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An Invariance Property of the Tridens Curve in the Isotropic Plane

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Abstract. The tridens curves of third order and their generalizations in the isotropic plane over $\mathbb{R}$ were studied by D. Palman [1] and H. Sachs [2,3]. For additional properties see [6,7]. In this paper we prove that for every such tridens curve T of third order there exists an inscribed triangle Δ with the property: T remains invariant under the correspondence of opposite angle points of Δ.

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1. The equation of every irreducible tridens curve T of third order in the isotropic plane $I_2(\mathbb{R})$ can be written in the form (see [6, Lemma, part (a)])

$$T(x,y) \equiv \frac{1}{R}\{y(x-\alpha) - Rx(x-a)(x-A)\}, \quad \text{with } \alpha, a, A, R \in \mathbb{R} \tag{1}$$

and

$$\alpha a A(\alpha - a)(\alpha - A)(a - A) \neq 0 \quad \text{and} \quad (2\alpha - a)(2\alpha - A)(2\alpha - a - A) \neq 0. \tag{2}$$

In the above selected affine x, y-coordinate system the absolute point of $I_2(\mathbb{R})$ is supposed to have the homogeneous coordinates $0:0:1$. Using the definitions

$$2\lambda R(2\alpha - a - A) = 1 \quad \text{and} \quad \lambda b := 2\alpha - A \tag{3}$$

of numbers λ and b we get instead of (1) with (2)

$$T(x,y) \equiv [x(x-a) - \lambda(\lambda b - a)y](a - x) + \lambda(\lambda b - a)(x - \lambda b)y = 0 \tag{4}$$

with

$$(A+\lambda b)aA(A+\lambda b-2a)(\lambda b-A)(a-A)\neq 0 \quad \text{and} \quad (A+\lambda b-a)\lambda b(\lambda b-a)\neq 0. \tag{5}$$

Hence the triangle Δ with the vertices

$$A_1 := (0,0), \quad A_2 := (a,0), \quad A_3 := (\lambda b, b) \tag{6}$$

is an *inscribed triangle* of the tridens curve T with the equation (4) with (5).

2. The correspondence of opposite angle points for an admissible triangle $\Delta = \Delta(A_1A_2A_3)$ (see [4, p.22]) of $I_2(\mathbb{R})$ is explained as follows. Let us denote with σ_i the line determined by the side of Δ which does not contain the vertex A_i and with ω_i the isotropic bisectrix of the straight lines σ_{i+1} and σ_{i+2} (in this order). For a point $P(x,y)$ we regard the line $P \vee A_i$ and its image line r_i under reflection at ω_i in the sense of the isotropic metric. The lines r_1, r_2, r_3 have a common point $P^*(x^*, y^*)$, the so called *opposite angle point* of $P(x,y)$ with respect to Δ. Basic properties of this involutory, quadratic correspondence were studied by K. Strubecker (see [4, p.528f]).
Referring us to the triangle Δ with the vertices (6) we have for the coordinates of the opposite angle points P and P^* the analytical expressions (see [5, p.158])

$$x = \lambda x^* \frac{\sigma_1(x^*,y^*)}{\kappa(x^*,y^*)}, \qquad y = (x^* - \lambda y^*)\frac{\sigma_1(x^*,y^*)}{\kappa(x^*,y^*)}. \tag{7}$$

Hereby we have

$$\kappa(x,y) \equiv x(x-a) - \lambda(\lambda b - a)y = 0 \tag{8}$$

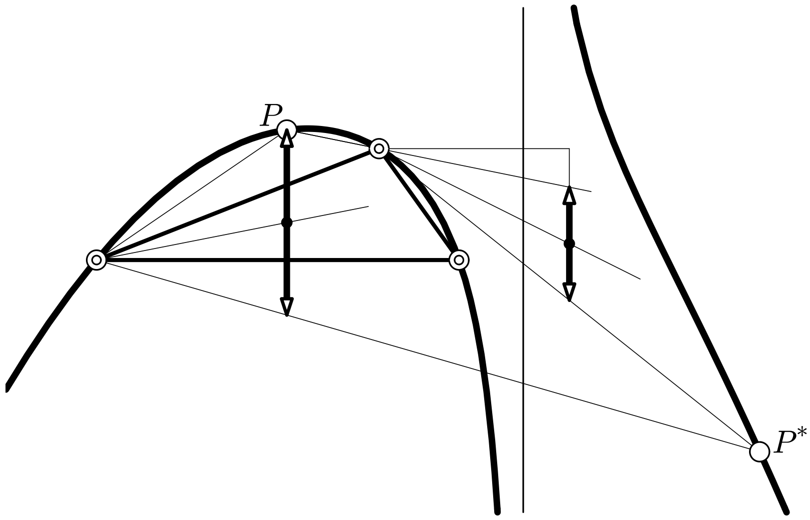
as the isotropic circumcircle of Δ and

$$\sigma_1(x,y) \equiv b(x-a) - (\lambda b - a)y = 0 \tag{9}$$

as the line determined by that side of Δ which is opposite to the vertex A_1. Using (7), a simple calculation leads to

$$\kappa(x^*,y^*)T(x,y) = \kappa(x,y)T(x^*,y^*). \tag{10}$$

Theorem. *For every irreducible tridens curve T of third order in the isotropic plane over $\mathbb{R}$ exists an inscribed triangle Δ with the property: T remains invariant under the correspondence of opposite angle points with respect to Δ.*



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