Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 2, 323-333.

Corner Cuts and their Polytopes

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Abstract. Corner cut polytopes (or staircase polytopes) were first defined by Shmuel Onn and Bernd Sturmfels in a computational commutative algebra context. They owe their name to the fact that their vertices are in one-to-one correspondence with certain partitions of natural numbers, so called corner cuts.
In this paper, we discuss some structural, nonetheless esthetic aspects of corner cut polytopes. In the 2-dimensional case, we draw a connection between a natural linear order on the vertices and a classical partial order on partitions. Furthermore, we explore the relationship between corner cuts and the face structure of corner cut polytopes.

1. Introduction

A *corner cut* of n in dimension d is a finite subset $\lambda \subset \mathbb{N}_0^d, |\lambda| = n$, that can be separated from its complement by an affine hyperplane disjoint from $\mathbb{N}_0^d$. For such a corner cut, define $p_\lambda := \sum_{\lambda_i \in \lambda} \lambda_i$. The convex hull of all points p_λ for λ a corner cut of n in dimension d is called the *corner cut polytope* or staircase polytope. In our paper, we analyse the geometric structure of corner cut polytopes.

Corner cuts and corner cut polytopes were first defined by S. Onn and B. Sturmfels [3]. Their work was motivated in a computational commutative algebra context. They show that the corner cuts of n in dimension d are in one-to-one correspondence with the Gröbner bases of a certain ideal in the polynomial ring $K[x_1, \dots, x_d], K$ an infinite field. This correspondence follows from the fact, that the vertices of the corner cut polyhedron $P_n^d := conv\left\{\sum_{\lambda_i \in \lambda} \lambda_i | \lambda \subset \mathbb{N}_0^d, |\lambda| = n\right\}$ are exactly the vertices of the corner cut polytope. For more details, see [3] and for the more general k-set polytopes see also [2].

There is various work on the number of corner cuts, which is equal to the numbers of vertices of corner cut polytopes. Onn and Sturmfels give an upper bound for the number of corner cuts for fixed dimension d in [3]. In [7] a lower and an upper bound are given, and for dimension 2 it is known (see [1]) that the number of corner cuts of n is equal to $\Theta(n \log n)$.

The focus of our paper is the geometric structure of corner cut polytopes. In Section 2 we recall basic definitions, mostly following [3].

In Section 3 we concentrate on 2-dimensional corner cut polytopes. We prove that the dominance order (see [6, p.288]), a classical partial order on 2-dimensional staircases, i.e. number partitions, is linear on the subset of corner cuts. This linear order coincides with the edge structure of the corner cut polytopes.

In Section 4 we characterize corner cut polytopes in arbitrary dimension which are pointed, i.e., which have a vertex on the diagonal $x_1 = x_2 = \cdots = x_d$.

In Section 5, we describe the one-skeleton of corner cut polytopes. Subsection 5.1 contains an alternative description of edges in dimension 2 corner cut polytopes in order to simplify the discussion for higher dimensional cases. This description also leads to a characterization of cover relations in the dominance order restricted to corner cuts: we show that with respect to the dominance order, there is a corner cut between two corner cuts λ and μ, if and only if the symmetric difference of λ and μ $(\lambda \oplus \mu)$ lies on a line.

In Subsection 5.2 we give a necessary condition for two corner cuts in dimension d to span an edge of the corner cut polytope.

2. Corner cut polytopes

In this section, we recall the most important definitions and properties. For further information see [3].

Definition 2.1. *A nonempty finite subset λ of $\mathbb{N}_0^d$ with the following property is called a staircase: if u is in λ and v is componentwise smaller than u, then v lies in λ as well.*

Staircases in dimension 2 correspond to number partitions written as Ferrer diagrams, (see [5, p.29]). Those in dimension 3 correspond to plane partitions. We recall the definition of plane partitions which is given in [4]:

Definition 2.2. *Let π be an array $\pi = \{\pi_{i,j}\}_{i,j \geq 1}$, of nonnegative integers $\pi_{i,j}$ with finite sum $|\pi| = \sum \pi_{i,j}$. If π is weakly decreasing in rows and columns, it is called a plane partition of $|\pi|$. Zero entries are omitted in this array.*

Definition 2.3. *Let λ be a staircase. If there exists an affine hyperplane which separates the points in λ from those in $\mathbb{N}_0^d \setminus \lambda$, λ is called a corner cut. The separating hyperplane is called a separator.*

Notation. Let $\binom{\mathbb{N}_0^d}{n}_{cut}$, and $\binom{\mathbb{N}_0^d}{n}_{stair}$, denote the set of corner cuts and staircases of cardinality n, respectively, in analogy of $\binom{\mathbb{N}_0^d}{n}$ denoting the set of n-element subsets of $\mathbb{N}_0^d$. Let λ be an element of $\binom{\mathbb{N}_0^d}{n}_{stair}$ with $\lambda = \{v_1, v_2, \ldots, v_n\}$. We write $\sum \lambda$ for $\sum_{i=1}^n v_i$.

Definition 2.4. *The polytope spanned by* $\sum \lambda$ *for* λ *in* $\binom{\mathbb{N}_0^d}{n}_{cut}$ *is called the corner cut polytope* Q_n^d*, i.e.,*

$$Q_n^d := conv\left\{\sum \lambda | \lambda \in \binom{\mathbb{N}_0^d}{n}_{cut}\right\}.$$

Definition 2.5. *The corner cut polytope* Q_n^d *has one vertex on each of the* d *coordinate axis. These vertices define a facet, which we will call the cover of* Q_n^d*.*

Lemma 2.6. *All points* $\sum \lambda$ *for* λ *in* $\binom{\mathbb{N}_0^d}{n}_{cut}$ *are vertices of* Q_n^d*.*

Proof. Let λ be a corner cut of n. It follows by Definition 2.3, that there exists a $w \in \mathbb{R}^d$ and a number $a \in \mathbb{R}$ with $vw < a$ for all $v \in \lambda$ and $vw > a$ for $v \in \mathbb{N}^d \setminus \lambda$. Hence, $\sum \lambda$ is the unique minimal argument of the linear function $\left\{\sum \mu | \mu \in \binom{\mathbb{N}_0^d}{n}_{cut}\right\} \to \mathbb{R}$, $\sum \mu \mapsto w \sum \mu$; i.e., $w \sum \lambda$ is minimal, and therefore $\sum \lambda$ is a vertex of Q_n^d. □

Remark. Analogous to the corner cut polytope, we can define the staircase polytope by

$$Q_n^d := conv\left\{\sum \mu | \mu \in \binom{\mathbb{N}_0^d}{n}_{stair}\right\}.$$

This is the original definition of [3, p.29]. It is shown in [3, p.31], that the two polytopes are the same, i.e., $\sum \mu$ is in the interior of the corner cut polytope Q_n^d for $\mu \in \binom{\mathbb{N}_0^d}{n}_{stair}$ but $\mu \notin \binom{\mathbb{N}_0^d}{n}_{cut}$.

3. A linear order on corner cuts in dimension 2

Although corner cuts in dimension 2 do not seem to retain much of a mystery, we want to touch on a certain surprising aspect. We show that the dominance order, a standard partial order on number partitions, is linear on corner cuts.
First of all, we recall the definition of the dominance order, (see [6, p.288]).

Definition 3.1. *Given two partitions of* n*,* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$*, and* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s > \mu_{s+1} = \cdots = \mu_n = 0$*,* λ_i*,* μ_i *denoting block lengths of partitions, respectively row lengths in Ferrer diagrams. Then,* $\lambda \preceq \mu$ *by dominance order iff* $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ *holds for all* $1 \leq i \leq n$*.*

Figure 1 shows the dominance order on partitions of 7, corner cuts depicted in black.

Theorem 3.2. *The dominance order is linear on corner cuts, i.e., any two corner cuts are comparable in the dominance order.*

Proof. Let λ and μ be partitions of n, λ a corner cut, μ a staircase that is not comparable to λ in the dominance order. Our goal is to show that μ can not be a corner cut.

If λ and μ can not be compared, there has to be a smallest index i with $\lambda_i < \mu_i$ (or $\lambda_i > \mu_i$, symmetrically). There is a second index $j > i$ where the sides change and $\lambda_1 + \lambda_2 + \cdots + \lambda_j$

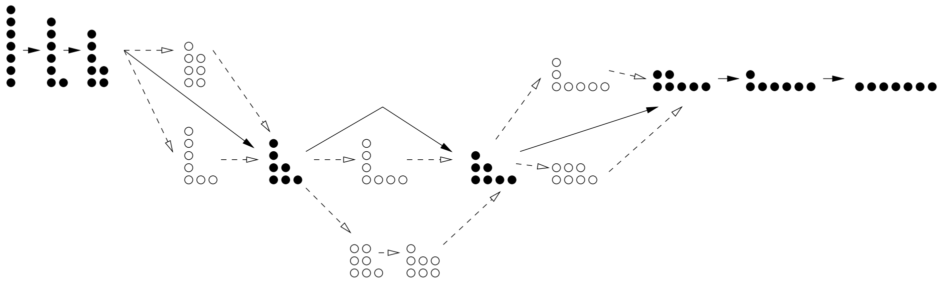

Figure 1: Partitions (corner cuts in black) of 7; arrows indicate "$\preceq$"

gets strictly bigger than $\mu_1 + \mu_2 + \cdots + \mu_j$ for the first time. For this, λ_j has to be strictly bigger than μ_j. Because both, λ and μ, are partitions of n, $\lambda_{j+1} + \cdots + \lambda_r$ has to be strictly smaller than $\mu_{j+1} + \cdots + \mu_s$. (In particular, there has to be at least one more row μ_{j+1} in the Ferrer diagram of μ.) Therefore, there exists an index $k > j$, $\lambda_k < \mu_k$ with λ_k possibly equal to 0. Figure 2 shows λ with its separator and the points of μ described above. It is obvious, that there is no separator for μ, and therefore μ can not be a corner cut. $\square$

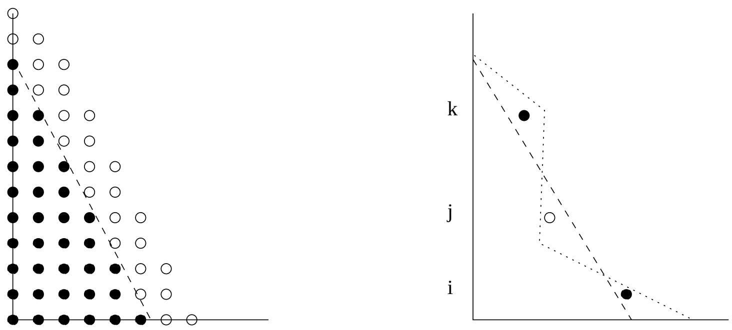

Figure 2: Corner cuts λ and μ with separating lines

It is easy to see, that the second component of the vertices $\sum \lambda, \lambda \in \binom{\mathbb{N}_0^2}{n}_{cut}$ of corner cut polytopes as well defines a linear order on the corner cuts. We show that the two orders coincide.

Theorem 3.3. *The reverse dominance order on corner cuts coincides with the order on vertices of corner cut polytopes by second component.*

Proof. Let λ and μ be two corner cuts with $\lambda \preceq \mu$, $\lambda \neq \mu$, by dominance order. We have to show that for the second components $(p_\lambda)_2$ and $(p_\mu)_2$ of the corresponding vertices of the corner cut polytope, $(p_\lambda)_2 > (p_\mu)_2$ in the natural linear order on $\mathbb{N}$. Hence, $\left(\binom{\mathbb{N}_0^2}{n}_{cut}, \preceq\right) \rightarrow (\mathbb{R}, \geq), \lambda \mapsto (p_\lambda)_2 = (\sum \lambda)_2$ is an order preserving bijection, which completes our proof.

By definition of the dominance order:

$$\begin{aligned}
(p_\lambda)_2 &= \lambda_2 + 2\lambda_3 + \cdots + (n-1)\lambda_n \\
&= (\lambda_2 + \lambda_3 + \cdots + \lambda_n) + (\lambda_3 + \cdots + \lambda_n) + \cdots + (\lambda_n) \\
&> (\mu_2 + \mu_3 + \cdots + \mu_n) + (\mu_3 + \cdots + \mu_n) + \cdots + (\mu_n) \\
&= \mu_2 + 2\mu_3 + \cdots + (n-1)\mu_n \\
&= (p_\mu)_2.
\end{aligned}$$

□

Remark. The edges of a corner cut polytope in dimension 2 can be read from the Ferrer diagrams of its vertices.

Also, there is a criterion for deciding if there is a corner cut between two others with respect to the dominance order, see Corollary 5.5.

4. Pointed corner cut polytopes

In this section, we characterize corner cut polytopes Q_n^d which have a (unique) vertex on the diagonal $x_1 = x_2 = \cdots = x_d$.

Definition 4.1. *For a fixed $k \in \mathbb{N}$, the hyperplane $\sum_{i=1}^d x_i = k$ is called a layer L_k of Q_n^d, if there is at least one vertex of Q_n^d coincident with this plane.*

There is a natural linear order on these layers:

Definition 4.2. *For L_{k_1}, L_{k_2} layers of Q_n^d, define: $L_{k_1} < L_{k_2}$, iff $k_1 < k_2$. Thus, the minimal layer is the layer with the smallest distance to the origin. We call the face on this minimal layer the central face.*

An interesting question is, whether a polytope has one or more vertices on the central face.

Definition 4.3. *If a corner cut polytope has only one vertex on its central face, we call the polytope pointed.*

First we characterise pointed 2-polytopes.

Definition 4.4. *A corner cut λ in dimension 2 is called symmetric, if the Ferrer diagram and its transpose are identical.*

Lemma 4.5. *Given a corner cut λ in dimension 2. Then, $\sum \lambda$ is a point on the diagonal if and only if λ is symmetric.*
In particular, each Q_n^2 can have at most one vertex on the diagonal, depending on n.

Proof. The "if-part", stating that $\sum \lambda$ has equal components if λ is symmetric, needs no proof. We show the "only-if-part": If λ is a non-symmetric corner cut, its transpose $(\lambda^\star \neq \lambda)$ is a corner cut as well. Hence, we have two different vertices on the same layer which are symmetric with respect to the diagonal. In particular, none of them lies on the diagonal. □

Lemma 4.6. *A corner cut polytope Q_n^2 is pointed if and only if n is of the form:*

$$n = \binom{k+1}{2} \text{ for some } k \geq 1, k \in \mathbb{N}.$$

Proof. Symmetric corner cuts have separators which are orthogonal to the diagonal. Therefore, the symmetric corner cuts are those with points in a isosceles right-angled triangle. Figure 3 shows this principle. Therefore, the parameters n allowing for a symmetric corner cut are integers a_k with $a_1 = 1$ and $a_k = a_{k-1} + k$, the explicit formula being $n = \frac{k(k+1)}{2}$. □

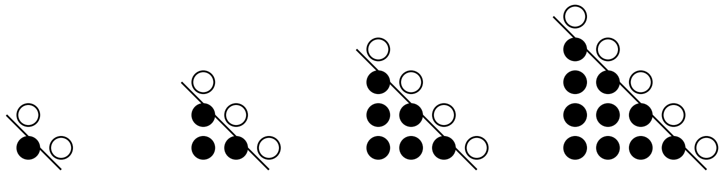

Figure 3: Symmetric corner cuts in dimension 2

For the d-dimensional case, we need the following notion of symmetry.

Definition 4.7. *Let λ be a staircase. If with p in λ, every permutation of the coordinates of p is in λ as well, λ is called totally symmetric.*

Lemma 4.8. *Given a corner cut λ in dimension d. Then, $\sum \lambda$ lies on the diagonal if and only if it is totally symmetric.*

Proof. Only the "only-if-part" is non-trivial:
Let λ be a corner cut in dimension d, which is not totally symmetric, but with $\sum \lambda$ on the diagonal. For each k, $\lambda|_{x_j=k}$ is a corner cut in dimension $(d-1)$. Choose a coordinate j such that the restrictions of the separator S to the planes $x_j = k$ ($S|_{x_j=k}$) are not for every k orthogonal to the diagonals $x_1 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_d, x_j = k$. Since λ is not totally symmetric, there is a smallest index i with the separator of $\lambda|_{x_j=i}$ not orthogonal to the restricted diagonal. To compensate this ($\sum \lambda$ is on the diagonal), there exists an index s with the separator of $\lambda|_{x_j=s}$ not orthogonal to this line either, and $S|_{x_j=s}$ not parallel to $S|_{x_j=i}$. Two disjoint $(d-2)$-dimensional affine subspaces in $\mathbb{R}^d$ which are not parallel cannot lie in a common $(d-1)$-dimensional affine hyperplane in $\mathbb{R}^d$ for $d \geq 3$. This is a contradiction to λ being a corner cut. □

It is easy to see, that we get totally symmetric d-dimensional corner cuts for corner cut sizes of the form $\binom{k+d-1}{d}$, namely the filled "d-pyramids" of integer points over the layer L_{k-1} of Q_n^d.

Theorem 4.9. *A corner cut polytope Q_n^d is pointed, if and only if n equals $\binom{k+d-1}{d}$ for some $k \geq 1$ in $\mathbb{N}$.*

5. The one-skeleton of corner cut polytopes

In this section, we study the one-skeleton of corner cut polytopes. Although we have already done this for dimension 2, we present a different aspect of the faces of 2-dimensional corner cut polytopes, providing the idea of how to generalise the description to higher dimensions. For dimensions 3 and higher, we obtain a necessary condition for two corner cuts to span an edge of the corner cut polytope.

5.1. Face structure in dimension 2 revisited

Definition 5.1. *For d-dimensional corner cuts λ and μ call the set of integer points $\lambda \setminus \mu \cup \mu \setminus \lambda$ the symmetric difference $\lambda \oplus \mu$ of λ and μ.*

Theorem 5.2. *Let λ and μ be 2-dimensional corner cuts, not both yielding cover points. Then, λ and μ correspond to vertices that span an edge of the corner cut polytope if and only if the points in $\lambda \oplus \mu$ lie on a line.*

To prove this theorem, we need the following lemmas:

Lemma 5.3. *Let λ and μ be two corner cuts with $\lambda \oplus \mu$ on a line g.*
Let λ and μ be given by $\lambda = \{v_1, \ldots, v_{n-k}, v_{n-k+1}, \ldots, v_n\}$, and $\mu = \{v_1, \ldots, v_{n-k}, v_{n+1}, \ldots, v_{n+k}\}$. Furthermore, let g be given by the equation $w_3 x = a_3$; obviously, $w_3 v_i \leq a_3$ for $i = 1, \ldots, n-k$.
If there exist integer points $p_1, \ldots, p_s \neq v_1, \ldots, v_{n-k}$ below g, then there exists a point $p \in \{p_1, \ldots, p_s\}$ with

$$2p - v_n \neq p_j, j = 1, \ldots, s \quad \text{or} \quad p_2 < \frac{(v_n)_2 + (v_{n+1})_2}{2},$$
$$\text{and}$$
$$2p - v_{n+1} \neq p_j, j = 1, \ldots, s \quad \text{or} \quad p_1 < \frac{(v_n)_1 + (v_{n+1})_1}{2}.$$

Proof. There are separators $w_1 x = a_1$ and $w_2 x = a_2$ for λ and μ, respectively:

$$w_1 v < a_1 \text{ for } v \in \{v_1, \ldots, v_n\}, w_1 v > a_1 \text{ for } v \in \{v_{n+1}, \ldots, v_{n+k}\},$$

$$w_2 v < a_2 \text{ for } v \in \{v_1, \ldots, v_{n-k}, v_{n+1}, \ldots, v_{n+k}\}, w_2 v > a_2$$
$$\text{for } v \in \{v_{n-k+1}, \ldots, v_n\},$$

$$w_3 v_{n-k+1} = \ldots = w_3 v_n = w_3 v_{n+1} = \ldots = w_3 v_{n+k} = a_3.$$

We prove Lemma 5.3 by contradiction. Suppose that for all points $p_i, i = 1, \ldots, s$, either

$$2p_i - v_n = p_j, \text{ for some } j \neq i \quad \text{and} \quad (p_i)_2 \geq \frac{(v_n)_2 + (v_{n+1})_2}{2},$$
$$\text{or}$$
$$2p_i - v_{n+1} = p_j, \text{ for some } j \neq i \quad \text{and} \quad (p_i)_1 \geq \frac{(v_n)_1 + (v_{n+1})_1}{2}.$$

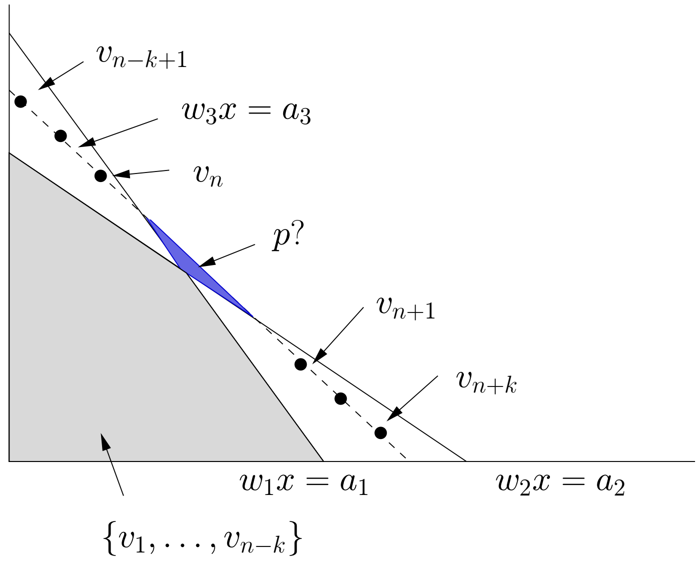

Figure 4: Corner cuts λ and μ with separators

Then we know that all points $p_i, i = 1, \ldots, s$ lie in one of the triangles Δ_1, depicted in Figure 5. The triangles are bounded by g but do not contain g. Because either $(2p_i - v_n)$ or $(2p_i - v_{n+1})$ coincides with another point p_j, which also lies in Δ_1, the points have to lie in the triangles Δ_2. This argument can be iterated. We conclude that the points p_i lie in the intersection of the triangles Δ_k for $k \geq 1$, which is empty, see Figure 5. We get a contradiction. $\square$

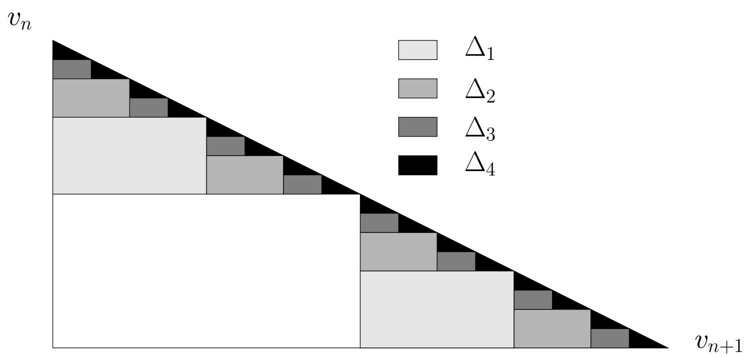

Figure 5: The series Δ_i, which leads to the empty set

Lemma 5.4. *There exists no integer point p different from $v_1, \ldots, v_{n-k}$ with $w_3p < a_3$.*

Proof. Again, we prove the lemma by contradiction:
Let $p_1, \ldots, p_s \neq v_1, \ldots, v_{n-k}$ be points with $w_3p_i < a_3$. We know that $w_1p_i > a_1, w_2p_i > a_2$, and $w_3p_i < a_3$ for $i = 1, \ldots, s$. We call regions above one of the lines $w_1x = a_1, w_2x = a_2$ and $w_3x = a_3$ and below one of the others, the forbidden regions. Forbidden, because there can not be other integer points in these regions than $v_{n-k+1}, \ldots, v_{n+k}$ and $p_1, \ldots, p_s$. Points

$(2p_i - v_n)$ and $(2p_i - v_{n+1})$ are integer points for all $i \in \{1, \dots, s\}$, and

$$\begin{array}{rcl} (2p_i - v_n)w_1 > a_1 & , & (2p_i - v_n)w_3 < a_3 \text{ and} \\ (2p_i - v_{n+1})w_2 > a_2 & , & (2p_i - v_{n+1})w_3 < a_3. \end{array}$$

Therefore, $(2p_i - v_n)$ and $(2p_i - v_{n+1})$ are integer points in the forbidden region, but possibly outside the first quadrant or $(2p_i - v_n) = p_j$ or $(2p_i - v_{n+1}) = p_j$ for some $j \neq i$. W.l.o.g. let $(v_n)_1 < (v_{n+1})_1$ and $(v_n)_2 > (v_{n+1})_2$. It follows, that $(v_n)_1 < (p_i)_1 < (v_{n+1})_1$ and $(v_n)_2 > (p_i)_2 > (v_{n+1})_2$. With Lemma 5.3, there exists a point $p \in \{p_1, \dots, p_s\}$ with

$$\begin{array}{c} 2p - v_n \neq p_j \text{ for } j = 1, \dots, s \quad \text{or} \quad p_2 < \dfrac{(v_n)_2 + (v_{n+1})_2}{2}, \\ \text{and} \\ 2p - v_{n+1} \neq p_j \text{ for } j = 1, \dots, s \quad \text{or} \quad p_1 < \dfrac{(v_n)_1 + (v_{n+1})_1}{2}. \end{array}$$

Case 1: $p_2 \geq \frac{(v_n)_2 + (v_{n+1})_2}{2}$ $\quad (\Rightarrow 2p - v_n \neq p_j$ for $j = 1, \dots, s)$.
Then $(2p - v_n)_1 > (v_n)_1 \geq 0$ and $(2p - v_n)_2 \geq (v_{n+1})_2 \geq 0$. Therefore, $(2p - v_n)$ is in the first quadrant and in the forbidden region, which gives us a contradiction.

Case 2: $p_1 \geq \frac{(v_n)_1 + (v_{n+1})_1}{2}$.
Then $(2p - v_{n+1})_1 \geq (v_n)_1 \geq 0$ and $(2p - v_{n+1})_2 > (v_{n+1})_2 \geq 0$, analogously.

Case 3: $p_1 < \frac{(v_n)_1 + (v_{n+1})_1}{2}$ and $p_2 < \frac{(v_n)_2 + (v_{n+1})_2}{2}$.
The point $q = ((2p - v_n)_1, (v_{n+1})_2)$ is an integer point in the first quadrant, and $q \neq v_{n-k+1}, \dots, v_{n+k}, p_1, \dots, p_s$.

$$\begin{array}{rcl} (2p - v_n)_2 & = & 2p_2 - (v_n)_2 < (v_{n+1})_2 \\ (2p - v_n)w_1 & > & a_1 \\ \Longrightarrow \quad qw_1 & = & (2p - v_n)_1 w_{11} + (v_{n+1})_2 w_{12} \\ & \overset{(w_{12} \geq 0)}{\geq} & (2p - v_n)_1 w_{11} + (2p - v_n)_2 w_{12} = (2p - v_n)w_1 > a_1 \end{array}$$

$$\begin{array}{rcl} (2p - v_n)_1 & < & (v_{n+1})_1 \\ v_{n+1}w_2 & < & a_2 \\ \Longrightarrow \quad qw_2 & = & (2p - v_n)_1 w_{21} + (v_{n+1})_2 w_{22} \\ & \overset{(w_{21} \geq 0)}{\leq} & (v_{n+1})_1 w_{21} + (v_{n+1})_2 w_{22} = v_{n+1}w_2 < a_2 \end{array}$$

Therefore, $(2p - v_n)$ is in the forbidden region, which leads to a contradiction. $\square$

Proof of Theorem 5.2. We start with the "only-if-part":
Let λ and μ be two adjacent corner cuts, not both on the cover. Let $\lambda \setminus \mu = \{\lambda_1, \dots, \lambda_s\}$ and $\mu \setminus \lambda = \{\mu_1, \dots, \mu_s\}$. There exists a $z \in \mathbb{R}^2_+$ with $z \sum \lambda = z \sum \mu$ the unique minimum on corner cuts, i.e., $z \sum \rho > z \sum \lambda$ for ρ a different corner cut, and $z \sum \rho \geq z \sum \lambda$ for ρ a

staircase. W.l.o.g., $z\lambda_1 \leq \ldots \leq z\lambda_s$ and $z\mu_1 \leq \ldots \leq z\mu_s$.
If we remove λ_s from λ and add μ_1 to it, we get a staircase, and because of the minimality, $z\mu_1 \geq z\lambda_s$. It follows, that $z\lambda_1 \leq \ldots \leq z\lambda_s \leq z\mu_1 \leq \ldots \leq z\mu_s$. Since $\sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \mu_i$, it follows that $z\lambda_1 = \cdots = z\lambda_s = z\mu_1 = \cdots = z\mu_s$. Hence, $\lambda_1, \ldots, \lambda_s$ and $\mu_1, \ldots, \mu_s$ lie on a common line.

Remark. The line g is parallel to a supporting hyperplane of the edge $(\sum \lambda, \sum \mu)$.

We prove the "if-part" using Lemma 5.4:
We know that there is no integer point $p \neq v_1, \ldots, v_{n-k}$ below g. It is also easy to see, that there can not be a corner cut $\rho \neq \lambda, \mu$ with $v_1, \ldots, v_{n-k} \in \rho$ and all points $p \in \rho$ with $p \neq v_1, \ldots, v_{n-k}$ on g. W.l.o.g., $w_3 v_1 \leq w_3 v_2 \leq \cdots \leq w_3 v_{n-k}$. We distinguish two cases:

Case 1: If ρ is a corner cut with $\{v_1, \ldots, v_{n-k}\}$ in ρ, at most $k-1$ points on g and at least one above g, then, $w_3 \sum \rho > w_3 \sum_{i=1}^{n-k} v_i + (k-1)a + a = w_3 \sum \lambda$. (Analogously, $w_3 \sum \rho > w_3 \sum \mu$).

Case 2: Let ρ be a corner cut with at most $\{v_1, \ldots, v_{n-k-1}\}$ in ρ, and the rest of the points on g or above. It is easy to see, that $w_3 \sum \rho \geq w_3 \sum_{i=1}^{n-k-1} v_i + (k+1)a_3 > w_3 \sum_{i=1}^{n-k-1} v_i + ka_3 + w_3 v_{n-k} = w_3 \sum \lambda$. Therefore, $\sum \lambda$ and $\sum \mu$ are the unique minima of the function $\left\{\sum \rho | \rho \in \binom{\mathbb{N}^2}{n}_{cut}\right\} \to \mathbb{R}$, $\sum \rho \to w_3 \sum \rho$. Hence, they form a face. $\square$

The following corollary is a consequence of Theorem 5.2.

Corollary 5.5. *In dimension* 2, *there is at least one corner cut* ρ *between two corner cuts* λ *and* μ *with respect to the dominance order, if and only if the points in* $\lambda \oplus \mu$ *span a plane.*

5.2. A necessary condition for edges

Theorem 5.6. *Let* λ *and* μ *be corner cuts in dimension* $d, d \geq 3$, *not both vertices of the cover. They can be adjacent as vertices of the corner cut polytopes only if the points in* $\lambda \oplus \mu$ *lie in a common* $(d-2)$*-dimensional affine subspace.*

Proof. Let λ and μ be adjacent corner cuts. Then, there is a $z \in \mathbb{R}_+^d$ with $z \sum \lambda = z \sum \mu$ the unique minimum of the function $\{\sum \rho | \rho \in \binom{\mathbb{N}^d}{n}_{cut}\} \to \mathbb{R}$, $\sum \rho \to z \sum \rho$. Hence, $z \sum \rho > z \sum \lambda$ for $\rho \neq \lambda, \mu$ a corner cut, and $z \sum \rho \geq z \sum \lambda$ for ρ a staircase. Let

$$\begin{aligned} \lambda \setminus \mu &= \{\lambda_1, \ldots, \lambda_s\}, \quad w.l.o.g., \quad z\lambda_1 \leq \ldots \leq z\lambda_s, \\ \mu \setminus \lambda &= \{\mu_1, \ldots, \mu_s\}, \quad w.l.o.g., \quad z\mu_1 \leq \ldots \leq z\mu_s. \end{aligned}$$

If we remove λ_s from λ, and add μ_1 to it, we get a staircase, and because of the minimality, $z\mu_1 \geq z\lambda_s$. Therefore we get

$$z\lambda_1 = \cdots = z\lambda_s = z\mu_1 = \cdots = z\mu_s = a,$$

and $\lambda \oplus \mu$ lie on an affine hyperplane E, orthogonal to z $(E : zv = a)$. Define p_1 to be the barycenter of $\lambda_1, \ldots, \lambda_s$, p_2 the barycenter of $\mu_1, \ldots, \mu_s$, and g the line coincident to p_1 and p_2; $(p_1, p_2) = g$. All the points below E lie in $\lambda \cap \mu$, since if not, there would be a smaller staircase, which is a contradiction. Also, all the points above E are neither in λ nor in μ, with an analogous argument. Let w be a vector orthogonal to g in E and define F to be a plane containing g and orthogonal to w. Assume the points $\lambda \oplus \mu$ span the whole $(d-1)$-dimensional affine hyperplane E, then we know that not all points lie on the hyperplane F. Take s points $p_1, \dots, p_s$ on E, with wp_i as small as possible. These points form a corner cut in the hyperplane E, and together with $\lambda \cap \mu$ a corner cut ρ in $\binom{\mathbb{N}^d}{n}$. Hence, λ, μ, and ρ are three different corner cuts with $z \sum \rho = z \sum \lambda = z \sum \mu$, which is a contradiction to the unique minimality. □

6. Conclusions and problems

The main results in this article are a linear order of corner cuts in dimension 2, a characterization of pointed corner cut polytopes, and a necessary condition for two corner cuts in any dimension to span an edge of the corner cut polytope.

It seems to be a quite intricate problem to prove a condition for edges that is necessary and sufficient.

Acknowledgements. My work on corner cuts and corner cut polytopes was initiated by Eva-Maria Feichtner. I would like to thank her for her outstanding support. Furthermore, I would like to thank Uli Wagner for helpful comments and discussions.

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Received April 11, 2002