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Covering the Unit Cube by Equal Balls

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Abstract. We give the minimal radius of 8 congruent balls, which cover the 4-dimensional unit cube.

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1. Introduction

We can read by Brass, Moser, Pach [3] about the problem of covering the d-dimensional unit cube by n equal minimal balls. In [3], [1] one can find numerous results in the case $d = 2$. G. Kuperberg and W. Kuperberg [4] found the optimal solution in $d = 3, n = 2, 3, 4, 8$. In higher-dimensions G. Kuperberg and W. Kuperberg [4] found the case $d \geq 4, n = 4$.

One can read results about covering by n equal balls a d-dimensional larger ball (Rogers [5], Verger-Gaugray [6]), and the d-dimensional crosspolytope (Böröczky, Jr., Fábián, Wintsche [2]).

2. Notations

Let $\mathbb{E}^d$ be the d-dimensional Euclidean space. Let $C^d := [0, 1]^d$ be the d-dimensional unit cube. Let $B^d(a, r)$ be the d-dimensional ball with centre a and radius r. $d(p, q)$ denotes the distance of the points p, q. Let $R_{a,b}$ be the ray with endpoint a and containing b. $poq\angle$ denotes the convex angle determined by the three points p, o, q in this order. Let $L_{a,b}$ be the straight line containing the different points a, b. $(E, F)\angle$ denotes the angle determined by the two rays E, F. Let $H(L, p)$ be the closed half plane bounded by the line L and containing the point p.

Let

$$o_1\left(\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{2},\tfrac{1}{6}\right),\ o_2\left(\tfrac{1}{6},\tfrac{1}{2},\tfrac{5}{6},\tfrac{1}{6}\right),\ o_3\left(\tfrac{1}{2},\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{6}\right),\ o_4\left(\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{2}\right),\ o_5\left(\tfrac{5}{6},\tfrac{5}{6},\tfrac{5}{6},\tfrac{1}{2}\right),$$
$$o_6\left(\tfrac{5}{6},\tfrac{1}{2},\tfrac{1}{6},\tfrac{5}{6}\right),\ o_7\left(\tfrac{1}{2},\tfrac{1}{6},\tfrac{5}{6},\tfrac{5}{6}\right),\ o_8\left(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2},\tfrac{5}{6}\right),\ m\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right),\ o_1^3\left(\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{2},0\right),$$
$$o_2^3\left(\tfrac{1}{6},\tfrac{1}{2},\tfrac{5}{6},0\right),\ o_3^3\left(\tfrac{1}{2},\tfrac{5}{6},\tfrac{1}{6},0\right),\ o_4^3\left(\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},0\right),\ o_5^3\left(\tfrac{5}{6},\tfrac{5}{6},\tfrac{5}{6},0\right).$$

Theorem. *The minimal radius of* 8 *congruent balls, which cover the* 4*-dimensional unit cube, is* $\sqrt{\frac{5}{12}}$.

3. Lemmas

Lemma 1. *The balls* $\mathbf{B}^4\left(o_i,\sqrt{\frac{5}{12}}\right)\subset\mathbb{E}^4$ $(i=1,2,\dots,8)$ *cover the cube* C^4.

Proof. Let $\mathbf{B}_i^4:=\mathbf{B}^4\left(o_i,\sqrt{\frac{5}{12}}\right)$ for $i=1,2,\dots,8$. Since $d(m,o_i)=\sqrt{\frac{1}{3}}<\sqrt{\frac{5}{12}}$ thus the center m of C^4 lies in $\mathbf{B}_i^4$ for $i=1,2,\dots,8$.

We will show that the balls $\mathbf{B}_i^4$ for $i=1,2,\dots,8$ cover every 3-dimensional face of the cube C^4. From this comes that $\mathbf{B}_i^4$ for $i=1,2,\dots,8$ cover C^4. (See Figure 1. The thick edges are the edges, which lie entirely in a ball $\mathbf{B}_i^4$.)

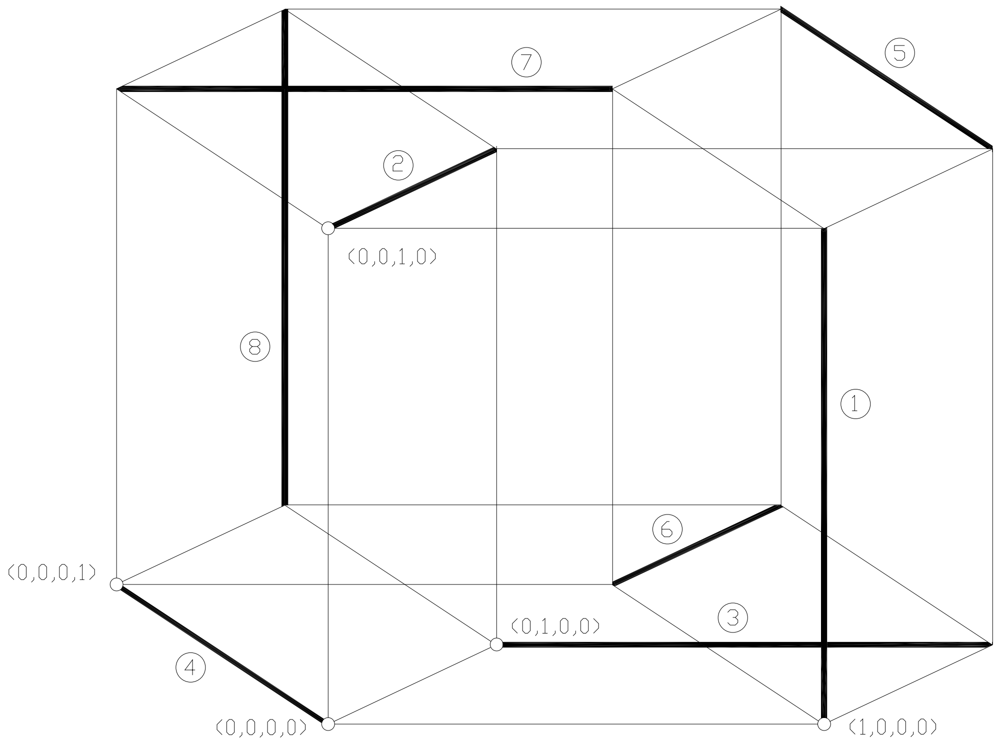

Figure 1. The four circle problem

We show that the balls $\mathbf{B}_i^4$ for $i=1,2,3,4,5$ cover the cube $[0,1]^3\times\{0\}$ (the cover of the other 3-dimensional faces of C^4 is similar).

The intersection of the 4-dimensional balls $\mathbf{B}_i^4$ for $i = 1, 2, 3, 4, 5$ and the hyperplane $x_4 = 0$ are the 3-dimensional balls $\mathbf{B}_i^3 := \mathbf{B}^3\left(o_i^3, \sqrt{\frac{7}{18}}\right)$ for $i = 1, 2, 3$ and $\mathbf{B}_i^3 := \mathbf{B}^3\left(o_i^3, \frac{1}{\sqrt{6}}\right)$ for $i = 4, 5$, resp.

Firstly we show that the cube $C_0^3 := [0, \frac{1}{2}]^3 \times \{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_i^3$ for $i = 1, 2, 3, 4$ (see Figure 2).

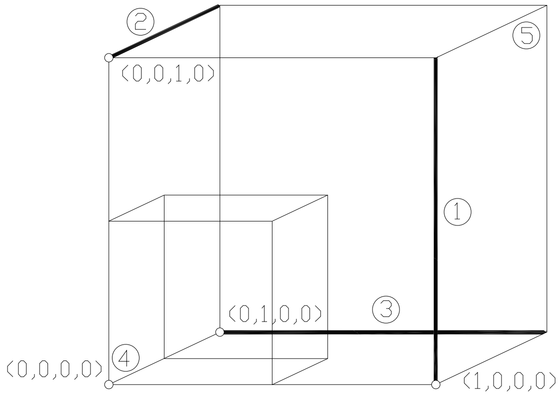

Figure 2.

Let $p(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. Since $d(p, o_i^3) = \sqrt{\frac{11}{36}} < \sqrt{\frac{7}{18}}$ for $i = 1, 2, 3$ and $d(p, o_4^3) = \frac{1}{\sqrt{12}} < \frac{1}{\sqrt{6}}$ thus $p \in \mathbf{B}_i^3$ for $i = 1, 2, 3, 4$. If we show that the 2-dimensional faces of the cube C_0^3 is covered by the balls $\mathbf{B}_i^3$ for $i = 1, 2, 3, 4$ then we get that the cube C_0^3 is covered by the balls $\mathbf{B}_i^3$ for $i = 1, 2, 3, 4$.

Let us see the 2-dimensional face with vertices $(0, 0, 0, 0), (\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0, 0)$. Since the region $\operatorname{conv}((0, 0, 0, 0), (\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, \frac{1}{3}, 0, 0), (\frac{1}{3}, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0, 0))$ is covered by $\mathbf{B}_4^3$, and the region $\operatorname{conv}\left((\frac{1}{2}, \frac{1}{3}, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0, 0)\right)$ is covered by $\mathbf{B}_3^3$ thus the above 2-dimensional face is covered by the balls $\mathbf{B}_3^3, \mathbf{B}_4^3$. Similarly the face with vertices $(0, 0, 0, 0), (\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, 0, \frac{1}{2}, 0), (0, 0, \frac{1}{2}, 0)$ is covered by the balls $\mathbf{B}_1^3, \mathbf{B}_4^3$, and the face with vertices $(0, 0, 0, 0), (0, \frac{1}{2}, 0, 0), (0, \frac{1}{2}, \frac{1}{2}, 0), (0, 0, \frac{1}{2}, 0)$ is covered by the balls $\mathbf{B}_2^3, \mathbf{B}_4^3$.

Let us consider the face with vertices $(\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}, 0)$. Since the region $\operatorname{conv}\left((\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, 0, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, 0), (\frac{1}{2}, \frac{1}{3}, 0, 0)\right)$ is covered by $\mathbf{B}_1^3$, and the region $\operatorname{conv}\left((\frac{1}{2}, \frac{1}{3}, 0, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0)\right)$ is covered by $\mathbf{B}_3^3$ thus the above face is covered by the balls $\mathbf{B}_1^3, \mathbf{B}_3^3$. Similarly the face with vertices $(0, 0, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}, 0)$ is covered by the balls $\mathbf{B}_1^3, \mathbf{B}_2^3$, and the face with vertices $(0, \frac{1}{2}, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}, 0)$ is covered by the balls $\mathbf{B}_2^3, \mathbf{B}_3^3$. Similarly the cube $[\frac{1}{2}, 1]^3 \times \{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3, \mathbf{B}_2^3, \mathbf{B}_3^3, \mathbf{B}_5^3$.

Secondly we show that the cube $[\frac{1}{2}, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times \{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3, \mathbf{B}_3^3$.

Since the region $\mathrm{conv}((1,0,0,0),(1,\frac{1}{2},0,0),(\frac{2}{3},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{3},0,0),(\frac{1}{2},0,0,0),$ $(\frac{1}{2},\frac{1}{2},\frac{1}{3},0),(\frac{1}{2},\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},0,\frac{1}{2},0),(1,0,\frac{1}{2},0),(1,\frac{1}{2},\frac{1}{2},0))$ is covered by $\mathbf{B}_1^3$, and the region $\mathrm{conv}\left((\frac{1}{2},\frac{1}{3},0,0),(1,\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)\right)$ is covered by $\mathbf{B}_3^3$ thus the above cube is covered. Similarly the cube $[\frac{1}{2},1]^2\times[0,\frac{1}{2}]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3,\mathbf{B}_3^3$, and the cube $[0,\frac{1}{2}]\times[\frac{1}{2},1]\times[0,\frac{1}{2}]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_2^3,\mathbf{B}_3^3$, and the cube $[0,\frac{1}{2}]\times[\frac{1}{2},1]^2\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_2^3,\mathbf{B}_3^3$, and the cube $[0,\frac{1}{2}]^2\times[\frac{1}{2},1]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3,\mathbf{B}_2^3$, and the cube $[\frac{1}{2},1]\times[0,\frac{1}{2}]\times[\frac{1}{2},1]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3,\mathbf{B}_2^3$. This implies that the cube $[0,1]^3\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_i^3$ for $i=1,2,3,4,5$, that is, $[0,1]^3\times\{0\}$ is covered by the 4-dimensional balls $\mathbf{B}_i^4$ for $i=1,2,3,4,5$.

Similarly the cube $[0,1]^3\times\{1\}$ is covered by the 4-dimensional balls $\mathbf{B}_4^4,\mathbf{B}_5^4,\mathbf{B}_6^4,$ $\mathbf{B}_7^4,\mathbf{B}_8^4$, and the cube $\{0\}\times[0,1]^3$ is covered by the 4-dimensional balls $\mathbf{B}_2^4,\mathbf{B}_3^4,\mathbf{B}_4^4,$ $\mathbf{B}_7^4,\mathbf{B}_8^4$, and the cube $\{1\}\times[0,1]^3$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_3^4,\mathbf{B}_5^4,$ $\mathbf{B}_6^4,\mathbf{B}_7^4$, and the cube $[0,1]\times\{0\}\times[0,1]^2$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_2^4,\mathbf{B}_4^4,\mathbf{B}_6^4,\mathbf{B}_7^4$, and the cube $[0,1]\times\{1\}\times[0,1]^2$ is covered by the 4-dimensional balls $\mathbf{B}_2^4,\mathbf{B}_3^4,\mathbf{B}_5^4,\mathbf{B}_6^4,\mathbf{B}_8^4$, and the cube $[0,1]^2\times\{0\}\times[0,1]$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_3^4,\mathbf{B}_4^4,\mathbf{B}_6^4,\mathbf{B}_8^4$, and the cube $[0,1]^2\times\{1\}\times[0,1]$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_2^4,\mathbf{B}_5^4,\mathbf{B}_7^4,\mathbf{B}_8^4$. Then the 3-dimensional faces of the cube C^4 are covered by the balls $\mathbf{B}_i^4$ for $i=1,2,\dots,8$, that is, the cube C^4 is covered by the balls $\mathbf{B}_i^4$ for $i=1,2,\dots,8$. □

Lemma 2. *Let $a_1,a_2\in\mathbf{B}^2(o,r)\subset\mathbb{E}^2$, $\left(\frac{1}{2}<r\right)$, $d(a_1,a_2)=1$. Let R_{a_1,b_1},R_{a_2,b_2} be two rays perpendicular to L_{a_1,a_2}. If $d(o,L_{a_1,a_2}),r$ are fixed numbers then*

$$\mathrm{diam}\left(R_{a_1,b_1}\cap\mathbf{B}^2(o,r)\right)+\mathrm{diam}\left(R_{a_2,b_2}\cap\mathbf{B}^2(o,r)\right)$$

is the greatest if $d(o,a_1)=d(o,b_1)$ and R_{a_1,b_1},R_{a_2,b_2} lie in a closed half plane bounded by L_{a_1,a_2} and containing o.

Proof. Let h_i be the point on the line L_{a_i,b_i} and not contained R_{a_i,b_i} for $i=1,2$. If L_{a_1,a_2} does not contain o and $R_{a_1,b_1}\not\subset H(L_{a_1,a_2},o)$ then $\mathrm{diam}\,(R_{a_1,b_1}\cap\mathbf{B}^2(o,r))<$ $\mathrm{diam}\,(R_{a_1,h_1}\cap\mathbf{B}^2(o,r))$. In this case we change R_{a_1,b_1} for R_{a_1,h_1} and we mark R_{a_1,h_1} with R_{a_1,b_1}. Similarly if L_{a_1,a_2} does not contain o and $R_{a_2,b_2}\not\subset H(L_{a_1,a_2},o)$ then we change R_{a_2,b_2} for R_{a_2,h_2} and we mark R_{a_2,h_2} with R_{a_2,b_2}. If L_{a_1,a_2} contains o and $H(L_{a_1,a_2},b_1)$ does not contain b_2 then we change R_{a_2,b_2} for R_{a_2,h_2} and we mark R_{a_2,h_2} with R_{a_2,b_2}. With these changes $\mathrm{diam}\,(R_{a_1,b_1}\cap\mathbf{B}^2(o,r))+$ $\mathrm{diam}\,(R_{a_2,b_2}\cap\mathbf{B}^2(o,r))$ does not decrease.

Let e be the straight line containing o and parallel L_{a_1,a_2} (Figure 3). Let c_1,c_2 be the intersection point of e and R_{a_1,b_1},R_{a_2,b_2}, resp. If o does not lie on the segment c_1c_2 and, say, $d(o,c_1)>d(o,c_2)$ then let $R_{a_1',b_1'}$ be the image of R_{a_1,b_1} under the reflection with respect to the line L_{a_2,b_2}. In this case $\mathrm{diam}\,(R_{a_1,b_1}\cap\mathbf{B}^2(o,r))+$ $\mathrm{diam}\,(R_{a_2,b_2}\cap\mathbf{B}^2(o,r))<\mathrm{diam}\left(R_{a_1',b_1'}\cap\mathbf{B}^2(o,r)\right)+\mathrm{diam}\,(R_{a_2,b_2}\cap\mathbf{B}^2(o,r))$. We use this method until o lies between the images of the rays. Thus, we can assume that o lies on the segment c_1c_2. Let d_1,d_2 be the intersection point of $\mathrm{bd}\,\mathbf{B}^2(o,r)$

and R_{a_1,b_1}, R_{a_2,b_2}, resp. Let $x := d(o, c_1)$ and

$$f(x) := \operatorname{diam}\left(R_{a_1,b_1} \cap \mathbf{B}^2(o,r)\right) + \operatorname{diam}\left(R_{a_2,b_2} \cap \mathbf{B}^2(o,r)\right) =$$

$$= d(a_1, d_1) + d(a_2, d_2) = 2d(o, L_{a_1,a_2}) + d(c_1, d_1) + d(c_2, d_2) =$$

$$= 2d(o, L_{a_1,a_2}) + \sqrt{r^2 - x^2} + \sqrt{r^2 - (1-x)^2}.$$

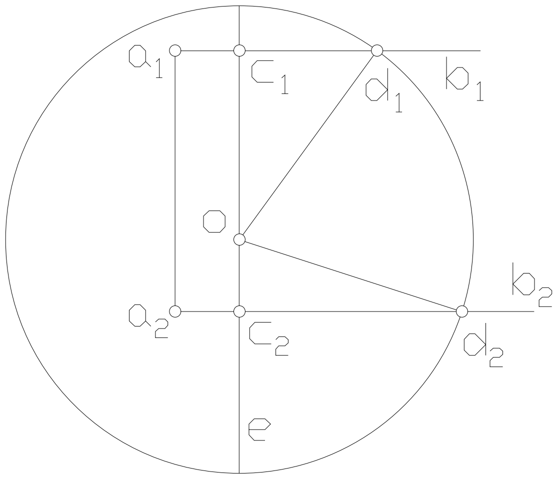

Figure 3.

By elementary calculus, the maximum value of $f(x)$ between 0 and 1 is achieved at $\frac{1}{2}$. This completes the proof of the lemma. □

Lemma 3. *Let* $R_{a,b_1}, R_{a,b_2}, R_{a,b_3} \subset \mathbb{E}^3$, $b_1ab_2\angle = \frac{\pi}{2}, b_1ab_3\angle = \frac{\pi}{2}, b_2ab_3\angle = \frac{\pi}{2}$ *and* $\mathbf{B}^3(o,r) \subset \mathbb{E}^3$. *Then*

$$\sum_{i=1,2,3} \operatorname{diam}\left(R_{a,b_i} \cap \mathbf{B}^3(o,r)\right) \le r\frac{3\sqrt{6}}{2}.$$

Proof. Let $c_i := R_{a,b_i} \cap \mathbf{B}^3(o,r)$ for $i = 1,2,3$. Of course,

$$d(a,c_1) + d(a,c_2) \le d(c_1,c_2)\sqrt{2},$$

$$d(a,c_2) + d(a,c_3) \le d(c_2,c_3)\sqrt{2},$$

$$d(a,c_1) + d(a,c_3) \le d(c_1,c_3)\sqrt{2}.$$

Thus

$$d(a,c_1) + d(a,c_2) + d(a,c_3) \le \frac{\sqrt{2}}{2}\left(d(c_1,c_2) + d(c_2,c_3) + d(c_1,c_3)\right).$$

Since

$$d(c_1,c_2) + d(c_2,c_3) + d(c_1,c_3) \le r3\sqrt{3}$$

thus

$$\sum_{i=1,2,3} \operatorname{diam}\left(R_{a,b_i} \cap \mathbf{B}^3(o,r)\right) = \sum_{i=1,2,3} d(a,c_i) \leq r\frac{3\sqrt{6}}{2}.$$

This completes the proof of the lemma. □

Note that if $R_{a,b_1} \cap \mathbf{B}^3(o,r) + R_{a,b_2} \cap \mathbf{B}^3(o,r) + R_{a,b_3} \cap \mathbf{B}^3(o,r) = r\frac{3\sqrt{6}}{2}$ then $a \in \operatorname{int} \mathbf{B}^3(o,r)$.

Lemma 4. *Let* $a_1, a_2 \in \mathbf{B}^4(o,r) \subset \mathbb{E}^4$, $\left(\frac{1}{2} < r < \sqrt{\frac{5}{12}}\right)$ *and* $d(a_1,a_2) = 1$. *Let* $R_{a_j,b_i^j}(i = 1,2,3; j = 1,2)$ *such rays that* $a_2 a_1 b_i^1 \angle = \frac{\pi}{2}, b_i^j a_j b_k^j \angle = \frac{\pi}{2}, b_i^1 \parallel b_i^2$ *and in the plane determined by the points* a_1, a_2, b_i^1 *the half plane* $H(L_{a_1,a_2}, b_i^1)$ *contains the point* b_i^2 *for any* $i,k \in \{1,2,3\} (i \neq k), j = 1,2$. *Then*

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4(o,r)\right) < 4.$$

Proof. The value of

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4(o,r)\right)$$

is smaller than

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4\left(o, \sqrt{\frac{5}{12}}\right)\right).$$

Let H be the hyper plane perpendicular to the segment $a_1 a_2$ containing o. If the projection of the rays $R_{a_j,b_i^j}(i = 1,2,3; j = 1,2)$ onto the hyper plane H is fixed then by Lemma 2

$$\sum_{i=1,2,3} \left(\operatorname{diam}\left(R_{a_1,b_i^1} \cap \mathbf{B}^4\left(o, \sqrt{\frac{5}{12}}\right)\right) + \operatorname{diam}\left(R_{a_2,b_i^2} \cap \mathbf{B}^4\left(o, \sqrt{\frac{5}{12}}\right)\right)\right)$$

is the greatest if $d(o,a_1) = d(o,a_2)$. Thus we can assume $d(o,a_1) = d(o,a_2)$.
Let H_1, H_2 be the hyper planes perpendicular to the segment $a_1 a_2$ containing a_1, a_2, resp. In this case $\operatorname{diam}\left(H_1 \cap \mathbf{B}^4\left(o, \sqrt{\frac{5}{12}}\right)\right) = \operatorname{diam}\left(H_2 \cap \mathbf{B}^4\left(o, \sqrt{\frac{5}{12}}\right)\right) = \frac{2}{\sqrt{6}}$.
By Lemma 3

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4\left(o, \sqrt{\frac{5}{12}}\right)\right) \leq 1 + 2\left(\frac{1}{\sqrt{6}} \frac{3\sqrt{6}}{2}\right) =$$

$$= 1 + 3 = 4.$$

This completes the proof of the lemma. □

4. Proof of the theorem

Theorem. *The minimal radius of* 8 *congruent balls, which cover the* 4*-dimensional unit cube, is* $\sqrt{\frac{5}{12}}$.

Proof. By Lemma 1 we have that 8 congruent balls with radius $\sqrt{\frac{5}{12}}$ can cover the cube C^4.

Let us assume that the 4-dimensional cube C^4 can cover 8 balls with radius $\frac{1}{2} < r < \sqrt{\frac{5}{12}}$ (of course, 8 balls with radius at most $\frac{1}{2}$ can not cover C^4).

Since in a ball with radius at most $\sqrt{\frac{5}{12}} < \frac{\sqrt{2}}{2}$ can not lie three vertices of C^4 thus in every ball lie exactly two vertices of C^4. By Lemma 4 the sum of the length of the edges of C^4 in a ball with radius $\frac{1}{2} < r < \sqrt{\frac{5}{12}}$ is smaller than 4, that is, 8 congruent balls with radius smaller than $\sqrt{\frac{5}{12}}$ can not cover the cube C^4 (the sum of the length of the edges of C^4 is 32); a contradiction. This completes the proof of the Theorem. □

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