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PARTIAL SUMS OF HARMONIC UNIVALENT CONVEX FUNCTIONS BY USING QUANTUM CALCULUS

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Abstract. In this paper, we introduce a new subclass of harmonic univalent convex functions of complex order and type alpha by using quantum calculus. In particular, we introduce coefficient estimates, partial sums, distortion bounds and convolution conditions for this subclass.

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1. Introduction

Quantum calculus (or $q-$calculus) is a theory of calculus where smoothness is not required. In 1908 and 1910, Jackson initiated in-depth study of $q-$calculus (see [11, 12, 13]) and developed the $q-$derivative and $q-$integral in a systematic way. Later, quantum calculus has been used in various branches of physics and mathematics, as for example, in the areas of ordinary fractional calculus, orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, the theory of relativity, optimal control problems, $q-$difference and $q-$integral equations and in $q-$transform analysis.

Throughout this paper, we shall assume that $q-$satisfies the condition $q \in (0,1)$.

Definition 1. *Let $q \in (0,1)$ and $\lambda \in \mathbb{C}$. The $q-$number, denoted by $[\lambda]_q$, is defined by*

$$[\lambda]_q = \frac{1-q^\lambda}{1-q}.$$

When $\lambda = n \in \mathbb{N}$, we obtain $[n]_q = 1 + q + q^2 + ... + q^{n-1}$, and when $q \to 1^-$, then $[n]_q = n$.

Applying the $q-$number and motivated by Jackson [11], $q-$derivative is defined below.

Definition 2. *The $q-$derivative (or $q-$difference operator) of a function f, defined on a subset of $\mathbb{C}$, is given by*

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0. \end{cases} \tag{1}$$

We note that $\lim_{q \to 1^-}(D_q f)(z) = f'(z)$ *if f is differentiable at z.*

For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}.$$

For definitions and properties of $q-$derivative and $q-$calculus, one may refer to [4, 5, 8, 9, 10, 15].

Let $\mathcal{A}$ denote the class of normalized functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1 \tag{2}$$

which are analytic in the open unit disk $\mathbb{D} := \{z : |z| < 1\}$. In view of (1) and (2), it follows that for any $f \in \mathcal{A}$, we have

$$D_q f(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \quad \text{and} \quad D_q(z D_q f(z)) = \sum_{n=1}^{\infty} [n]_q^2 a_n z^{n-1},$$

where $q \in (0,1)$.

In order to define a subclass of harmonic univalent functions associated with $q-$calculus, we first need some notations and terminology of harmonic univalent functions.

Let $\mathcal{H}$ denote the family of complex-valued harmonic functions $f = h + \overline{g}$ in the unit disc $\mathbb{D}$, where h and g are analytic and have the following series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{3}$$

Note that $f = h + \overline{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)/h'(z)| < 1$ in $\mathbb{D}$ (see [6]). Also, let $\mathcal{S}_{\mathcal{H}}$ be a subclass of functions f in $\mathcal{H}$ that are univalent in $\mathbb{D}$. We observe that for $g(z) \equiv 0$ in $\mathbb{D}$, the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions in $\mathbb{D}$.

We also recall that convolution of two complex-valued harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \quad \text{and} \quad F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n} \tag{4}$$

is defined by

$$f(z) * F(z) = (f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n z^n}, \quad (z \in \mathbb{D}).$$

A comprehensive study for the theory of harmonic univalent functions may be found in Duren [7]. One may also refer to the survey articles [1] and [2].

Let the function $f = h + \overline{g}$ be defined by (3) with $b_1 = 0$. Then, the sequences of partial sums of functions f are given by

$$f_m(z) = z + \sum_{n=2}^{m} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n},$$

$$f_s(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{s} \overline{b_n z^n}.$$

Study of partial sums of starlike functions and convex functions was first initiated by Silverman [18] and Silvia [19]. Later, Porwal [16], Porwal and Dixit [17] derived some results of partial sums for harmonic univalent functions.

Using $q-$difference operator, we define the class $\mathcal{C}_{\mathcal{H}_q}(b, \alpha)$ of $q-$harmonic univalent convex functions of complex order and type alpha:

Definition 3. *A function $f = h + \overline{g}$ given by (3) is in the class $\mathcal{C}_{\mathcal{H}_q}(b, \alpha)$ if*

$$Re\left[1 + \frac{1}{b}\left(\frac{zD_q(zD_q f(z))}{zD_q f(z)} - 1\right)\right] \geq \alpha, \tag{5}$$

where $b \in \mathbb{C} \setminus \{0\}$, $q \in (0, 1)$, $\alpha \in [0, 1)$ and $z \in \mathbb{D}$. Such functions are called $q-$harmonic univalent convex functions of complex order and type alpha.

For various parameters, we obtain some known subclasses as special cases:

a) Setting $b = 1$, we get the class $\mathcal{C}_{\mathcal{H}_q}(1, \alpha) \equiv \mathcal{C}_{\mathcal{H}_q}(\alpha)$, [3].

b) Setting $b = 1$ and $q \to 1^-$, we get the class $\lim_{q \to 1^-} \mathcal{C}_{\mathcal{H}_q}(1, \alpha) \equiv \mathcal{C}_{\mathcal{H}}(\alpha)$, [14].

c) Setting $b = 1$, $\alpha = 0$ and $q \to 1^-$, we get the class $\lim_{q \to 1^-} \mathcal{C}_{\mathcal{H}_q}(1, 0) \equiv \mathcal{C}_{\mathcal{H}}$, [6].

In this paper, we first obtain sufficient coefficient estimates of the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. Making use of these coefficient estimates, we explore some ratios of partial sums of the functions in the class class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ by

$$Re\left\{\frac{f(z)}{f_m(z)}\right\},\quad Re\left\{\frac{f_m(z)}{f(z)}\right\},\quad Re\left\{\frac{f(z)}{f_s(z)}\right\},\quad Re\left\{\frac{f_s(z)}{f(z)}\right\}.$$

We finally introduce distortion theorems, covering theorem and convolution conditions for the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

2. Coefficient Bounds and Univalence Criteria

We first prove sufficient coefficient condition and show the univalence criteria of the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

Theorem 1. *Let $b \in \mathbb{C}\setminus\{0\}$, $q \in (0,1)$, $\alpha \in [0,1)$, $z \in \mathbb{D}$, and let $f = h + \overline{g} \in \mathcal{H}$ be defined by* (3)*. If*

$$\sum_{n=2}^{\infty}\lambda_n|a_n| + \sum_{n=1}^{\infty}\nu_n|b_n| \leq |b|(1-\alpha), \tag{6}$$

where

$$\lambda_n = [n]_q([n]_q - 1 + |b|(1-\alpha)), \quad (n \geq 2), \tag{7}$$

$$\nu_n = [n]_q([n]_q + 1 - |b|(1-\alpha)), \quad (n \geq 1), \tag{8}$$

then f is sense-preserving and univalent function in $\mathbb{D}$; thus, $f \in \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

Proof. In order to show that the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ is univalent, we will show that $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. Then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1 - \left|\frac{\sum\limits_{n=1}^{\infty} b_n(\overline{z}_1^n - \overline{z}_2^n)}{(z_1-z_2) + \sum\limits_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right| \\ &> 1 - \frac{\sum\limits_{n=1}^{\infty} n|b_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|}\end{aligned}$$

$$\geq 1 - \frac{\sum_{n=1}^{\infty} \frac{[n]_q\left([n]_q+1-|b|(1-\alpha)\right)}{|b|(1-\alpha)}|b_n|}{1-\sum_{n=2}^{\infty} \frac{[n]_q\left([n]_q-1+|b|(1-\alpha)\right)}{|b|(1-\alpha)}|a_n|} \geq 0.$$

This proves univalence of f. Also, f is sense-preserving in $\mathbb{D}$ because

$$\begin{aligned}
|D_q h(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} &> 1 - \sum_{n=2}^{\infty} \frac{[n]_q\big([n]_q - 1 + |b|(1-\alpha)\big)}{|b|(1-\alpha)}|a_n| \\
&\geq \sum_{n=1}^{\infty} \frac{[n]_q\big([n]_q + 1 - |b|(1-\alpha)\big)}{|b|(1-\alpha)}|b_n| \\
&> \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |D_q g(z)|.
\end{aligned}$$

We recall that $Re(\omega) > \alpha$ if and only if $|1-\alpha+\omega| > |1+\alpha-\omega|$. In view of (5), we will show that

$$\begin{aligned}
&\left|1-\alpha+\frac{(b-1)zD_qf(z)+zD_q(zD_qf(z))}{bzD_qf(z)}\right| \\
&-\left|1+\alpha-\frac{(b-1)zD_qf(z)+zD_q(zD_qf(z))}{bzD_qf(z)}\right| \geq 0.
\end{aligned} \tag{9}$$

The series expansions of $zD_qf(z)$ and $zD_q(zD_qf(z))$ can be written as

$$\begin{aligned}
zD_qf(z) &= zD_qh(z) - zD_qg(z) \\
&= z + \sum_{n=2}^{\infty}[n]_q a_n z^n - \sum_{n=1}^{\infty}[n]_q b_n \overline{z^n}
\end{aligned}$$

and

$$\begin{aligned}
zD_q(zD_qf(z)) &= zD_q(zD_qh(z)) + zD_q(zD_qg(z)) \\
&= z + \sum_{n=2}^{\infty}[n]_q^2 a_n z^n + \sum_{n=1}^{\infty}[n]_q^2 b_n \overline{z^n}.
\end{aligned}$$

Substituting these series expansions into the left side of (9), we observe that

$$\begin{aligned}
&= \left|b(2-\alpha)z + \sum_{n=2}^{\infty}[n]_q([n]_q - 1 + b(2-\alpha))a_n z^n + \sum_{n=1}^{\infty}[n]_q([n]_q + 1 - b(2-\alpha))b_n\overline{z^n}\right| \\
&\quad - \left|b\alpha z - \sum_{n=2}^{\infty}[n]_q([n]_q - 1 - b\alpha)a_n z^n - \sum_{n=1}^{\infty}[n]_q([n]_q + 1 + b\alpha))b_n\overline{z^n}\right|
\end{aligned}$$

$$\begin{aligned}
&\geq |b|(2-\alpha)|z| - \sum_{n=2}^{\infty}[n]_q([n]_q - 1 + |b|(2-\alpha))|a_n||z|^n \\
&\quad - \sum_{n=1}^{\infty}[n]_q([n]_q + 1 - |b|(2-\alpha))|b_n||z|^n \\
&\quad - |b|\alpha|z| - \sum_{n=2}^{\infty}[n]_q([n]_q - 1 - |b|\alpha)|a_n||z|^n - \sum_{n=1}^{\infty}[n]_q([n]_q + 1 + |b|\alpha))|b_n||z|^n \\
&\geq 2|b|(1-\alpha)|z| - 2\sum_{n=2}^{\infty}[n]_q([n]_q - 1 + |b|(1-\alpha))|a_n||z|^n \\
&\quad - 2\sum_{n=1}^{\infty}[n]_q([n]_q + 1 - |b|(1-\alpha))|b_n||z|^n \\
&\geq |b|(1-\alpha)|z|\left(1 - \sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n||z|^{n-1}\right) \geq 0,
\end{aligned}$$

by (6). This proves that $f \in \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. The function defined by

$$f(z) = z + \sum_{n=2}^{\infty}\frac{|b|(1-\alpha)}{\lambda_n}x_n z^n + \sum_{n=1}^{\infty}\frac{|b|(1-\alpha)}{\nu_n}\overline{y_n z^n},$$

where $\sum\limits_{n=2}^{\infty}|x_n| + \sum\limits_{n=1}^{\infty}|y_n| = 1$, show that the estimate in (6) is sharp.

In order to establish that (6) is also necessary condition, we need to define a class $\mathcal{TH}$ containing the functions $f = h + \overline{g}$, where

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \quad \text{and} \quad g(z) = -\sum_{n=1}^{\infty}|b_n|z^n \quad (z \in \mathbb{D}). \tag{10}$$

The bound in (6) is necessary condition for the class $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ defined by $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha) := \mathcal{TH} \cap \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. We prove the next coefficient characterization.

Theorem 2. *If $f = h + \overline{g}$ defined by* (10) *belongs to $\mathcal{TH}$, then $f \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\lambda_n|a_n| + \sum_{n=1}^{\infty}\nu_n|b_n| \leq |b|(1-\alpha), \tag{11}$$

where λ_n and ν_n given, respectively, by (7) *and* (8)*, and where $b \in \mathbb{C}\setminus\{0\}$, $q \in (0,1)$ and $\alpha \in [0,1)$.*

Proof. Due to $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha) \subset \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$, we need to show that "only if" part of this theorem. Using the functions $f = h+\overline{g}$ defined by (10), we observe that the condition (5) can be written by

$$Re\left\{\frac{(b-1)zD_qf(z) + zD_q(zD_qf(z))}{bzD_qf(z)} - \alpha\right\} \geq 0,$$

which is equivalent to

$$Re\left\{\frac{b(1-\alpha)z - \sum\limits_{n=2}^{\infty}\lambda_n|a_n|z^n - \sum\limits_{n=1}^{\infty}\nu_n|b_n|\overline{z}^n}{z - \sum\limits_{n=2}^{\infty}[n]_q|a_n|z^n + \sum\limits_{n=1}^{\infty}[n]_q|b_n|\overline{z}^n}\right\} \geq 0. \tag{12}$$

The expression given in (12) must satisfy for all $z \in \mathbb{D}$, ($|z| = r < 1$). If we choose the values of z on the positive real axis, we must have

$$\frac{|b|(1-\alpha) - \sum\limits_{n=2}^{\infty}\lambda_n|a_n|r^{n-1} - \sum\limits_{n=1}^{\infty}\nu_n|b_n|r^{n-1}}{1 - \sum\limits_{n=2}^{\infty}[n]_q|a_n|r^{n-1} + \sum\limits_{n=1}^{\infty}[n]_q|b_n|r^{n-1}} \geq 0. \tag{13}$$

The numerator in (13) is negative for r sufficiently close to 1 if the bound in (11) does not satisfy. Hence, there is a point $z_0 = r_0$ between $(0,1)$ for which the quotient in (13) is negative. Due to contradiction, we say that $f \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$.

3. Partial Sums

We now establish some ratios of partial sums of the functions in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

Theorem 3. *Let $f = h + \overline{g}$ defined by* (3) *be in $\mathcal{H}$ with $b_1 = 0$. If f holds the condition given by* (6) *and*

$$\lambda_n \geq \begin{cases} |b|(1-\alpha), & n=2,3,\dots,m \\ \lambda_{m+1}, & n=m+1,m+2,\dots, \end{cases}$$

then

$$i)\quad Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{|b|(1-\alpha)}{\lambda_{m+1}}, \tag{14}$$

$$ii)\quad Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{\lambda_{m+1}}{\lambda_{m+1} + |b|(1-\alpha)}. \tag{15}$$

These estimates are sharp for

$$f(z)=z+\frac{|b|(1-\alpha)}{\lambda_{m+1}}z^{m+1}. \tag{16}$$

Proof. i) Since $f\in\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$, by (6) we have

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=1}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|\leq 1,$$

where λ_n and ν_n given, respectively, by (7) and (8). For proving (14), we consider

$$\begin{aligned}\omega_1(z)&=\frac{\lambda_{m+1}}{|b|(1-\alpha)}\left\{\frac{f(z)}{f_m(z)}-\left(1-\frac{|b|(1-\alpha)}{\lambda_{m+1}}\right)\right\}\\&=1+\frac{\lambda_{m+1}}{|b|(1-\alpha)}\left(\frac{f(z)-f_m(z)}{f_m(z)}\right)\\&=1+\frac{\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}a_nz^n}{z+\sum\limits_{n=2}^{m}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}},\end{aligned} \tag{17}$$

which is analytic in $\mathbb{D}$ with $\omega_1(0)=1$. It is sufficent to show that $Re(\omega_1(z))>0$ or equivalently

$$\left|\frac{\omega_1(z)-1}{\omega_1(z)+1}\right|\leq 1.$$

Substituting (17) into this inequality, we get

$$\left|\frac{\omega_1(z)-1}{\omega_1(z)+1}\right|\leq\frac{\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2(\sum\limits_{n=2}^{m}|a_n|+\sum\limits_{n=2}^{\infty}|b_n|)-\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}.$$

The last inequality is bounded by 1 if and only if

$$\sum_{n=2}^{m}|a_n|+\sum_{n=2}^{\infty}|b_n|+\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum_{n=m+1}^{\infty}|a_n|\leq 1. \tag{18}$$

It suffices to show that left side of (18) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|,$$

which is equivalent to

$$\sum_{n=2}^{m}\frac{\lambda_n-|b|(1-\alpha)}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n-|\tau|(1-\alpha)}{|b|(1-\alpha)}|b_n|+\sum_{n=m+1}^{\infty}\frac{\lambda_n-\lambda_{m+1}}{|b|(1-\alpha)}|a_n|\geq 0.$$

If we take $f(z)=z+\frac{|b|(1-\alpha)}{\lambda_{m+1}}z^{m+1}$ with $z=re^{i\pi/m}$ and r approaches to 1 from left, then we get

$$\frac{f(z)}{f_m(z)}=1+\frac{|b|(1-\alpha)}{\lambda_{m+1}}z^m\to 1-\frac{|b|(1-\alpha)}{\lambda_{m+1}}.$$

ii) Similarly, for proving (15) we consider

$$\begin{aligned}\omega_2(z)&=\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left\{\frac{f_m(z)}{f(z)}-\left(1-\frac{|b|(1-\alpha)}{\lambda_{m+1}+|b|(1-\alpha)}\right)\right\}\\&=1+\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left(\frac{f_m(z)-f(z)}{f(z)}\right)\\&=1-\frac{\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}a_nz^n}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}},\end{aligned}$$

which is analytic in $\mathbb{D}$ with $\omega_2(0)=1$. Therefore

$$\left|\frac{\omega_2(z)-1}{\omega_2(z)+1}\right|\leq\frac{\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2(\sum\limits_{n=2}^{m}|a_n|+\sum\limits_{n=2}^{\infty}|b_n|)-\frac{\lambda_{m+1}-|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}\leq 1$$

if and only if

$$\sum_{n=2}^{m}|a_n|+\sum_{n=2}^{\infty}|b_n|+\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum_{n=m+1}^{\infty}|a_n|\leq 1. \tag{19}$$

Since left side of (19) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|,$$

this completes the proof.

Theorem 4. *Let $f = h + \overline{g}$ defined by (3) be in $\mathcal{H}$ with $b_1 = 0$. If f holds the condition given by (6) and*

$$\nu_n \geq \begin{cases} |b|(1-\alpha), & n=2,3,\dots,s \\ \nu_{s+1}, & n=s+1,s+2,\dots, \end{cases}$$

then

$$i) \quad Re\left(\frac{f(z)}{f_s(z)}\right) \geq 1 - \frac{|b|(1-\alpha)}{\nu_{s+1}}, \tag{20}$$

$$ii) \quad Re\left(\frac{f_s(z)}{f(z)}\right) \geq \frac{\nu_{s+1}}{\nu_{s+1} + |b|(1-\alpha)}. \tag{21}$$

These estimates are sharp for

$$f(z) = z + \frac{|b|(1-\alpha)}{\nu_{s+1}}\overline{z}^{s+1}. \tag{22}$$

Proof. i) Since $f \in \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$, by (6) we have

$$\sum_{n=2}^{\infty} \frac{\lambda_n}{|b|(1-\alpha)}|a_n| + \sum_{n=1}^{\infty} \frac{\nu_n}{|b|(1-\alpha)}|b_n| \leq 1,$$

where λ_n and ν_n given, respectively, by (7) and (8). For proving (20), we consider

$$\begin{aligned} \omega_3(z) &= \frac{\nu_{s+1}}{|b|(1-\alpha)}\left\{\frac{f(z)}{f_s(z)} - \left(1 - \frac{|b|(1-\alpha)}{\nu_{s+1}}\right)\right\} \\ &= 1 + \frac{\nu_{s+1}}{|b|(1-\alpha)}\left(\frac{f(z)-f_s(z)}{f_s(z)}\right) \\ &= 1 + \frac{\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty} b_n\overline{z^n}}{z + \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=2}^{s} \overline{b_n z^n}}. \end{aligned}$$

which is analytic in $\mathbb{D}$ with $\omega_3(0) = 1$. It is sufficent to show that $Re(\omega_3(z)) > 0$, or

$$\left|\frac{\omega_3(z)-1}{\omega_3(z)+1}\right| \leq \frac{\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}|b_n|}{2 - 2(\sum\limits_{n=2}^{\infty}|a_n| + \sum\limits_{n=2}^{s}|b_n|) - \frac{\nu_{s+1}}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}|b_n|} \leq 1$$

if and only if

$$\sum_{n=2}^{\infty}|a_n|+\sum_{n=2}^{s}|b_n|+\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum_{n=s+1}^{\infty}|b_n|\leq 1. \tag{23}$$

It suffices to show that left side of (23) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|$$

which is equivalent to

$$\sum_{n=2}^{\infty}\frac{\lambda_n-|b|(1-\alpha)}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{s}\frac{\nu_n-|b|(1-\alpha)}{|b|(1-\alpha)}|b_n|+\sum_{n=s+1}^{\infty}\frac{\nu_n-\nu_{s+1}}{|b|(1-\alpha)}|a_n|\geq 0.$$

To prove that $f(z)=z+\frac{|b|(1-\alpha)}{\nu_{s+1}}\overline{z}^{s+1}$ gives the sharp result, we observe that for $z=re^{i\pi/s+2}$ we have

$$\frac{f(z)}{f_s(z)}=1+\frac{|b|(1-\alpha)}{\nu_{s+1}}r^s e^{-i(s+2)\frac{\pi}{s+2}}\to 1-\frac{|b|(1-\alpha)}{\nu_{s+1}},$$

when $r\to 1^-$.

ii) Similarly, we obtain (21). Therefore, the proof is omitted.

4. Distortion Bounds and Covering Theorem

Distortion bounds for the functions belonging to the class $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ can be proven as below. This result also yields the covering theorem.

Theorem 5. *If $f\in\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$, then for $|z|=r<1$ we have*

$$|f(z)|\leq(1+|b_1|)r+\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2, \tag{24}$$

and

$$|f(z)|\geq(1-|b_1|)r-\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2. \tag{25}$$

Proof. Let $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$. Then

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\le (1+|b_1|)r + \frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))} \\
&\quad \sum_{n=2}^{\infty}\frac{[2]_q([2]_q-1+|b|(1-\alpha))}{|b|(1-\alpha)}\big(|a_n|+|b_n|\big)r^2 \\
&\le (1+|b_1|)r + \frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))} \\
&\quad \sum_{n=2}^{\infty}\left(\frac{[n]_q([n]_q-1+|b|(1-\alpha))}{|b|(1-\alpha)}|a_n| + \frac{[n]_q([n]_q+1-|b|(1-\alpha))}{|b|(1-\alpha)}|b_n|\right)r^2 \\
&\le (1+|b_1|)r + \frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2.
\end{aligned}$$

This proves (24). Similarly, we get the proof of (25).

From inequality (25), we obtain the covering result as follows:

Corollary 6. *If $f \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$, then*

$$\left\{w : |w| < \left(1-\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\right) + \left(\frac{2-|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}-1\right)|b_1|\right\} \subset f(\mathbb{D}).$$

5. Convolution Conditions

By using the definition of convolution, we show that $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ is closed under convolution.

Theorem 7. *For $0 \le \beta \le \alpha < 1$, suppose $f \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ and $F \in \mathcal{TC}_{\mathcal{H}_q}(b,\beta)$, then $f * F \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha) \subset \mathcal{TC}_{\mathcal{H}_q}(b,\beta)$.*

Proof. By using (10), the harmonic functions f and F given by (4) can be written as

$$f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\overline{z}^n$$

and

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n - \sum_{n=1}^{\infty} |B_n| \overline{z}^n.$$

Due to the definition of convolution of two harmonic functions, we write

$$(f * F)(z) = z + \sum_{n=2}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_n||B_n| \overline{z}^n.$$

Since $F \in \mathcal{TC}_{\mathcal{H}_q}(b, \beta)$, from Theorem 2 we observe that $|A_n| \leq 1$ and $|B_n| \leq 1$. Thus, we obtain

$$\sum_{n=2}^{\infty} \frac{[n]_q([n]_q - 1 + |b|(1-\alpha))}{|b|(1-\alpha)} |a_n||A_n| + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + 1 - |b|(1-\alpha))}{|b|(1-\alpha)} |b_n||B_n|$$
$$\leq \sum_{n=2}^{\infty} \frac{[n]_q([n]_q - 1 + |b|(1-\alpha))}{|b|(1-\alpha)} |a_n| + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + 1 - |b|(1-\alpha))}{|b|(1-\alpha)} |b_n| \leq 1.$$

Due to Theorem 2, we prove that $f * F \in \mathcal{TC}_{\mathcal{H}_q}(b, \alpha) \subset \mathcal{TC}_{\mathcal{H}_q}(b, \beta)$.

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