible and the representation of $\Gamma \cdot L_{\mathbb{H}}$ on $\mathcal{D} \times V \times U$ is trivial, we see that there is a canonical isomorphism

$$H^r(\Gamma \cdot L_{\mathbb{H}}, \mathbb{C}) = H^r(\mathcal{T}, \mathbb{C}).$$

On the other hand, as was mentioned in Section 4, the representation of $\Gamma \ltimes L$ on $\mathcal{D} \times V$ is also trivial; hence by the universal coefficient theorem we have

$$H^p(\Gamma \ltimes L, \wedge^q(\mathbb{C}^d)) = H^p(\Gamma \ltimes L, \mathbb{C}) \otimes \wedge^q(\mathbb{C}^d).$$

Using this and the fact that $\mathcal{D} \times V$ is contractible, we obtain

$$H^p(\Gamma \ltimes L, \wedge^q(\mathbb{C}^d)) = H^p(Y, \mathbb{C}) \otimes \wedge^q(\mathbb{C}^d),$$

and therefore the theorem follows. □

Remark 5.3. A result similar to Theorem 5.2 was obtained in [9] for a circle bundle over a Kuga fiber variety, or a twisted torus bundle over an arithmetic variety, associated to a group G that is not necessarily a subgroup of a symplectic group.

We now define an action of Γ on the space $\mathcal{D} \times \wedge^q(\mathbb{C}^{2n})^*$ by

$$\gamma \cdot (z, \phi) = (\gamma z, {}^t \wedge^q (\rho(\gamma)^{-1})\phi)$$

for all $\gamma \in \Gamma$, $z \in \mathcal{D}$ and $\phi \in \wedge^q(\mathbb{C}^{2n})^*$, and denote the associated quotient space by

$$L_q = \Gamma \backslash \mathcal{D} \times \wedge^q(\mathbb{C}^{2n})^*.$$

Then the natural projection map $\mathcal{D} \times \wedge^q(\mathbb{C}^{2n})^* \to \mathcal{D}$ induces the map $\varpi_q : L_q \to X = \Gamma \backslash \mathcal{D}$ which provides L_q a structure of a vector bundle over X with fiber $\wedge^q(\mathbb{C}^{2n})^*$. We denote by $\mathcal{L}_q$ the sheaf of sections of the vector bundle L_q over X.

Theorem 5.4. *For each nonnegative integer r the r-th complex cohomology group of $\mathcal{T}$ has the decomposition*

$$H^r(\mathcal{T}, \mathbb{C}) = \bigoplus_{j+k+\ell=r} H^j(X, \mathcal{L}_k) \otimes \wedge^\ell(\mathbb{C}^d)$$

in terms of cohomology of the arithmetic variety X.

Proof. Let $\{E_r\}$ be the spectral sequence satisfying (4.5). It is known that this spectral sequence degenerates at E_2 (see [7, Theorem II.3.12]); hence for each $\ell \geq 0$ we have a decomposition of the form

$$H^\ell(\Gamma \ltimes L, \mathbb{C}) = \bigoplus_{j+k=\ell} H^j(\Gamma, H^k(L, \mathbb{C})) = \bigoplus_{j+k=\ell} H^j(\Gamma, \wedge^k(L \otimes \mathbb{C})^*),$$

where we used (4.6). Since $\mathcal{D}$ is contractible and the action of an element $\gamma \in \Gamma$ on $\wedge^q(L \otimes \mathbb{C})^*$ is multiplication by the matrix in (4.8), we obtain the canonical isomorphism

$$H^j(\Gamma, \wedge^k(L \otimes \mathbb{C})^*) = H^j(X, \mathcal{L}_k)$$

for each $j \geq 0$. Using this and the identification $H^\ell(\Gamma \ltimes L, \mathbb{C}) = H^\ell(Y, \mathbb{C})$, we obtain

$$H^\ell(Y, \mathbb{C}) = \bigoplus_{j+k=\ell} H^j(\Gamma, \mathcal{L}_k).$$

Now the theorem follows by combining this with (5.1). □

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