Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 48 (2007), No. 2, 435-442.

Splitting Classes in Categories of Groups

H. G. Grundman **D. Soltis**

Department of Mathematics, Bryn Mawr College
Bryn Mawr, PA 19010, USA
e-mail: grundman@brynmawr.edu

Interactive Telecommunications Program
Tisch School of the Arts, New York University
New York, NY 10003, USA
e-mail: ds1935@nyu.edu

Abstract. The ideas behind splitting classes were introduced by Freyd and Scedrov in [1] and expanded by Lippincott in [2]. In the latter work, Lippincott proves that there are exactly six splitting class pairs in the category of sets, but uncountably many in the category of groups. In this paper, we prove much more generally that any category containing the category of finite abelian p-groups as a full subcategory, for some prime p, has uncountably many splitting class pairs.

1. Introduction

A commuting square

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

in an arbitrary category $\mathcal{C}$ is said to *split* if there exists a morphism $\varepsilon : X \to Y$ such that the diagram

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \arrow[dl, "\varepsilon"'] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

commutes, in other words, such that $\varepsilon\alpha = \varphi$ and $\beta\varepsilon = \psi$. (Note: By *commuting square* we will always mean one that is oriented, that is, one in which the horizontal maps and the vertical maps are determined as such.) A morphism $\alpha : W \to X$ *splits over* $\beta : Y \to Z$ (equivalently, β *splits under* α) if every commuting square

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

splits.

Following [2], given a class of morphisms A in $\mathcal{C}$, we let A^* denote the class of all morphisms that split over all morphisms of A and we let A_* denote the class of all morphisms that split under all morphisms of A. It is easy to show that the sequence obtained by applying these two operators alternately stabilizes after the first step, that is ${(A^*)_*}^* = A^*$ and ${(A_*)^*}_* = A_*$. A *splitting class pair*, T/B is a pair of classes of morphisms such that $T_* = B$ and $B^* = T$. In this case, T is called a *top class* and B is called a *bottom class*.

Each of the $*$-operators reverses inclusion on classes of morphisms. Hence the operators can be viewed as contravariant functors on the category of classes of morphisms in $\mathcal{C}$ with containment as morphisms.

Top and bottom classes always contain all isomorphisms in $\mathcal{C}$ and are closed under composition. Hence, they are subcategories of $\mathcal{C}$.

Splitting classes arise in logic and category theory in results concerning "satisfaction" of Q-trees and diagrammatic sentences. The following two theorems demonstrate that satisfaction in a given category is closely tied to the splitting classes there. The first [1] says that morphisms in bottom classes both preserve and reflect satisfaction of Q-sequences and Q-trees.

Theorem 1. (Freyd, Scedrov) *Given a Q-sequence in a class A, then the morphisms in A_* preserve and reflect satisfaction of the Q-sequence. That is, if $A_0 \to A_1 \to \cdots \to A_n$ is a Q-sequence with all morphisms in A and $B \to B' \in A_*$ then $A_0 \to B$ satisfies the Q-sequence if and only if $A_0 \to B \to B'$ does so. More generally, given a Q-tree in a class A, then the morphisms in A_* preserve and reflect satisfaction of the Q-tree.*

Working instead in the language of diagrammatic sentences (constructs almost identical to Q-trees), Lippincott [2] proves a partial converse to this theorem in which the category is slightly restricted.

Theorem 2. (Lippincott) *Let $\mathcal{C}$ be a locally small category with finite co-limits. Let Z be an object and S a set of objects. Let T be a top class of morphisms. If every diagrammatic sentence with universal arrows in T that is satisfied by every structure in S is also satisfied by Z, then there is a morphism in T_* from some $Y \in S$ to Z.*

Lippincott developed most of the terminology used herein along with basic properties of splitting morphisms and classes. She also examined the splitting classes in two specific categories: $\mathcal{S}$, the category of sets and functions, and $\mathcal{G}$, the category of groups and group homomorphisms. Specifically, she proved that there are exactly six splitting class pairs in $\mathcal{S}$, but uncountably many such pairs in $\mathcal{G}$.

In Section 2, we present these results with brief indications of their proofs. In Section 3, we prove the main theorem of this work (Theorem 5) that for any prime p there are uncountably many splitting class pairs in $\mathcal{F}_p$, the category of finite abelian p-groups. Finally, in Section 4, we extend the results of Theorem 5 to a wide range of categories of groups and examine the limitations of this extension.

2. Sets and groups

The categories of sets and of groups provide an excellent illustration of the variation in the manifestations of splitting classes. In $\mathcal{S}$ there are only six splitting class pairs while in $\mathcal{G}$ there are uncountably many. For details of the proofs indicated below, see [2].

Theorem 3. (Lippincott) *There are exactly six splitting class pairs in $\mathcal{S}$.*

Let $N = \{f | \text{dom } f = \phi \text{ and cod } f \neq \phi\}$. The lattice of splitting class pairs in $\mathcal{S}$ is given in Figure 1. It is straightforward to verify that the given classes do form splitting class pairs. To see that there are no other splitting classes, let T be an arbitrary top class. Lippincott proves that if T contains any non-injection, then it contains all surjections; if it contains any non-surjection, then it contains all injections not in N; and if it contains any element of N, then it contains all injections. The result then follows using the fact that top classes are closed under function composition.

Theorem 4. (Lippincott) *There are uncountably many splitting class pairs in $\mathcal{G}$.*

Let S be a nonempty set of prime numbers. Consider the two classes

$$\begin{aligned} B_S &= \{f | \forall a \in \ker f, \text{ord } a \notin S\}; \\ T_S &= \{f | f \text{ is surjective and } \forall H \subseteq \ker f, \\ &\qquad H \lhd G \Rightarrow \exists a \in \ker f - H \text{ s.t. } \forall p \text{ prime } (p | \text{ord } a \Rightarrow p \in S)\}. \end{aligned}$$

Lippincott proves that for each S, T_S/B_S is a splitting class pair. Since distinct S yield distinct B_S (consider the homomorphisms $\mathbf{Z}/p\mathbf{Z} \to \{0\}$), this provides an uncountable collection of splitting class pairs.

$$\begin{tikzcd}[column sep=tiny]
& \{\text{functions}\}/\{\text{bijections}\} \arrow[dl, dash] \arrow[dr, dash] & & \\
\{\text{injections}\}/\{\text{surjections}\} \arrow[dr, dash] & & (\{\text{functions}\}-N)\ /(\{\text{bijections}\} \cup N) \arrow[dl, dash] \arrow[dr, dash] & \\
& (\{\text{injections}\}-N)\ /(\{\text{surjections}\} \cup N) \arrow[dr, dash] & & \{\text{surjections}\}/\{\text{injections}\} \arrow[dl, dash] \\
& & \{\text{bijections}\}/\{\text{functions}\} &
\end{tikzcd}$$

Figure 1. Splitting classes in $\mathcal{S}$

3. Finite abelian *p*-groups

In this section, we prove that for each prime p there are uncountably many splitting class pairs in the category of finite abelian p-groups. It is significant to note that the construction given in the proof of Theorem 4 does not work in $\mathcal{F}_p$ since it depends on there being infinitely many primes dividing the orders of the groups. The construction given here, however, does hold in the category of groups and indeed provides an infinite family of uncountable collections of splitting class pairs in $\mathcal{G}$.

Theorem 5. *There are uncountably many splitting class pairs in $\mathcal{F}_p$, for any prime p.*

Proof. Let S be a nonempty set of positive integers. Consider the two classes

$$\begin{aligned} B_S &= \{f | \forall s \in S, \forall y \in \operatorname{cod} f \text{ s.t. } \operatorname{ord} y \leq p^s, \exists x \in f^{-1}(y) \text{ s.t. } \operatorname{ord} x \leq p^s\}; \\ T_S &= \{f | \exists \text{ an isomorphism } \delta : \operatorname{cod} f \to \operatorname{dom} f \oplus \mathbf{Z}/p^{r_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{r_k}\mathbf{Z}, \\ &\qquad \text{s.t. } \forall i, r_i \in S \text{ and } \forall x \in \operatorname{dom} f, \delta f(x) = (x, 0, 0, \ldots, 0)\} \end{aligned}$$

We will show that for each S, T_S/B_S is a splitting class pair and that each S defines a distinct pair. Therefore $\{T_S/B_S\}$ is an uncountable collection of splitting class pairs in $\mathcal{F}_p$.

Let $S \neq S'$. Without loss of generality, let $r \in S' - S$. Let

$$S_- = \{s \in S | s < r\} \quad \text{and} \quad S_+ = \{s \in S | s > r\}.$$

So $S = S_- \cup S_+$. Let $m = \max S_-$ and $n = \min S_+$, if they exist. Define

$$\gamma : \mathbf{Z}/p^m\mathbf{Z} \oplus \mathbf{Z}/p^n\mathbf{Z} \to \mathbf{Z}/p^r\mathbf{Z}$$

by

$$\gamma(1,0) = p^{r-m} \quad \text{and} \quad \gamma(0,1) = 1.$$

(If either m or n does not exist, omit that factor of the domain. The rest of the proof goes through.)

We first show that $\gamma \in B_S$. Let $s \in S$ and let $y \in \operatorname{cod} f$ with ord $y \leq p^s$. If $s \in S_-$, then $1 \leq s \leq m$, and so y is in the subgroup of order p^m in $\mathbf{Z}/p^r\mathbf{Z}$. But this subgroup is generated by $\gamma(1,0)$ with each element of $\mathbf{Z}/p^m\mathbf{Z} \oplus \{0\}$ mapping to an element of the same order. So there is some $x \in f^{-1}(y)$ with ord $x =$ ord $y \leq p^s$ as desired. Further, if $s \in S_+$, then $n \leq s$. But γ is surjective, since $n > r$, and every element in the domain has order less than or equal to p^n. Thus for every $y \in \operatorname{cod} f$ there exists an $x \in f^{-1}(y)$ with ord $x \leq p^n \leq p^s$. Since the defining condition of B_S is satisfied by each $s \in S$, $\gamma \in B_S$.

Now notice that every element of the preimage of $1 \in \mathbf{Z}/p^r\mathbf{Z}$ is of order p^n which is greater than p^r. Hence the defining condition of $B_{S'}$ is not satisfied by r and so $\gamma \notin B_{S'}$. Therefore $\gamma \in B_S - B_{S'}$ and hence there are uncountably many distinct B_S's.

To see that T_S/B_S is a splitting class pair, first, let $\alpha \in T_S$ and $\beta \in B_S$. Let φ and ψ be given such that the diagram

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

commutes. Let $\delta : X \to W \oplus \mathbf{Z}/p^{r_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{r_k}\mathbf{Z}$ be as in the definition of T_S.

For each i, $1 \leq i \leq k$, let

$$e_i = (0_W, 0, 0, \ldots, 1, \ldots, 0) \in W \oplus \mathbf{Z}/p^{r_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{r_k}\mathbf{Z},$$

where the 1 is in the $i+1$st entry in the $k+1$-tuple. Let $z_i = \psi\delta^{-1}(e_i)$. Note that since e_i is of order p^{r_i}, z_i is of order less than or equal to p^{r_i}. By the definition of B_S, there exists an element of $\beta^{-1}(z_i)$ of order less than or equal to p^{r_i}. Choose one such element, y_i, for each i.

Define $\varepsilon' : W \oplus \mathbf{Z}/p^{r_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{r_k}\mathbf{Z} \to Y$ by $\varepsilon'(w, 0, 0, \ldots, 0) = \varphi(w)$ and $\varepsilon'(e_i) = y_i$. Define $\varepsilon : X \to Y$ by $\varepsilon = \varepsilon'\delta$.

Using ε, we now show that the above square splits. Note that for each $w \in W$, $\varepsilon\alpha(w) = \varepsilon'\delta\alpha(w) = \varepsilon'(w, 0, 0, \ldots, 0) = \varphi(w)$. Let $x \in X$ be arbitrary and let $\delta(x) = (w, n_1, \ldots, n_k)$. Then $\beta\varepsilon(x) = \beta\varepsilon'(w, n_1, \ldots, n_k) = \beta(\varphi(w) + \sum n_i y_i) = \psi\alpha(w) + \sum n_i z_i = \psi\delta^{-1}\delta\alpha(w) + \sum n_i(\psi\delta^{-1}(e_i)) = \psi\delta^{-1}(w, n_1, \ldots, n_k) = \psi(x)$. So $\varepsilon\alpha = \varphi$ and $\beta\varepsilon = \psi$. Thus α splits over β.

Next, let $\beta : Y \to Z$ split under every element of T_S. We will show that β must be in B_S. Let $s \in S$ and let $z_0 \in Z$ be any element of order less than or equal to p^s. Note that the homomorphism $\{0\} \to \mathbf{Z}/p^s\mathbf{Z}$ is in T_S and so β splits under it. Consider the diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathbf{Z}/p^s\mathbf{Z} \arrow[d, "\psi"] \\ \mathrm{Y} \arrow[r, "\beta"] & \mathrm{Z} \end{tikzcd}$$

in which $\psi(1) = z_0$. Let ε be a splitting homomorphism for this square. Then $\varepsilon(1)$ is an element of $\beta^{-1}(z_0)$ of order less than or equal to p^s. Thus $\beta \in B_S$ and it follows that $B_S = (T_S)_*$.

Finally, suppose that $\alpha : W \to X$ splits over every element of B_S. We will show that α must be in T_S. Note that the zero map from any group to the trivial group is in B_S. Therefore, the diagram

$$\begin{tikzcd} \mathrm{W} \arrow[r, "\alpha"] \arrow[d, "1_W"'] & \mathrm{X} \arrow[d] \\ \mathrm{W} \arrow[r] & \{0\} \end{tikzcd}$$

splits. Hence, α has a left inverse. Thus there is an isomorphism $\delta : X \to W \oplus W'$ for some W' with $\delta\alpha(w) = (w, 0)$ for all $w \in W$.

It remains to show that W' is of the required form. Let $\mathbf{Z}/p^r\mathbf{Z}$ be a fixed summand in the expansion of W' as a sum of cyclic groups. We show that $r \in S$, as required.

Suppose that $r \notin S$. Then $\gamma : \mathbf{Z}/p^m\mathbf{Z} \oplus \mathbf{Z}/p^n\mathbf{Z} \to \mathbf{Z}/p^r\mathbf{Z}$, as defined above, is in B_S. Let $\psi' : W \oplus W' \to \mathbf{Z}/p^r\mathbf{Z}$ be projection onto the fixed summand of W'. Let $\psi = \psi'\delta$. Then $\psi\alpha = 0$ and so the diagram

$$\begin{tikzcd} \mathrm{W} \arrow[r, "\alpha"] \arrow[d, "0"'] & \mathrm{X} \arrow[d, "\psi"] \\ \mathbf{Z}/p^m\mathbf{Z} \oplus \mathbf{Z}/p^n\mathbf{Z} \arrow[r, "\gamma"] & \mathbf{Z}/p^r\mathbf{Z} \end{tikzcd}$$

commutes. Again, since α splits over every element of B_S, the square splits. But $\psi^{-1}(1)$ consists of elements of order p^r in $X \cong W \oplus W'$ while $\gamma^{-1}(1)$ consists of elements of order p^n. Since $r < n$, there is no possible $\varepsilon : X \to \mathbf{Z}/p^m\mathbf{Z} \oplus \mathbf{Z}/p^n\mathbf{Z}$ with $\gamma\varepsilon = \psi$. Thus, contrary to the supposition, $r \in S$ and hence W' is of the required form. So $\alpha \in T_S$ and we have $T_S = (B_S)^*$.

Therefore, $\{T_S/B_S | S \subseteq \mathbf{Z}^+, S \neq \emptyset\}$ is an uncountable collection of splitting class pairs in $\mathcal{F}_p$. $\square$

4. Generalization

In this section, we examine extensions and limitations of Theorem 5. We start with a general lemma about top and bottom classes in full subcategories.

Lemma 6. *Given a category $\mathcal{C}$ with a full subcategory $\mathcal{D}$, any top (resp. bottom) class of $\mathcal{D}$ is the intersection of a top (resp. bottom) class of $\mathcal{C}$ with the class of morphisms in $\mathcal{D}$.*

Proof. Let T/B be a splitting class pair in $\mathcal{D}$. Let T_* be the bottom class in $\mathcal{C}$ determined by T. Let $(T_*)^*$ be the associated top class in $\mathcal{C}$. It's easy to verify that $T \subseteq (T_*)^*$.

Let α be a morphism in $\mathcal{D}$ with $\alpha \in (T_*)^*$. By definition, every element of B splits in $\mathcal{D}$ under every element of T. Since $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, every element of B also splits in $\mathcal{C}$ under every element of T. This means that $B \subseteq T_*$. But then since $\alpha \in (T_*)^*$, it splits in $\mathcal{C}$ (and hence in $\mathcal{D}$) over every element of B. Thus $\alpha \in T$.

Therefore T is the intersection of $(T_*)^*$ with the class of morphisms in $\mathcal{D}$. The proof for bottom classes is virtually identical. □

Corollary 7. *Given a full subcategory $\mathcal{D}$ of a category $\mathcal{C}$, $\mathcal{C}$ has at least as many splitting class pairs as does $\mathcal{D}$.*

Combining Theorem 5 and Corollary 7, the following is immediate.

Corollary 8. *Let p be any prime. There are uncountably many splitting class pairs in every category in which $\mathcal{F}_p$ is a full subcategory.*

So there are uncountably many splitting class pairs in the category of finitely generated groups, the category of 2-groups, the category of abelian groups, etc. In many of these categories, Theorem 5 actually provides an infinite family (indexed by p) of uncountable collections of splitting class pairs.

It is important to note that although the result in Theorem 5 generalizes to a wide range of categories of groups, it certainly does not extend to all such categories. We illustrate this with the category of groups of exponent 2. The lattice of splitting class pairs in this category is given in Figure 2. The proof of the following theorem follows the same reasoning as that of Theorem 3 described in Section 2.

Theorem 9. *There are exactly four splitting class pairs in the category of groups of exponent 2.*

$$\begin{array}{ccc} & \{\text{functions}\}/\{\text{bijections}\} & \\ \{\text{injections}\}/\{\text{surjections}\} & & \{\text{surjections}\}/\{\text{injections}\} \\ & \{\text{bijections}\}/\{\text{functions}\} & \end{array}$$

Figure 2. Splitting classes in the category of groups of exponent 2

References

[1] Freyd, P.; Scedrov, A.: *Categories, Allegories.* North-Holland, Amsterdam 1990. Zbl 0698.18002

[2] Lippincott, L.: *Properties preserved by certain classes of morphisms.* Preprint.

Received June 26, 2006