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Variations of (para-)Hodge Structures and their Period Maps in tt^*-geometry

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Abstract. We introduce the notion of variations of Hodge structures (VHS) in para-complex geometry and define the associated period map. Moreover, we construct VHS from special (para-)complex and (para-) Kähler manifolds and prove that they provide solutions of (metric) tt^*-bundles (cf. [3] for the complex case). In the case of odd weight we relate the period map to the (para-)pluriharmonic maps associated to tt^*-bundles (cf. [18], [19]).

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1. Introduction

In complex geometry it is known that (metric) tt^*-bundles provide a generalization of variations Hodge structures (cf. [3]). Moreover one [18, D] can associate to any metric tt^*-bundle (E, D, S, g) a pluriharmonic map into $GL(r, \mathbb{R})/O(p, q)$ where (p, q) with $r = p + q$ is the signature of the metric g. In this paper we relate for a variation of Hodge structures of odd weight this pluriharmonic map to the period map of the variation of Hodge structures.

More recently the author [19] introduced the para-complex notion of tt^*-bundles. Examples of such structures on the tangent bundle of a special para-Kähler manifold were given in the same reference. In the complex setting special Kähler manifolds carry a polarized variation of Hodge structures of weight one.

This is one way to see that they provide tt^*-structures. The described information suggest to study the question if one can generalize VHS and their period maps to para-complex geometry, if the tangent bundle of special para-Kähler manifold carries such VHS, if these VHS provide para-tt^*-bundles and if one can identify the related para-pluriharmonic maps. This program is carried out in this paper.

2. Para-complex differential geometry

We shortly recall some notions and facts of para-complex differential geometry. For a more complete source we refer to [7].

In para-complex geometry one replaces the complex structure J with $J^2 = -\mathbb{1}$ (on a finite dimensional vector space V) by the para-complex structure $\tau \in \mathrm{End}\,(V)$ satisfying $\tau^2 = \mathbb{1}$ and one requires that the ± 1-eigenspaces have the same dimension. An *almost para-complex structure* on a smooth manifold M is an endomorphism-field τ, which is a point-wise para-complex structure. If the eigen-distributions $T^{\pm}M$ are integrable τ is called *para-complex structure on M* and M is called a *para-complex manifold*. As in the complex case, there exists a tensor, also called *Nijenhuis tensor*, which is the obstruction to the integrability of the para-complex structure.

The real algebra, which is generated by 1 and by the *para-complex unit* e with $e^2 = 1$, is called the *para-complex numbers* and denoted by C. For all $z = x + ey \in C$ with $x, y \in \mathbb{R}$ we define the *para-complex conjugation* as $\bar{\cdot} : C \to C, x + ey \mapsto x - ey$ and the *real* and *imaginary parts* of z by $\Re(z) := x, \Im(z) := y$. The free C-module C^n is a para-complex vector space where its para-complex structure is just the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \to C^n,\ v \mapsto \bar{v}$.

Note, that $z\bar{z} = x^2 - y^2$. Therefore the algebra C is sometimes called the *hypercomplex* numbers. The circle $\mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \,|\, x^2 + y^2 = 1\}$ is replaced by the four hyperbola $\{z = x + ey \in C \,|\, x^2 - y^2 = \pm 1\}$. We define $\tilde{\mathbb{S}}^1$ to be the hyperbola given by the one parameter group $\{z(\theta) = \cosh(\theta) + e\sinh(\theta) \,|\, \theta \in \mathbb{R}\}$.

A para-complex vector space (V, τ) endowed with a pseudo-Euclidean metric g is called *para-hermitian vector space*, if g is τ-anti-invariant, i.e. $\tau^* g = -g$. The *para-unitary group of V* is defined as the group of automorphisms

$$U^\pi(V) := \mathrm{Aut}(V, \tau, g) := \{L \in GL(V) | [L, \tau] = 0 \text{ and } L^* g = g\}$$

and its Lie-algebra is denoted by $\mathfrak{u}^\pi(V)$. For $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ the *standard para-hermitian structure* is defined by the above para-complex structure and the metric $g = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$ (cf. Example 7 of [7]). The corresponding para-unitary group is given by (cf. Proposition 4 of [7]):

$$U^\pi(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \,|\, A, B \in \mathrm{End}(\mathbb{R}^n),\, A^T A - B^T B = \mathbb{1}_n,\ A^T B - B^T A = 0 \right\}. \tag{2.1}$$

There exist two bi-gradings on the exterior algebra: The one is induced by the splitting in $T^{\pm}M$ and denoted by $\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M$ and induces an obvious bi-grading on exterior forms with values in a vector bundle E. The second is induced by the decomposition of the *para-complexified* tangent bundle $TM^C = TM \otimes_{\mathbb{R}} C$ into the subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ which are defined as the $\pm e$-eigenbundles of the para-complex linear extension of τ. This induces a bi-grading on the C-valued exterior forms noted $\Lambda^k T^*M^C = \bigoplus_{k=p+q} \Lambda^{p,q}\, T^*M$ and finally on the C-valued differential forms on M $\Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M)$. In the case $(1,1)$ and $(1+,1-)$ the two gradings induced by τ coincide, in the sense that $\Lambda^{1,1}\, T^*M = (\Lambda^{1+,1-}\, T^*M) \otimes C$. The bundles $\Lambda^{p,q}\, T^*M$ are para-complex vector bundles in the following sense: A *para-complex vector bundle* of rank r over a para-complex manifold (M,τ) is a smooth real vector bundle $\pi : E \to M$ of rank $2r$ endowed with a fiber-wise para-complex structure $\tau^E \in \Gamma(\mathrm{End}\,(E))$. We denote it by (E,τ^E). In the following text we always identify the fibers of a para-complex vector bundle E of rank r with the free C-module C^r. One has a notion of para-holomorphic vector bundles [16], too. In Proposition 2 of the same reference we have shown, that a para-complex connection with vanishing $(0,2)$-curvature on a para-complex vector bundle E induces a para-holomorphic structure on E. This generalizes a well-known theorem of complex geometry.

Let us transfer some notions of hermitian linear algebra (cf. [21]): A *para-hermitian sesquilinear scalar product* is a non-degenerate sesquilinear form $h : C^r \times C^r \to C$, i.e. it satisfies (i) h is non-degenerate: Given $w \in C^r$ such that for all $v \in C^r$ $h(v,w) = 0$, then it follows $w = 0$, (ii) $h(v,w) = \overline{h(w,v)}$, $\forall\, v,w \in C^r$, and (iii) $h(\lambda v, w) = \lambda h(v,w)$, $\forall\, \lambda \in C$; $v,w \in C^r$. The *standard para-hermitian sesquilinear scalar product* is given by

$$(z,w)_{C^r} := z \cdot \bar{w} = \sum_{i=1}^{r} z^i \bar{w}^i, \text{ for } z = (z^1,\ldots,z^r), w = (w^1,\ldots,w^r) \in C^r.$$

The *para-hermitian conjugation* is defined by $C \mapsto C^h = \bar{C}^t$ for $C \in \mathrm{End}\,(C^r) = \mathrm{End}\,_C(C^r)$ and C is called *para-hermitian* if and only if $C^h = C$. We denote by $\mathrm{herm}(C^r)$ the set of para-hermitian endomorphisms and by $\mathrm{Herm}(C^r) = \mathrm{herm}(C^r) \cap GL(r,C)$. We remark, that there is no notion of para-hermitian signature, since from $h(v,v) = -1$ for an element $v \in C^r$ we obtain $h(ev,ev) = 1$.

Proposition 1. *Given an element C of End(C^r) then it holds $(Cz,w)_{C^r} = (z, C^h w)_{C^r}$, $\forall z,w \in C^r$. The set herm(C^r) is a real vector space. There is a bijective correspondence between Herm(C^r) and para-hermitian sesquilinear scalar products h on C^r given by $H \mapsto h(\cdot,\cdot) := (H\cdot,\cdot)_{C^r}$.*

A *para-hermitian metric* h on a para-complex vector-bundle E over a para-complex manifold (M,τ) is a smooth fiber-wise para-hermitian sesquilinear scalar product.

To unify the complex and the para-complex case we introduce some notations: First we note J^ϵ where $J^{\epsilon^2} = \epsilon 1\!\!1$ with $\epsilon \in \{\pm 1\}$. The ϵcomplex unit is denoted by $\hat{i}$, i.e. $\hat{i} := e$, for $\epsilon = 1$, and $\hat{i} = i$, for $\epsilon = -1$. Further we introduce $\mathbb{C}_\epsilon$ with $\mathbb{C}_1 = C$ and $\mathbb{C}_{-1} = \mathbb{C}$. In the rest of this work we extend our language by the following ϵ-notation: If a word has a prefix ϵ with $\epsilon \in \{\pm 1\}$, i.e. is of the form ϵX, this expression is replaced by

$$\epsilon X := \begin{cases} \text{X, for } \epsilon = -1, \\ \text{para-X, for } \epsilon = 1. \end{cases}$$

The ϵunitary group and its Lie-algebra are

$$U^\epsilon(p,q) := \begin{cases} U^\pi(C^r), & \text{for } \epsilon = 1, \\ U(p,q), & \text{for } \epsilon = -1 \end{cases} \quad \text{and } \mathfrak{u}^\epsilon(p,q) := \begin{cases} \mathfrak{u}^\pi(C^r), & \text{for } \epsilon = 1, \\ \mathfrak{u}(p,q), & \text{for } \epsilon = -1, \end{cases}$$

where in the complex case (p,q) for $r = p+q$ is the hermitian signature.

Further we use the notation

$$\mathrm{Herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon) := \begin{cases} \mathrm{Herm}(C^r), & \text{for } \epsilon = 1, \\ \mathrm{Herm}_{p,q}(\mathbb{C}^r), & \text{for } \epsilon = -1, \end{cases}$$

$$\mathrm{herm}^\epsilon_{p,q}(\mathbb{C}^r_\epsilon) := \begin{cases} \mathrm{herm}(C^r), & \text{for } \epsilon = 1, \\ \mathrm{herm}_{p,q}(\mathbb{C}^r), & \text{for } \epsilon = -1, \end{cases}$$

where, for $p+q=r$, $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ are the hermitian matrices of hermitian signature (p,q) and $\mathrm{herm}_{p,q}(\mathbb{C}^r)$ are the hermitian matrices with respect to the standard hermitian product of hermitian signature (p,q) on $\mathbb{C}^r$. The standard ϵhermitian sesquilinear scalar product is $(z,w)_{\mathbb{C}^r_\epsilon} := z \cdot \bar{w} = \sum_{i=1}^r z^i \bar{w}^i$, for $z = (z^1,\dots,z^r), w = (w^1,\dots,w^r) \in \mathbb{C}^r_\epsilon$ and we note

$$\cos_\epsilon(x) := \begin{cases} \cos(x), \text{ for } \epsilon = -1, \\ \cosh(x), \text{ for } \epsilon = 1 \end{cases} \quad \text{and} \quad \sin_\epsilon(x) := \begin{cases} \sin(x), \text{ for } \epsilon = -1, \\ \sinh(x), \text{ for } \epsilon = 1. \end{cases}$$

3. Variations of ϵHodge structures

3.1. ϵHodge structures and their variations

In this section we introduce the notion of variations of ϵHodge structures in para-complex geometry and recall variations of Hodge structures which are classical objects in complex geometry. We follow the notations of [2] which is a reference and contains references for further study of variations of Hodge structures. The para-complex version seems to be new.

Definition 1.

(a) *A real ϵHodge structure of weight $w \in \mathbb{N}$ is a real vector space H on the ϵcomplexification of which there is a decomposition into ϵcomplex vector spaces*

$$H^{\mathbb{C}_\epsilon} = \bigoplus_{w=p+q} H^{p,q} \text{ with } p,q \in \mathbb{N} \tag{3.1}$$

and where

$$\overline{H^{p,q}} = H^{q,p} \text{ with } p,q \in \mathbb{N}. \tag{3.2}$$

The ϵcomplex conjugation $\bar{\cdot}$ is relative to the real structure on $H^{\mathbb{C}_\epsilon} = H \otimes \mathbb{C}_\epsilon$.

(b) *Suppose, that an ϵHodge structure of weight w carries a bilinear form $b : H \times H \to \mathbb{R}$ which satisfies the following Riemannian bilinear relations*

(i) *The $\mathbb{C}_\epsilon$-linear extension of the bilinear form b, also denoted by b, satisfies $b(x, y) = 0$ if $x \in H^{p,q}$ and $y \in H^{r,s}$ for $(r, s) \neq (w - p, w - q) = (q, p)$,*

(ii) *The bilinear form b defines an ϵhermitian sesquilinear scalar product (compare Section 2) on $H^{p,q}$ by $h(x, y) = (-1)^{w(w-1)/2}\hat{\imath}^{p-q} b(x, \bar{y})$.*

Then we call this ϵHodge structure weakly polarized.

(c) *Suppose, that a (complex) Hodge structure of weight w carries a bilinear form $b : H \times H \to \mathbb{R}$ which satisfies the first Riemannian bilinear relation* (i) *and in addition*

(ii) *The bilinear form b defines a positive definite hermitian sesquilinear form on $H^{p,q}$ by $h(x, y) = (-1)^{w(w-1)/2} i^{p-q} b(x, \bar{y})$.*

Then we call this Hodge structure strongly polarized.

(d) *An ϵHodge structure of weight w is called polarized if it is weakly polarized or strongly polarized.*

Closely related to the ϵHodge decomposition is the following filtration

$$F^p = \bigoplus_{a \geq p} H^{a,b},\ p = 0, \ldots, w, \tag{3.3}$$

which satisfies for an ϵHodge structure of weight w the relation

$$H^{\mathbb{C}_\epsilon} = F^p \oplus \overline{F^{w-p+1}},\ p = 1, \ldots, w. \tag{3.4}$$

Any filtration which obeys equation (3.4) is called an *ϵHodge filtration.* Such as an ϵHodge decomposition induces an ϵHodge filtration we obtain from an ϵHodge filtration an ϵHodge decomposition by $H^{p,q} = F^p \cap \overline{F^q}$, with $p + q = w$. This ϵHodge decomposition satisfies the relation (3.3).

We remark further, that the first Riemannian bilinear relation (cf. Definition 1) is equivalent to $(F^p)^\perp = F^{w-p+1}$, $p = 1, \ldots, w$, where $\perp$ is taken with respect to the bilinear from b. Now we are going to consider deformations of these structures:

Definition 2. *A (real) variation of ϵHodge structures (ϵVHS) is a triple (E, ∇, F^p), where E is a real vector bundle over an (connected) ϵcomplex base manifold (M, J^ϵ), ∇ is a flat connection and F^p is a filtration of $E^{\mathbb{C}_\epsilon}$ by ϵholomorphic sub-bundles of $E^{\mathbb{C}_\epsilon}$, which is a point-wise ϵHodge structure satisfying the infinitesimal period relation or the Griffiths transversality*

$$\nabla_\chi F^p \subset F^{p-1},\ \forall \chi \in T^{1,0}M. \tag{3.5}$$

A polarization of a variation of ϵHodge structures (E, ∇, F^p) consists of a non-degenerate bilinear form $b \in \Gamma(E^* \otimes E^*)$ having the following properties

(i) *b induces a polarization on each fiber obeying the first and the second bilinear relation,*

(ii) *b is parallel with respect to* ∇.

Remark 1. In complex geometry VHS roughly arise on the Hodge-decomposition of the cohomology of smoothly varying families of Kähler manifolds X_t where t is the parameter of the variation (cf. [2] Chapter 4 for details). To ensure that the Hodge-numbers $h^{p,q}(X_t)$ are constant in t one needs a result, which states that the kernel of a family D_t of elliptic differential operators depends upper-semi-continuously on t. Unfortunately this does not generalize to para-complex geometry for the following reason: If we consider an (almost) para-complex manifold M^{2n} endowed with a para-hermitian metric g the metric is forced to have split signature, i.e. signature (n, n). As a consequence the naturally associated differential operators are no longer elliptic and we are not able to use the above cited theory.

A class of ϵVHS which is related to the special geometry of (Euclidean) supersymmetry is discussed in the next subsection.

3.2. ϵVHS and special ϵKähler manifolds

Each fiber of the ϵcomplex tangent bundle

$$TM^{\mathbb{C}_\epsilon} = T^{1,0}M \oplus T^{0,1}M$$

carries a natural ϵHodge structure of weight 1 :

$$0 = F^2_x \subset F^1_x = T^{1,0}_x M \subset F^0_x = T^{\mathbb{C}_\epsilon}_x M. \tag{3.6}$$

We recall that *an affine special ϵKähler manifold* (M, J, ∇, g) (cf. [1, 15, 7]) is an ϵKähler manifold endowed with a flat torsion-free connection ∇, such that (∇, J^ϵ) is *special*, i.e. ∇J^ϵ is symmetric and $\nabla\omega = 0$, where ω is the ϵKähler form. The complex version of the next lemma and proposition was proved in [3] and we generalize it to the para-complex case.

Lemma 1. *Let ∇ be a torsion-free flat connection on the ϵcomplex manifold (M, J^ϵ). Then $F^1 = T^{1,0}M$ is an ϵholomorphic subbundle of $F^0 = T^{\mathbb{C}_\epsilon}M$ with respect to the ϵholomorphic structure defined by ∇ (compare Section 2) if and only if (∇, J^ϵ) is special, i.e. ∇J^ϵ is symmetric.*

Proof. The condition of F^1 to be ϵholomorphic is equivalent to

$$\nabla_{\bar{Y}} X = 0 \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M)$$

and the condition of (∇, J^ϵ) to be special is equivalent to

$$(\nabla_X J^\epsilon)(\bar{Y}) = (\nabla_{\bar{Y}} J^\epsilon)(X) \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M),$$

due to the following short argument:
Let $X, Y \in \Gamma(T^{1,0}M)$ or $X, Y \in \Gamma(T^{0,1}M)$

$$(\nabla_X J^\epsilon)(Y) = \nabla_X J^\epsilon Y - J^\epsilon \nabla_X Y = \pm\hat{i}\nabla_X Y - J^\epsilon \nabla_X Y,$$

which is symmetric as one sees by choosing vector fields X and Y such that $[X,Y]=0$. Let now $X,Y\in\Gamma(T^{1,0}M)$ be ϵholomorphic vector fields, i.e. $\mathcal{L}_X(J^\epsilon)=0$ where $\mathcal{L}$ is the Lie-derivative. Then it holds

$$\begin{aligned} 0 &= \mathcal{L}_X(J^\epsilon)\bar{Y} = [X,J^\epsilon\bar{Y}] - J^\epsilon[X,\bar{Y}] \\ &= \nabla_X J^\epsilon\bar{Y} - \nabla_{J^\epsilon\bar{Y}}X - J^\epsilon\nabla_X\bar{Y} + J^\epsilon\nabla_{\bar{Y}}X \\ &= (\nabla_X J^\epsilon)\bar{Y} - (\nabla_{\bar{Y}}J^\epsilon)X + \nabla_{\bar{Y}}J^\epsilon X - \nabla_{J^\epsilon\bar{Y}}X \\ &= [(\nabla_X J^\epsilon)\bar{Y} - (\nabla_{\bar{Y}}J^\epsilon)X] + 2\hat{i}\nabla_{\bar{Y}}X. \end{aligned}$$

This finishes the proof. □

From the lemma we obtain:

Proposition 2. *Let (M,J^ϵ) be an ϵcomplex manifold, ∇ be a torsion-free flat connection and $F^\bullet$ defined as in equation* (3.6).

1. *Then (M,J^ϵ,∇) is an affine special ϵcomplex manifold if and only if ∇ and $F^\bullet$ give a variation of ϵHodge structures of weight* 1 *on $TM^{\mathbb{C}_\epsilon}$.*
2. *Then (M,J^ϵ,∇,g) is an affine special ϵKähler manifold if and only if ∇, $F^\bullet$ and $\omega(\cdot,\cdot)=g(J^\epsilon\cdot,\cdot)$ give a variation of polarized ϵHodge structures of weight* 1 *on $TM^{\mathbb{C}_\epsilon}$.*

In [8] the following notion of a *conical special ϵKähler manifold* $(M,J^\epsilon,g,\nabla,\zeta)$ is introduced, i.e. an affine special ϵKähler manifold (M,J^ϵ,g,∇) endowed with a vector field ζ, such that

$$\nabla\zeta = D\zeta = Id, \tag{3.7}$$

where D is the Levi-Civita connection of g. In the same reference it is shown, that

$$\mathcal{L}_\zeta J^\epsilon = 0. \tag{3.8}$$

This implies that the distribution $\mathcal{D}=\mathrm{span}\{\zeta,J^\epsilon\zeta\}$ is integrable. The space of *leaves*, i.e. integral manifolds of $\mathcal{D}$ is denoted by $\bar{M}$. If $(M,J^\epsilon,g,\nabla,\zeta)$ is a *projective special ϵKähler manifold* (cf. [8]) of (real) dimension $2n+2$ then the canonical quotient map $\pi: M\to\bar{M}$ is an ϵholomorphic submersion onto an ϵcomplex manifold of (real) dimension $2n$. The manifold $\bar{M}$ inherits an ϵKähler metric $\bar{g}$ from the metric g such that π is a pseudo-Riemannian submersion. In this case it holds in particular $g(\zeta,\zeta)=-\epsilon g(J^\epsilon\zeta,J^\epsilon\zeta)\neq 0$. The affine geometry of conical special para-Kähler manifolds was studied in [9].

In the remaining part of this section we shortly discuss the polarized variation of Hodge structure of weight 3 on $V=TM\to\bar{M}$ related to a projective special ϵKähler manifold $(M,J^\epsilon,g,\nabla,\zeta)$:

Let us consider the real line bundle L, which is generated by ζ. We use $g(\zeta,\zeta)=-\epsilon g(J^\epsilon\zeta,J^\epsilon\zeta)\neq 0$ to obtain $TM=L\oplus J^\epsilon(L)\oplus L'$, such that $L'\cong T\bar{M}$ is the orthogonal complement of $L\oplus J^\epsilon(L)$ with respect to the pseudo-metric g. From condition (3.7) we conclude

$$\nabla L_{|L'} = L'. \tag{3.9}$$

Now we can define the Hodge filtration: We set $F^0 = TM^{\mathbb{C}_\epsilon}$. The relation (3.8) implies that $\zeta + \epsilon\,\hat{i} J^\epsilon \zeta$ generates the ϵholomorphic line bundle $F^3 = L^{1,0} \subset TM^{\mathbb{C}_\epsilon}$. From the Riemannian bilinear relation it follows $(F^3)^\perp = F^1$, where $\perp$ is taken with respect to the ϵKähler form ω of g which is extended $\mathbb{C}_\epsilon$-bilinearly. It remains to define $F^2 = T^{1,0}M$. The Griffiths transversality $\nabla F^3 \subset F^2$ is a consequence of equation (3.9) and $L' \cong T\bar{M}$. The condition $\nabla F^2 \subset F^1$ follows from equation (3.9) by similiar arguments as in [5]. This means we have defined a variation of ϵHodge structures of weight 3 by

$$F^3 = L^{1,0} \subset F^2 = T^{1,0}M \subset F^1 = (F^3)^\perp \subset F^0 = TM^{\mathbb{C}_\epsilon},$$

which is polarized by the ϵKähler form ω.

4. Period domains of variations of ϵHodge structures

We recall some information about period domains of variations of ϵHodge structures and have a closer look at the description of these either as homogeneous spaces or as flag manifolds, since this is crucial to understand our later results. A reference for the complex case is the book [2]. Again the complex case is classical and the para-complex case is new.

We introduce the period domain parameterizing the set of polarized ϵHodge structures on a fixed real vector space H having a fixed weight w and fixed ϵHodge numbers $h^{p,q}$. Such an ϵHodge structure is determined by specifying a flag $F^w \subset F^{w-1} \subset \ldots \subset F^0$ of fixed type satisfying the two bilinear relations. The set of such flags satisfying the first bilinear relation is usually called $\tilde{D}$ and can be described in a homogeneous model $G_{\mathbb{C}_\epsilon}/B$ where $G_{\mathbb{C}_\epsilon}$ is the group of automorphisms of $H^{\mathbb{C}_\epsilon}$ fixing the polarization b and B is the stabilizer of some given reference structure $F_o^\bullet$.

Proposition 3. *The set $\tilde{D}$ classifying ϵHodge decompositions of weight w with fixed ϵHodge numbers $h^{p,q}$ which obey the first bilinear relation is a flag manifold of type $(f_w, \ldots, f_v)$, $f_p = \dim F^p$, $v = \left[\frac{w+1}{2}\right]$, such that*

- (i) *in the case of even weight $w = 2v$ each F^p, for $p = w, \ldots, v+1$, is isotropic with respect to the bilinear form b,*
- (ii) *in the case of odd weight $w = 2v-1$ each F^p, for $p = w, \ldots, v$, is isotropic with respect to the bilinear form b.*

It can also be identified with the homogeneous manifold $G_{\mathbb{C}_\epsilon}/B$.

Proof. (i) In the case of even weight we recover the spaces F^p, for $p = 0, \ldots, (w - v + 1) = v + 1$, from F^p, for $p = w, \ldots, v$, by using the decomposition

$$H^{\mathbb{C}_\epsilon} = F^p \oplus_\perp \overline{F^{w-p+1}},$$

where $\perp$ is taken with respect to the non-degenerate ϵhermitian sesquilinear form $b(\cdot, \bar{\cdot})$. The condition on F^p, for $p = w, \ldots, v+1$, to be isotropic is the first Riemannian bilinear relation.

(ii) In fact, for odd weight, one can recover the whole flag from F^p for $p = w, \dots, v$, by using the decomposition

$$H^{\mathbb{C}_\epsilon} = F^p \oplus_\perp \overline{F^{w-p+1}},$$

where $\perp$ is taken with respect to the non-degenerate ϵhermitian sesquilinear form $b(\cdot, \bar{\cdot})$. The condition on F^p, for $p = w, \dots, v$, to be isotropic is in the case of odd weight w inherited from the first Riemannian bilinear relation. □

In the complex case B is a parabolic subgroup. There seems to be no equivalent para-complex notion in the literature. The subset of $\tilde{D}$ classifying ϵHodge structures which also satisfy the second bilinear relation is called D. As a non-degeneracy or a positivity condition the second bilinear relation defines an open subset of $\tilde{D}$.

Proposition 4. *The period domain D classifying ϵHodge filtrations $F^\bullet$ of fixed dimension $f^p = \dim F^p$ satisfying both bilinear relations is an open subset of $\tilde{D}$ and it is a homogeneous manifold $D = G/V$, where G is the group of linear automorphisms of H preserving b and $V = G \cap B$.*

We consider the case of Hodge structures which are strongly polarized. Given the space G/V, we call G/K where K is the maximal compact subgroup of G the 'associated symmetric space' and denote the canonical map by $\pi : G/V \to G/K$.

The case of odd weight

Now we have a glance at the groups G, V and K and the associated flag manifolds for ϵHodge structures of odd weight. Using this we describe for strongly polarized variations of Hodge structures the map π at the level of flag manifolds. This description is needed later to relate the (classical) period map to the ϵpluriharmonic maps appearing in ϵtt^*-geometry.

In the case of odd weight $w = 2l+1$ for $l = v-1$ the form b is anti-symmetric due to the first Riemannian bilinear relation and hence a symplectic form on H. In particular the real dimension of H is even. Hence the group G is the symplectic group $Sp(H, b) \cong Sp(\mathbb{R}^r)$ with $r = \dim_{\mathbb{R}} H \in 2\mathbb{N}$. The maximal compact subgroup of $Sp(\mathbb{R}^r)$ is $K = U(r)$.

We define the b-isotropic ϵcomplex vector space $\mathcal{L} = \bigoplus_{p=0}^{l} H^{w-p,p} = F^{w-l} = F^v$. One sees by equation (3.4)

$$H^{\mathbb{C}_\epsilon} = \mathcal{L} \oplus \overline{\mathcal{L}}. \tag{4.1}$$

Since they have the same dimension, $\mathcal{L}$ and $\overline{\mathcal{L}}$ are, by the first bilinear relation, Lagrangian subspaces. We further fix a reference structure $F_o^\bullet$. Taking successively ϵunitary bases[1]

$$\{f^i\}_{i=1}^{\dim(\mathcal{L})}$$

and

$$\{f_o^i\}_{i=1}^{\dim(\mathcal{L}_o)} \tag{4.2}$$

[1]This means a basis with $h(f_i, f_j) = \pm\delta_{ij}$.

with respect to the ϵhermitian sesquilinear scalar product $h(\cdot,\cdot) = (-1)^{w(w-1)/2}\hat{\imath}^{p-q}$ $b(\cdot,\bar{\cdot})$ of the flags

$$H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \cdots \subset \mathfrak{L}$$

and

$$H_o^{w,0} \subset H_o^{w,0} \oplus H_o^{w-1,1} \subset \cdots \subset \mathfrak{L}_o$$

and extending these with $\{\bar{f}^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ and $\{\bar{f}_o^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ on $\overline{\mathfrak{L}}$ and $\overline{\mathfrak{L}}_o$ to symplectic bases one sees that $Sp(\mathbb{R}^r)$ acts transitively by change of the basis from $\{f_o^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$ to $\{f^i\}_{i=1}^{\dim(\mathfrak{L}_o)}$.

(i) First we discuss the complex case. If we have a strongly polarized variation of Hodge structures, then the stabilizer of $F_o^\bullet$ is the group $V = \Pi_{p=0}^l U(\, h^{w-p,p})$. The map $\pi : G/V \to G/K$ is at this level nothing else than the forgetful map from the flag $H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \cdots \subset \mathfrak{L}$ to the subspace $\mathfrak{L}$. We remark, that the stabilizer of $\mathfrak{L}_o$ is contained in the group $U(r)$, if we assume the variation of Hodge structures to be strongly polarized.

If we consider a weakly polarized variation of Hodge structures, then the stabilizer of $F_o^\bullet$ is the group $V = \Pi_{p=0}^l U(k_p, l_p)$, where (k_p, l_p), with $h^{p,q} = k_p + l_p$, is the hermitian signature of h restricted to $H^{w-p,p}$ with $q = w - p$.

The stabilizer of $\mathfrak{L}_o$ is in this case an element of the group $U(k,l)$, where $r = 2(k+l)$ and (k,l) is the hermitian signature of h on $\mathfrak{L}_o$, i.e. $k = \sum k_p$ and $l = \sum l_p$.

Given a variation of Hodge structures of odd weight over the complex base manifold (M,J) we denote by L the (holomorphic) map

$$\begin{aligned} L : M &\to Sp(\mathbb{R}^r)/U(k,l), && (4.3)\\ x &\mapsto \mathfrak{L}_x. && (4.4)\end{aligned}$$

The Grassmannian of Lagrangian subspaces, on which h has signature (k,l) will be denoted by $\mathrm{Gr}_0^{k,l}(\mathbb{C}^r)$ and on which h is positive definite will be denoted by $\mathrm{Gr}_0(\mathbb{C}^r) = \mathrm{Gr}_0^{r,0}(\mathbb{C}^r)$.

(ii) In the para-complex case the stabilizer of $\mathfrak{L}_o$ is the group $U^\pi(C^n)$, with $r = 2n$, compare equation (2.1). As before given a variation of para-Hodge structures of odd weight w over the para-complex base manifold (M,τ) we denote by L the (para-holomorphic) map

$$\begin{aligned} L : M &\to Sp(\mathbb{R}^r)/U^\pi(C^n), && (4.5)\\ x &\mapsto \mathfrak{L}_x. && (4.6)\end{aligned}$$

The associated Grassmannian of Lagrangian subspaces will be denoted by Gr_0^n (C^{2n}) with $r = 2n$.

5. ϵtt^*-bundles and associated ϵpluriharmonic maps

In this section we recall the notion of (metric) ϵtt^*-bundles and explain the correspondence between metric ϵtt^*-bundles and ϵpluriharmonic maps, which was given in [18, 19].

Definition 3. *An ϵtt^*-bundle (E, D, S) over an ϵcomplex manifold (M, J^ϵ) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$ which satisfy the ϵtt^*-equation*

$$R^\theta = 0 \quad \textit{for all} \quad \theta \in \mathbb{R}, \tag{5.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + \cos_\epsilon(\theta) S_X + \sin_\epsilon(\theta) S_{J^\epsilon X} \quad \textit{for all} \quad X \in TM. \tag{5.2}$$

A metric ϵtt^*-bundle (E, D, S, g) is an ϵtt^*-bundle (E, D, S) endowed with a possibly indefinite D-parallel fiber metric g such that for all $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \textit{for all} \qquad X, Y, Z \in T_pM. \tag{5.3}$$

Remark 2. 1) If (E, D, S) is an ϵtt^*-bundle then (E, D, S^θ) is an ϵtt^*-bundle for all $\theta \in \mathbb{R}$, where $S^\theta := D^\theta - D = \cos_\epsilon(\theta) S + \sin_\epsilon(\theta) S_{J^\epsilon}$. The same remark applies to metric ϵtt^*-bundles.

2) The flatness of the connection D^θ can be expressed in a set of equations on D and S which can be found in [18, 19].

Given a metric ϵtt^*-bundle (E, D, S, g), we consider the flat connection D^θ for a fixed $\theta \in \mathbb{R}$. Any D^θ-parallel frame $s = (s_1, \dots, s_r)$ of E defines a map

$$G = G^{(s)} : M \to \operatorname{Sym}_{p,q}(\mathbb{R}^r); \quad x \mapsto G(x) := (g_x(s_i(x), s_j(x))), \tag{5.4}$$

where (p, q) is the signature of the metric g.

Let G/K be a pseudo-Riemannian symmetric space with associated symmetric decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We recall that a map $f : (M, J^\epsilon) \to G/K$ is said to be *admissible*, if the ϵcomplex linear extension of its differential maps $T^{1,0}_x M$ (respectively $T^{0,1}_x M$) to an Abelian subspace of $\mathfrak{p}^{\mathbb{C}_\epsilon} = \mathfrak{p} \otimes \mathbb{C}_\epsilon$ for all $x \in M$.

If M is simply-connected then it was shown in [18, 19], that $G : M \to Sym_{p,q}(\mathbb{R}^r)$ is ϵpluriharmonic and that it induces an admissible ϵpluriharmonic map $\tilde{G} : M \xrightarrow{G} Sym_{p,q}(\mathbb{R}^r) \xrightarrow{\tilde{}} S(p, q)$.

Conversely, we constructed in [18, 19] a metric ϵtt^*-bundle $(E = M \times \mathbb{R}^{2r},\ D = \partial - \epsilon S,\ S = \epsilon d\tilde{G}, g = < G\cdot, \cdot >_{\mathbb{R}^{2r}})$ over a simply-connected manifold from an admissible ϵpluriharmonic map $\tilde{G} : M \to S(p, q)$. If M is not simply-connected, then we have to replace the maps G and $\tilde{G}$ by twisted ϵpluriharmonic maps (cf. [19] Theorems 5 and 6).

6. Variations of ϵHodge structures as ϵtt^*-bundles

In this section we recall the result of Hertling [3] that variations of Hodge structures give solutions of metric tt^*-bundles and generalize it to para-complex geometry and symplectic ϵtt^*-bundles. Our presentation differs from that of [3], since we give this result in the language of real differential geometry. Again, the para-complex version seems to be new.

Let (E, ∇, F^p) be a (real) variation of ϵHodge structures of weight w. The ϵcomplexified connection of ∇ on $E^{\mathbb{C}_\epsilon} = E \otimes \mathbb{C}_\epsilon$ will be denoted by ∇^c. Griffiths transversality and the ϵholomorphicity of the subbundles F^p gives

$$\nabla^c : \Gamma(F^p) \to \Lambda^{1,0}(F^{p-1}) + \Lambda^{0,1}(F^p) \tag{6.1}$$

and ϵcomplex conjugation yields

$$\nabla^c : \Gamma(\overline{F}^p) \to \Lambda^{0,1}(\overline{F}^{p-1}) + \Lambda^{1,0}(\overline{F}^p). \tag{6.2}$$

Summarizing one obtains with $H^{p,w-p} = F^p \cap \overline{F}^{w-p}$

$$\begin{aligned} \nabla^c : \Gamma(H^{p,w-p}) \;\; \to \;\; & \underbrace{\Lambda^{1,0}(H^{p,w-p}) + \Lambda^{0,1}(H^{p,w-p})}_{D} + \\ & \underbrace{\Lambda^{1,0}(H^{p-1,w+1-p}) + \Lambda^{0,1}(H^{p+1,w-1-p})}_{S}. \end{aligned} \tag{6.3}$$

Using the decomposition induced by the ϵHodge structure and by the bi-degree of differential forms, one can find, that the curvature of ∇^c vanishes if and only if (E^c, D, S) defines an ϵtt^*-bundle. In addition the ϵcomplex conjugation $\kappa = \bar{\cdot}$ respects the ϵHodge decomposition and it is $\nabla^c \kappa = 0$. Again the decomposition induced by the ϵHodge structure and the bi-degree of differential forms imply that $D\kappa = 0$, i.e. D leaves E invariant and that $S\kappa = \kappa S$, i.e. S leaves E invariant, too.

If b is a polarization of the above variation of ϵHodge structures (E, ∇, F^p), then $\nabla b = 0$ and $\nabla^c \kappa = 0$ yield after decomposing with respect to the ϵHodge structure the equations $Dg = 0$ and $g(S\cdot, \cdot) = g(\cdot, S\cdot)$ with $g = \mathrm{Re}\, h$. Concluding we obtain the proposition:

Proposition 5. *Let (E, ∇, F^p) be a (real) variation of ϵHodge structures of weight w with a polarization b, then $(E^{\mathbb{C}_\epsilon}, D, S, g = Re\, h)$ and $(E, D, S, g = Re\, h)$ with D and S as defined in equation* (6.3) *are metric ϵtt^*-bundles.*

The above consideration holds for $\Omega = \mathrm{Im} h$, too. This implies $D\Omega = 0$ and $\Omega(S\cdot, \cdot) = \Omega(\cdot, S\cdot)$. Hence we have proven

Proposition 6. *Let (E, ∇, F^p) be a (real) variation of ϵHodge structures of weight w with a polarization b, then $(E^{\mathbb{C}_\epsilon}, D, S, \Omega = Im\, h)$ and $(E, D, S, \Omega = Im\, h)$ with D and S as defined in equation* (6.3) *are symplectic ϵtt^*-bundles.*

7. The period map of a variation of ϵHodge structures

Like period domains describe ϵHodge structures, ϵholomorphic maps into period domains describe variations of ϵHodge structures, in the sense of the following proposition which is in the complex case due to Griffiths (cf. [2] Chapter 4.5). We only consider the simply connected case:

Proposition 7. *Let (M, J^ϵ) be a simply connected ϵcomplex manifold and G/V the period domain classifying polarized ϵHodge structures of given weight and ϵHodge numbers, then giving a variation of ϵHodge structures is equivalent to giving an ϵholomorphic map from M to G/V which satisfies the Griffiths transversality condition. Such maps are called period maps.*

Let (E, ∇, F^p) be a variation of ϵHodge structures of odd weight w over the ϵcomplex base manifold (M, J^ϵ) endowed with a polarization b where E has rank r and where $f_p = \dim F_p$. Denote by (E, D, S, g) the corresponding ϵtt^*-bundle constructed in proposition 5. We suppose, that M is simply connected.

Like in Section 5 we examine the metric g in a $D^0 = \nabla$-parallel frame s of E. The flat frame is chosen as constructed in Section 4. The metric g defines a smooth map

$$G : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{Mat}(\mathbb{R}^r) \mid A = A^t \text{ and } A \text{ has signature } (p, q)\}. \tag{7.1}$$

In the complex case $(p, q) = (2k, 2l)$ is the symmetric signature of g. We remark that for a variation of para-Hodge structures the metric g is forced to have split signature $(p, q) = (n, n)$ with $n = \frac{1}{2}\dim_{\mathbb{R}} H$.

The map G will be called the *fundamental matrix* of the variation of ϵHodge structures (E, ∇, F^p) and as above $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ is identified with the pseudo-Riemannian symmetric space $GL(r, \mathbb{R})/O(p, q)$.

We recall that for odd weight each fiber of E has the structure of a symplectic vector space and consequently it holds $\mathrm{rk}_{\mathbb{R}} E = r = 2n \in 2\mathbb{N}$.

Theorem 1. *Let (E, ∇, F^p) be a polarized variation of ϵHodge structures of odd weight w with polarization b over the ϵcomplex base manifold (M, J^ϵ). Let $r = 2n$ be the real rank of E.*

Then the fundamental matrix G takes values in the totally geodesic submanifold

$$i \;:\; Gr_0^{k,l}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n})/U(k,l) \to GL(r, \mathbb{R})/O(2k, 2l), \text{ for } \epsilon = -1, \tag{7.2}$$
$$i \;:\; Gr_0^{n}(C^{2n}) = Sp(\mathbb{R}^{2n})/U^{\pi}(C^{n}) \to GL(r, \mathbb{R})/O(n, n), \text{ for } \epsilon = 1 \tag{7.3}$$

and coincides with the map L, i.e. $G = i \circ L : M \to GL(r, \mathbb{R})/O(p, q)$.

Proof. Given a point $x \in M$ we put $V = E_x^{\mathbb{C}_\epsilon}$ and $V^{\mathbb{R}} = E_x \cong \mathbb{R}^r$. To any polarized ϵHodge structure F^p of odd weight w with polarization b the map L associated a Lagrangian subspace $\mathfrak{L} \in \mathrm{Gr}_0^{k,l}(V)$ in the complex and a Lagrangian subspace $\mathfrak{L} \in \mathrm{Gr}_0^{n}(V)$ in the para-complex case (see Section 4). We define a scalar product $g^{\mathfrak{L}} = \mathrm{Re}\, h|_{\mathfrak{L}}$ on $\mathfrak{L} \subset V$. The projection onto the real points

$$\mathrm{Re} : V \to V^{\mathbb{R}} \tag{7.4}$$

induces an isomorphism $\mathfrak{L} \cong V^{\mathbb{R}}$. Its inverse we denote by $\Phi = \Phi_{\mathfrak{L}} : V^{\mathbb{R}} \to \mathfrak{L}$.

Claim:

$$i(\mathfrak{L}) = \Phi_{\mathfrak{L}}^* g^{\mathfrak{L}} =: G^{\mathfrak{L}}. \tag{7.5}$$

We first show the $Sp(\mathbb{R}^r)$-equivariance of the map

$$\mathcal{L} \mapsto G^{\mathcal{L}}. \tag{7.6}$$

From the definition of $\Phi_{\mathcal{L}}$ we obtain with $\Lambda \in Sp(\mathbb{R}^r)$:

$$\Phi_{\Lambda\mathcal{L}} = \Lambda \circ \Phi_{\mathcal{L}} \circ \Lambda^{-1}_{|\mathbb{R}^r} \tag{7.7}$$

and from this the transformation law of $G^{\mathcal{L}}$

$$G^{\Lambda\mathcal{L}} = \Phi^*_{\Lambda\mathcal{L}} g^{\Lambda\mathcal{L}} = (\Lambda^{-1})^* \Phi^*_{\mathcal{L}} \Lambda^* g^{\Lambda\mathcal{L}} = (\Lambda^{-1})^* \Phi^*_{\mathcal{L}} g^{\mathcal{L}} = (\Lambda^{-1})^* G^{\mathcal{L}} = \Lambda \cdot G^{\mathcal{L}}. \tag{7.8}$$

Let F^p_o be the reference flag of $V^{\mathbb{C}_\epsilon}_o$ with $\dim F^p_o = f_p$. We calculate $G^{\mathcal{L}_o}$ in the basis $\{f^i_o\}_{i=1}^{\dim(\mathcal{L}_o)}$ constructed in equation (4.2)

$$(G^{\mathcal{L}_0}(\mathrm{Re} f^i_o, \mathrm{Re} f^j_o)) = \not{\mathbb{1}}_{p,q}, \text{after permutation.} \tag{7.9}$$

This yields

$$\Phi^*_{\mathcal{L}_0} g^{\mathcal{L}_0} = \not{\mathbb{1}}_{p,q}. \tag{7.10}$$

The proof is finished, since $G(x) = G^{L(x)} = i(L(x))$. □

Corollary 1. *Let (E, ∇, F^p) be a polarized variation of ϵHodge structures of odd weight w with polarization b over the ϵcomplex base manifold (M, J^ϵ). Then the map $L : M \to Gr^{k,l}_0(\mathbb{C}^r_\epsilon) = Sp(\mathbb{R}^r)/U^\epsilon(k, l)$ is ϵpluriharmonic.*

Proof. This follows from the ϵpluriharmonicity of the fundamental matrix G : $M \to \mathrm{GL}(r, \mathbb{R})/O(p, q)$, since $G = i \circ L$, where i is a totally geodesic immersion and consequently, by a well-known result about ϵpluriharmonic maps (cf. [18, 19]), the ϵpluriharmonicity of L is equivalent to that of G. □

The last theorem and the last corollary can be specialized for variations of Hodge structures (this means $\epsilon = -1$.), which are strongly polarized:

Theorem 2. *Let (E, ∇, F^p) be a strongly polarized variation of Hodge structures of odd weight w with polarization b over the complex base manifold (M, J). Then the fundamental matrix G takes values in the totally geodesic submanifold*

$$i : Gr_0(\mathbb{C}^r) = Gr^{r,0}_0(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U(r) \to GL(r, \mathbb{R})/O(r) \tag{7.11}$$

and coincides with the map $L = \pi \circ \mathcal{P} : M \to G/K$, i.e. $G = i \circ L : M \to GL(r, \mathbb{R})/O(r)$.

With the same argument as before, we obtain the

Corollary 2. *Let (E, ∇, F^p) be a strongly polarized variation of Hodge structures of odd weight w with polarization b over the complex base manifold (M, J). Then the map $L : M \to Gr_0(\mathbb{C}^r) = Gr^{r,0}_0(\mathbb{C}^r) = Sp(\mathbb{R}^r)/U(r)$ is pluriharmonic.*

This means our results generalize the following result for strongly polarized complex variations of Hodge structures of odd weight:

Theorem 3. (cf. [2] Theorem 14.4.1) *Let $f : M \to G/V$ be a period mapping and $\pi : G/V \to G/K$, as defined in Section 4 the canonical map to the associated locally symmetric space. Then $\pi \circ f$ is pluriharmonic.*

References

[1] Alekseevsky, D. V.; Cortés, V.; Devchand, C.: *Special complex manifolds*. J. Geom. Phys. **42** (2002), 85–105. Zbl 1004.53038

[2] Carlson, J.; Müller-Stach, S.; Peters, C.: *Period Mappings and Period Domains*. Cambridge University Press 2003. Zbl 1030.14004

[3] Hertling, C.: *tt* geometry, Frobenius manifolds, their connections, and the construction for singularities*. J. Reine Angew. Math. **555** (2003), 77–161. Zbl 1040.53095

[4] Bertram, W.: *The Geometry of Jordan and Lie-Structures.* Lecture Notes in Mathematics **1754**, Springer-Verlag, Berlin 2000. Zbl 1014.17024

[5] Bryant, R. L.; Griffiths, P. A.: *Some observations on the infinitesimal period relations for regular threefolds with trivial canonical bundle.* Arithmetic and geometry, Vol. II, Prog. Math. **36** (1983), Birkhäuser Boston, 77–102 . Zbl 0543.14005

[6] Cecotti, S.; Vafa, C.: *Topological-antitopological fusion.* Nuclear Physics **B 367** (1991), 351–461.

[7] Cortés, V.; Mayer, C.; Mohaupt, T.; Saueressig, F.: *Special Geometry of Euclidean Supersymmetry I: Vectormultiplets*. J. High Energy Phys. JHEP03 (2004) 028, hep-th/0312001 2004.

[8] Cortés, V.: *Projective special (para)-Kähler manifolds*. Part of work in preparation with T. Mohaupt.

[9] Cortés, V.; Lawn, M.-A.; Schäfer, L.: *Parabolic spheres and special para-Kähler manifolds*. Int. Journal of Geometric Methods in Modern Physics **3**(5,6) (2006), 995–1009.

[10] Cortés, V.; Schäfer, L.: *Topological-antitopological fusion equations, pluriharmonic maps and special Kähler manifolds*. Progress in Mathematics **234** (2005), 59–74 Zbl 1077.53060

[11] Cruceanu, V.; Fortuny, P.; Gadea, P. M.: *A survey on paracomplex geometry*. Rocky Mt. J. Math. **26**(1) (1996), 83–115. Zbl 0856.53049

[D] Dubrovin, B.: *Geometry and integrability of topological-antitopological fusion*. Commun. Math. Phys. **152** (1993), 539–564. Zbl 0771.53042

[12] Eells, J.; Sampson, J. H.: *Harmonic mappings of Riemannian manifolds.* Am. J. Math. **86** (1964), 109–160. Zbl 0122.40102

[13] Erdem, S.: *Paraholomorphic structures and the connections of vector bundles over paracomplex manifolds*. New Zealand J. Math., Vol. **30**(1) (2001), 41–50. Zbl 1071.53522

[14] Etayo, F.; Santamaria, R.; Trias, U. R.: *The geometry of a Bi-Lagrangian manifold*. Differ. Geom. Appl. **24**(1) (2006), 33–59. Zbl 1101.53047

[15] Freed, D. S.: *Special Kähler manifolds*. Commun. Math. Phys. **203**(1) (1999), 31–52. Zbl 0940.53040

[16] Lawn, M.-A.; Schäfer, L.: *Decompositions of para-complex vector-bundles and para-complex affine immersions.* Result. Math. vol. **48**(3/4) (2005), 246–274. Zbl 1116.53020

[17] Libermann, P.: *Sur les structures presque paracomplexes.* C. R. Acad. Sci., Paris **234** (1952), 2517–2519. Zbl 0046.15601

[18] Schäfer, L.: *tt^*-bundles and pluriharmonic maps.* Ann. Global Anal. Geom. **28**(3) (2005), 285–300. Zbl 1082.58011

[19] Schäfer, L.: *tt^*-bundles in para-complex geometry, special para-Kähler manifolds and para-pluriharmonic maps.* Differ. Geom. Appl. **24**(1) (2006), 60–89. Zbl 1093.53046

[20] Schäfer, L.: *Harmonic bundles, topological-antitopological fusion and the related puriharmonic maps.* J. Geom. Phys. **56**(5) (2006), 830–842. Zbl 1089.58011

[21] Schäfer, L.: *Harmonic bundle solutions of topological-antitopological fusion in para-complex geometry.* To appear in Diff. Geom. and Appl.

[22] Simpson, C. T.: *Higgs bundles and local systems.* Publ. Math., Inst. Hautes Étud. Sci. **75** (1992), 5–95. Zbl 0814.32003

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