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SUBCLASS OF BI-UNIVALENT FUNCTIONS DEFINED BY CHEBYSHEVS POLYNOMIAL ASSOCIATED WITH LAMBDA PSEUDO STARLIKE FUNCTION

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Abstract. In this paper we have obtained the coefficient bounds by using chebyshev polynomial associated with λ pseudo starlike functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ and given by Taylor Maclaurin series :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ Let $\mathcal{S}$ be the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in U. An analytic function f is subordinate to an analytic function g can be written as $f(z) \prec g(z)$, $(z \in U)$ provided there is schwartz function ϕ which is analytic in U with $\phi(0) = 0$ and $| \phi(z) |< 1, \quad (z \in U)$ such that $f(z) = g(\phi(z)), \ \ (z \in U)$.
By Koebe one quarter theorem [8]ensures that the image of Uunder every function $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. So every $f \in \mathcal{S}$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$ $\left(| w |< r_0(f); r_0(f) \geq \dfrac{1}{4}\right)$ also the inverse function $g = f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \ . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions in U given by(1.1). For various properties of subclasses of bi-univalent functions one can refer[[1],[4],[5],[6],[10],[11],[12],[15],[16]]
Recently babalola [3] defined the class $L_\lambda(\beta)$ of λ-pseudo starlike functions of order β as follows, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real , A function $f \in \mathcal{A}$ given by (1.1)belongs to the class $L_\lambda(\beta)$ of order β in the unit disc U if and only if $R(\dfrac{z(f'(z))^\lambda}{f(z)}) > \beta \quad (z \in U)$
One of the important tools in numerical analysis ,from both theoritical and practical point of view is chebyshev polynomials. The majority of research papers dealing with specific orthogonal polynomials of chebyshev family are first and second kind $T_m(t)$ and $U_m(t)$ [[2],[7],[9],[13]]
The chebyshev polynomial of the first kind second kinds are well-known. In case of real variable t on (-1,1) they are defined by

$$T_m(t) = cosm\theta,$$
$$U_m(t) = \frac{sin(m+1)\theta}{sin\theta},$$

where subscript m denotes the polynomial degree and $t = cos\theta$.

Definition 1. *For $\lambda \geq 1$, $t \in (\frac{1}{2}, 1]$ and $0 \leq \mu < 1$ a function $f \in \sum$ given by* (1.1)*is said to be in the class $LG_\Sigma(\lambda, \mu, t)$if the following conditions are satisfied.*

$$\frac{z[f'(z)]^\lambda}{(1-\mu)f(z) + \mu z f'(z)} \prec H(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in U) \tag{1.3}$$

and

$$\frac{w[g'(w)]^\lambda}{(1-\mu)g(w) + \mu w g'(w)} \prec H(w,t) = \frac{1}{1 - 2tw + w^2} \quad (w \in U) \tag{1.4}$$

where the function $g = f^{-1}$is defined by equation (1.2).

We note that if $t = cos\alpha$ where $\alpha \in (-\frac{\pi}{3}, \frac{\pi}{3})$ then

$$H(z,t) = \frac{1}{1 - 2cos\alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n \quad (z \in U).$$

Thus

$$H(z,t) = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \ldots \quad (z \in U).$$

We can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + ... \quad (z \in U, t \in (-1,1)),$$

Where

$$U_{n-1} = \frac{sin(narccost)}{\sqrt{1-t^2}} \ (n \in N)$$

are the chebyshev polynomials of second kind and we have

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), U_1(t) = 2t, U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t \qquad (1.5)$$

The generating function of the first kind of chebyshev polynomial $T_n(t)$, $t \in [-1,1]$is given by

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \ (z \in U).$$

The first kind of chebyshev polynomial $T_n(t)$ and second kind of the chebyshev polynomial $U_n(t)$ are connected by

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); T_n(t) = U_n(t) - tU_{n-1}(t); 2T_n(t) = U_n(t) - U_{n-2}(t)$$

In the present paper we use the chebyshev polynomial expansion to provide the initial coefficient of bi-univalent functions in $LG_\Sigma(\lambda, \mu, t)$ also we have determine fekete-Szego problem for function in this class.

2. Coefficient estimates for the functions in the class $LG_\Sigma(\lambda, \mu, t)$

Theorem 1. *For $\lambda \geq 1$, $0 \leq \mu < 1$ and $t \in (\frac{1}{2}, 1]$ let the function $f \in \Sigma$ given by(1.1) be in the class $LG_\Sigma(\lambda, \mu, t)$ then*

$$| a_2 | \leq \frac{2t\sqrt{2t}}{\sqrt{4t^2(2\lambda\mu - 2\lambda^2 + 3\lambda - 2\mu - 1) + (2\lambda - \mu - 1)^2}} \qquad (2.1)$$

and

$$| a_3 | \leq \frac{4t^2}{(2\lambda - \mu - 1)^2} + \frac{2t}{3\lambda - 2\mu - 1} \qquad (2.2)$$

Proof. Let the function $f \in \Sigma$ given by (1.1)be in the class $LG_{\Sigma}(\lambda, \mu, t)$ from the equation (1.3) and (1.4) we have

$$\frac{z[f'(z)]^{\lambda}}{(1-\mu)f(z)+\mu zf'(z)} = 1 + U_1(t)p(z) + U_2(t)p^2(z) + ... \tag{2.3}$$

and

$$\frac{w[g'(w)]^{\lambda}}{(1-\mu)g(w)+\mu wg'(w)} = 1 + U_1(t)q(w) + U_2(t)q^2(w) + ... \tag{2.4}$$

for some analytic functions

$$p(z) = c_1(z) + c_2z^2 + c_3z^3 + ...\ (z \in U) \tag{2.5}$$

and

$$q(w) = d_1(w) + d_2w^2 + d_3w^3 + ...\ (w \in U) \tag{2.6}$$

such that $p(0) = q(0) = 0$, $\mid p(z) \mid < 1$, $z \in U$ and $\mid q(w) \mid < 1$, $w \in U$ It is known that if $\mid p(z) \mid < 1$ and $\mid q(w) \mid < 1$ then

$$\mid c_j \mid \leq 1 and \mid d_j \mid \leq 1\ \forall j \in N\ \ where\ \ N = 1, 2, 3... \tag{2.7}$$

from (2.3),(2.4),(2.5),(2.6) we get

$$\frac{z[f'(z)]^{\lambda}}{(1-\mu)f(z)+\mu zf'(z)} = 1 + U_1(t)c_1z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + ... \tag{2.8}$$

and

$$\frac{w[g'(w)]^{\lambda}}{(1-\mu)g(w)+\mu wg'(w)} = 1 + U_1(t)d_1w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + ... \tag{2.9}$$

equating the coefficients of (2.8),(2.9) we get

$$(2\lambda - \mu - 1)a_2 = U_1(t)c_1 \tag{2.10}$$

$$(2\lambda^2 + \mu^2 - 2\lambda\mu - 4\lambda + 2\mu + 1)a_2^2 + (3\lambda - 2\mu - 1)a_3 = U_1(t)c_2 + U_2(t)c_1^2 + ... \tag{2.11}$$

$$-(2\lambda - \mu - 1)a_2 = U_1(t)d_1 \tag{2.12}$$

$$(2\lambda^2 + \mu^2 - 2\lambda\mu + 2\lambda - 2\mu - 1)a_2^2 + (2\mu - 3\lambda + 1)a_3 = U_1(t)d_2 + U_2(t)d_1^2 + ... \tag{2.13}$$

from (2.10) and (2.12)

$$c_1 = -d_1 \tag{2.14}$$

and

$$2(2\lambda-\mu-1)^2a_2^2=U_1^2(t)(c_1^2+d_1^2) \tag{2.15}$$

Also by using (2.11),(2.13), we have

$$(4\lambda^2+2\mu^2-4\lambda\mu-2\lambda)a_2^2=U_1(t)(c_2+d_2)+U_2(t)(c_1^2+d_1^2) \tag{2.16}$$

By using (2.15) in (2.16) we get

$$a_2^2[(4\lambda^2+2\mu^2-4\lambda\mu-2\lambda)-\frac{U_2(t)}{U_1^2(t)}2(2\lambda-\mu-1)^2]=U_1(t)(c_2+d_2) \tag{2.17}$$

From (1.5),(2.7)and (2.17) we get

$$\mid a_2\mid\leq\frac{2t\sqrt{2t}}{\sqrt{4t^2(2\lambda\mu-2\lambda^2+3\lambda-2\mu-1)+(2\lambda-\mu-1)^2}} \tag{2.18}$$

Now for a_3 subtract (2.13) from (2.11)

$$(-6\lambda+4\mu+2)a_2^2+(6\lambda-4\mu-2)a_3=U_1(t)(c_2-d_2)+U_2(t)(c_1^2-d_1^2) \tag{2.19}$$

$$a_3=\frac{U_1(t)(c_2-d_2)}{2(3\lambda-2\mu-1)}+\frac{U_1^2(t)(c_1^2+d_1^2)}{2(2\lambda-\mu-1)^2} \tag{2.20}$$

using (1.5),(2.7)and (2.20) We get

$$\mid a_3\mid\leq\frac{4t^2}{(2\lambda-\mu-1)^2}+\frac{2t}{3\lambda-2\mu-1} \tag{2.21}$$

3. Fekete-Szegö inequalities for the class $LG_\Sigma(\lambda,\mu,t)$

Theorem 2. *Let f given by(1.1) be in the class $LG_\Sigma(\lambda,\mu,t)$ and $\delta\in\mathbb{R}$. Then*

$$\mid a_3-\delta a_2^2\mid\leq\begin{cases}\dfrac{2t}{(3\lambda-\mu-1)}; \\ \qquad\qquad for\mid\delta-1\mid\leq\left|\dfrac{4t^2(2\lambda\mu-2\lambda^2+3\lambda-2\mu-1)+(2\lambda-\mu-1)^2}{4t^2(3\lambda-\mu-1)}\right| \\ \dfrac{8t^3\mid(1-\delta)\mid}{|4t^2(2\lambda\mu-2\lambda^2+3\lambda-2\mu-1)+(2\lambda-\mu-1)^2|}; \\ \qquad\qquad for\mid\delta-1\mid\geq\left|\dfrac{4t^2(2\lambda\mu-2\lambda^2+3\lambda-2\mu-1)+(2\lambda-\mu-1)^2}{4t^2(3\lambda-\mu-1)})\right|\end{cases}$$

Proof. For Fekete-Szegö inequality for the function class $LG_{\Sigma}(\lambda,\mu,t)$ we have from equation (2.19),(2.16),(2.7),(1.5) and for some $\delta \in \mathbb{R}$, we get

$$| a_3 - \delta a_2^2 | \leq \begin{cases} \dfrac{2t}{(3\lambda-\mu-1)}; \\ \qquad\qquad 0 \leq | h(\delta) | \leq \dfrac{1}{(3\lambda-\mu-1)} \\ 2 \mid h(\delta) \mid t; \\ \qquad\qquad | h(\delta) | \geq \dfrac{1}{(3\lambda-\mu-1)} \end{cases}$$

Where

$$h(\delta) = \frac{U_1^2(t)(1-\delta)}{U_1^2(t)(2\lambda^2+\mu^2-2\lambda\mu-\lambda) - U_2(t)(2\lambda-\mu-1)^2}$$

If we choose $\delta = 1$ we get the following corollary

Corollary 3. *If $f \in LG_{\Sigma}(\lambda,\mu,t)$, then*

$$| a_3 - a_2^2 | \leq \frac{2t}{3\lambda-\mu-1}. \tag{3.22}$$

By specializing the parameters we get results of [14].

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