CERTAIN INTEGRAL TRANSFORMS OF THE GENERALIZED K-STRUVE FUNCTION

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Abstract. The aim of this paper is to the study of Struve type functions. Using k-Struve functions, we derive various integral transforms, including Euler transform, Laplace transform, Whittakar transform, $\mathcal{K}$-transform and Fractional Fourier transform. The transform images are expressed in terms of the generalized Wright function. Interesting special cases of the main result are also considered.

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1. Introduction and Preliminaries

Integral transforms have been widely used in several problems of mathematical physics and applied mathematics. Integral transforms with such special functions as (for example) the Bessel functions, Mittag-Leffler functions, Struve functions, hypergeometric functions have been played important roles in solving various applied problems. This information has inspired the study of several integral transforms with verity of special functions (see [2, 6, 14, 16, 18, 19, 24, 25]). The present paper deals with the evaluation of the Euler transform, Laplace transform, Whittakar transform, $\mathcal{K}$-transform and Fractional Fourier transform of the k-Struve function. Special cases of the results are also pointed out briefly. For the convenience of the reader, we give here the basic definitions and related notations which is necessary for the understanding of this study.

The Struve function $H_\nu(x)$ and modified Struve function $L_\nu(x)$ possess power series representation of the form [27] as

$$H_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n + v + {}^3\!/\!{}_2\right)\Gamma\left(n + {}^3\!/\!{}_2\right)} \left(\frac{x}{2}\right)^{2n+\nu+1}, \tag{1}$$

and

$$L_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n + v + 3/2\right)\Gamma\left(n + 3/2\right)} \left(\frac{x}{2}\right)^{2n+\nu+1}, \tag{2}$$

which is a particular solution of the non-homogeneous Bessel differential equation and ordinary differential equation ([17], p. 288)

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = \frac{4\left(\frac{x}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu + 1/2\right)},$$

$$x^2 y'' + xy' - \left(x^2 + \nu^2\right) y = \frac{x^{\nu+1}}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu + 1/2\right)}.$$

respectively.

The k-gamma function defined in [7] as

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-t^k/k} dt \quad (z \in \mathbb{C}), \tag{3}$$

and

$$\Gamma_k(z) = k^{(z/k)-1}\Gamma\left(z/k\right). \tag{4}$$

Recently, k-Struve function $S_{v,c}^k(.)$ introduced by Nisar et al. [15] as in the form

$$S_{v,c}^k(x) := \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k\left(nk + v + 3k/2\right)\Gamma\left(n + 3/2\right)} \left(\frac{x}{2}\right)^{2n+v/k+1}, \tag{5}$$

is a solution of second-order non-homogeneous ordinary differential equation

$$x^2 y'' + xy' + \frac{1}{k^2}\left(ckx^2 - \nu^2\right) y = \frac{4\left(\frac{x}{2}\right)^{\frac{\nu}{k}+1}}{k\Gamma_k\left(\nu + k/2\right)\Gamma\left(\frac{1}{2}\right)}.$$

For the detailed definition of the Struve functions and its more generalization, the interested reader may refer to the research papers (Bhow-mick [3, 4], Kanth [9], Nisar and Atangana [13], Singh [20, 21]).

For the convenience of the reader, we provide here the basic definitions and related notations which is necessary for the understanding of this study.

Definition 1. ***Euler Transform*** *(Sneddon [22]).*
The Euler transform of a function $f(z)$ was defined as

$$B\left\{f(z); a, b\right\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\, dz \quad a, b \in \mathbb{C}, \Re\left(a\right) > 0, \Re\left(b\right) > 0. \tag{6}$$

Definition 2. Laplace Transform *(Sneddon [22]).*
The Laplace transform of a function $f(t)$, denoted by $F(s)$, was defined by the equation

$$F(s) = (Lf)(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)\, dt \quad \Re(s) > 0, \tag{7}$$

provided the integral (7) is convergent and that the function $f(t)$, is continuous for $t > 0$ and of exponential order as $t \to \infty$, (7) may be symbolically written as

$$F(s) = L\{f(t); s\} \, or f(t) = L^{-1}\{F(s); t\}. \tag{8}$$

Definition 3. Whittakar Transform *(Whittakar and Watson [26]).*

$$\int_0^\infty t^{\zeta-1} e^{-\frac{t}{2}} \mathcal{W}_{\tau,\omega}(t)\, dt = \frac{\Gamma\left(\frac{1}{2} + w + \zeta\right) \Gamma\left(\frac{1}{2} - w + \zeta\right)}{\Gamma(1 - \tau + \zeta)} \tag{9}$$

where $\Re(w \pm \zeta) > -1/2$ and $\mathcal{W}_{\tau,\omega}(t)$ is the Whittakar confluent hypergeometric function

$$\mathcal{W}_{\omega,\zeta}(t) = \frac{\Gamma(-2\omega)}{\Gamma\left(\frac{1}{2} - \tau - \omega\right)} M_{\tau,\omega}(t) + \frac{\Gamma(2\omega)}{\Gamma\left(\frac{1}{2} + \tau + \omega\right)} M_{\tau,-\omega}(t) \tag{10}$$

where $M_{\tau,\omega}(z)$ is defined by

$$M_{\tau,\omega}(t) = t^{1/2+\omega} e^{-1/2t} {}_1F_1\left(\frac{1}{2} + \omega - \tau; 2\omega + 1; t\right). \tag{11}$$

Definition 4. $\mathcal{K}$-Transform *(Erdélyi et al. [8]).*
This transform was defined by the following integral equation

$$\Re_v[f(x); p] = g[p; v] = \int_0^\infty (px)^{1/2} \mathcal{K}_v(px) f(x)\, dx \tag{12}$$

where $\Re(p) > 0; \mathcal{K}_v(x)$ is the Bessel function of the second kind defined by ([8], p. 332)

$$\mathcal{K}_v(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathcal{W}_{0,v}(2z)$$

where $W_{0,v}(.)$ is the Whittakar function defined in equation (9).
The following result given in Mathai et al. ([10], p. 54, Eq. 2.37) will be used in evaluating the integrals

$$\int_0^\infty t^{\rho-1} \mathcal{K}_v(at)\, dt = 2^{\rho-2} a^{-\rho} \Gamma\left(\frac{\rho \pm v}{2}\right); \Re(a) > 0; \Re(\rho \pm v) > 0. \tag{13}$$

Definition 5. *Fractional Fourier Transform* ([12]).
The fractional Fourier transform of order α, $0 < \alpha \leq 1$ is defined by

$$\widehat{u}_\alpha (\omega) = \Im_\alpha [u] (\omega) = \int_R e^{i\omega^{(1/\alpha)}t} u(t)\, dt. \tag{14}$$

When $\alpha = 1$, equation (14) reduces to the conventional Fourier transforms with some properties can be found in papers [1, 5, 11].

For our purpose, we recall the generalized Wright hypergeometric function $_p\psi_q(z)$ (see, for detail, Srivastava and Karlsson [23]), for $z, a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_i \in \mathbb{R}$, with $\alpha_i, \beta_j \neq 0$ $(i = 1, 2, ..., p; j = 1, 2, ..., q)$ defined as follows

$$_p\psi_q(z) = \,_p\psi_q \left[\begin{array}{c} (a_i, \alpha_i)_{1,p}; \\ (b_j, \beta_j)_{1,q}; \end{array} z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k) k!}. \tag{15}$$

The generalized Wright function was introduced by Wright [28] in the form of (15) under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{16}$$

2. Integral Transforms of $S_{\nu,c}^k(t)$

Further, we evaluate the following Euler transform, Laplace transform, Whittakar transform, $\mathcal{K}$-transform and Fractional Fourier transform of k-Struve functions.

Theorem 1. *Let $k \in \Re$, ν, c, r, s are complex numbers with $\nu > \frac{3}{2}k$, $\Re(r) > 0$ and $\Re(s) > 0$, then the following result holds true*

$$\int_0^1 t^{r-1} (1-t)^{s-1} S_{\nu,c}^k \left(x^{1/2} t^\sigma \right) dt = k^{-\frac{\nu}{k} - \frac{1}{2}} \left(\frac{\sqrt{x}}{2} \right)^{\frac{\nu}{k}+1} \Gamma(s)$$

$$\times \,_2\psi_3 \left[\begin{array}{c} \left(r + \sigma + \frac{\sigma\nu}{k}, 2\sigma \right), (1,1); \\ \left(\frac{\nu}{k} + \frac{3}{2}, 1 \right), \left(\frac{3}{2}, 1 \right), \left(r + \sigma + s + \frac{\sigma\nu}{k}, 2\sigma \right); \end{array} \frac{-cx}{4k} \right]. \tag{17}$$

Proof. Applying Eq. (5), on the left hand side of Theorem 1 denoted by I_1, we obtain

$$I_1 = \int_0^1 t^{r-1} (1-t)^{s-1} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k \left(nk + \nu + \frac{3}{2}k \right) \Gamma \left(n + \frac{3}{2} \right)} \left(\frac{x^{1/2} t^\sigma}{2} \right)^{2n + \frac{\nu}{k} + 1} dt$$

$$= \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n \int_0^1 t^{r+2\sigma n+\frac{\sigma\nu}{k}+\sigma-1}(1-t)^{s-1}\,dt,$$

By applying definition of Beta function, we have

$$I_1 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n B\left(r+2\sigma n+\frac{\sigma\nu}{k}+\sigma, s\right)$$

$$= \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{\Gamma\left(r+2\sigma n+\frac{\sigma\nu}{k}+\sigma\right)\Gamma(n+1)\Gamma(s)}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(r+2\sigma n+\frac{\sigma\nu}{k}+\sigma+s\right)n!} \left(\frac{-cx}{4}\right)^n,$$

In accordance with Eq. (4) and Eq. (15), we obtain the result (17). This completes the proof of the theorem.

Corollary 2. *For $k = c = 1$, Theorem 1 reduces in the following form*

$$\int_0^1 t^{r-1}(1-t)^{s-1} H_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} \Gamma(s)$$

$$\times \, _2\psi_3 \left[\begin{array}{c} (r+\sigma\nu+\sigma, 2\sigma),(1,1)\,; \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right)(r+\sigma\nu+\sigma+s,2\sigma)\,; \end{array} \frac{-x}{4} \right]. \tag{18}$$

Corollary 3. *For $k = 1$ and $c = -1$, Theorem 1 reduces in the following form*

$$\int_0^1 t^{r-1}(1-t)^{s-1} L_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} \Gamma(s)$$

$$\times \, _2\psi_3 \left[\begin{array}{c} (r+\sigma\nu+\sigma, 2\sigma),(1,1)\,; \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right)(r+\sigma\nu+\sigma+s,2\sigma)\,; \end{array} \frac{x}{4} \right]. \tag{19}$$

Theorem 4. *Suppose that $k \in \Re$, ν, c, s, σ are complex numbers with $\nu > \frac{3}{2}k$ and $\Re(s) > 0$, then the following result holds true*

$$\int_0^\infty e^{-st} S_{\nu,c}^k\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} k^{-\frac{\nu}{k}-\frac{1}{2}} s^{-\left(\frac{\sigma\nu}{k}+\sigma+1\right)}$$

$$\times \, _2\Psi_2 \left[\begin{array}{c} \left(1+\sigma+\frac{\sigma\nu}{k}, 2\sigma\right),(1,1)\,; \\ \left(\frac{\nu}{k}+\frac{3}{2},1\right),\left(\frac{3}{2},1\right)\,; \end{array} \frac{-cx}{4ks^{2\sigma}} \right]. \tag{20}$$

Proof. The left-hand side of Theorem 4 denoted by I_2. Using the definition of k-Struve function Eq. (5), we have

$$I_2 = \int_0^\infty e^{-st} \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma_k\left(nk + v + \frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{x^{\frac{1}{2}}t^\sigma}{2}\right)^{2n+\frac{\nu}{k}+1} dt$$

$$= \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^\infty \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n \int_0^\infty t^{2\sigma n+\frac{\sigma\nu}{k}+\sigma} e^{-st} dt$$

$$= \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^\infty \frac{\Gamma\left(2\sigma n+\frac{\sigma\nu}{k}+\sigma+1\right)}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n \ell\left\{\frac{t^{2\sigma n+\frac{\sigma\nu}{k}+\sigma}}{\Gamma\left(2\sigma n+\frac{\sigma\nu}{k}+\sigma+1\right)}\right\},$$

Now using the definition of Laplace transform, we have

$$I_2 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^\infty \frac{\Gamma\left(2\sigma n+\frac{\sigma\nu}{k}+\sigma+1\right)\Gamma(n+1)}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)n!} \left(\frac{-cx}{4s^{2\sigma}}\right)^n s^{-\left(\frac{\sigma\nu}{k}+\sigma+1\right)},$$

In view of Eq. (4) and Eq. (15), we arrive at the desired result Eq. (20).

Corollary 5. *On setting $k = c = 1$, Theorem 4 reduces in the following form*

$$\int_0^\infty e^{-st} H_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} s^{-(\sigma\nu+\sigma+1)}$$

$$\times {}_2\Psi_2 \left[\begin{array}{c} (\sigma\nu+\sigma+1,2\sigma),(1,1); \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right); \end{array} \frac{-x}{4s^{2\sigma}}\right]. \tag{21}$$

Corollary 6. *On setting $c = -1$ and $k = 1$, Theorem 4 reduces in the following form*

$$\int_0^\infty e^{-st} L_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} s^{-(\sigma\nu+\sigma+1)}$$

$$\times {}_2\Psi_2 \left[\begin{array}{c} (\sigma\nu+\sigma+1,2\sigma),(1,1); \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right); \end{array} \frac{x}{4s^{2\sigma}}\right]. \tag{22}$$

Theorem 7. *Suppose that $k \in \Re$, ν, c, λ are complex numbers with $\nu > \frac{3}{2}k$, $\Re(\omega \pm \lambda) > -1/2$ and $\Re(e) > |\Re(\omega)| > -1/2$, then the following integral holds true*

$$\int_0^\infty t^{\lambda-1} e^{-\frac{\alpha t}{2}} \, \mathcal{W}_{\tau,\omega}(\alpha t) S_{\nu,c}^k\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \alpha^{-\left(\frac{\sigma\nu}{k}+\lambda+\sigma\right)} k^{-\frac{\nu}{k}-\frac{1}{2}}$$

$$\times {}_3\psi_3 \left[\begin{array}{c} \left(\omega+\lambda+\sigma+\frac{\sigma\nu}{k}+\frac{1}{2},2\sigma\right),\left(-\omega+\lambda+\sigma+\frac{\sigma\nu}{k}+\frac{1}{2},2\sigma\right),(1,1); \\ \left(\frac{\nu}{k}+\frac{3}{2},1\right),\left(\frac{3}{2},1\right),\left(-\tau+\lambda+\sigma+1+\frac{\sigma\nu}{k},2\sigma\right); \end{array} \frac{-cx}{4k\alpha^{2\sigma}}\right]. \tag{23}$$

Proof. The left-hand side of Theorem 7 denoted by I_3. Let $\alpha t = y$, after interchanging the integration and summation, we obtain

$$I_3 = \alpha^{-\left(\lambda+\sigma\left(\frac{\nu}{k}+1\right)\right)} \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4\alpha^{2\sigma}}\right)^n$$

$$\times \int_0^\infty y^{\lambda+2\sigma n+\frac{\sigma\nu}{k}+\sigma-1} e^{-\frac{\lambda}{2}}\, \mathcal{W}_{\tau,\omega}(y)\, dy,$$

Now using the Whittakar transformation Eq. (9), we arrive at

$$I_3 = \alpha^{-\left(\lambda+\sigma\left(\frac{\nu}{k}+1\right)\right)} \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{\Gamma\left(\omega+\lambda+\sigma+2\sigma n+\frac{\sigma\nu}{k}+\frac{1}{2}\right)}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)}$$

$$\times \frac{\Gamma\left(-\omega+\lambda+\sigma+2\sigma n+\frac{\sigma\nu}{k}+\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\lambda-\tau+\frac{\sigma\nu}{k}+\sigma+1+2\sigma n\right) n!} \left(\frac{-cx}{4\alpha^{2\sigma}}\right)^n,$$

In view of Eq. (4) and Eq. (15), we arrive at the desired result Eq. (23).

Corollary 8. *If we set $k = c = 1$, then the formula Eq. (23) reduces in the following form*

$$\int_0^\infty t^{\lambda-1} e^{-\frac{\alpha t}{2}}\, \mathcal{W}_{\tau,\omega}(\alpha t)\, H_\nu\left(x^{1/2}t^\sigma\right) dt = \alpha^{-(\lambda+\sigma(\nu+1))} \left(\frac{\sqrt{x}}{2}\right)^{\nu+1}$$

$$\times {}_3\psi_3 \left[\begin{array}{c} \left(\omega+\lambda+\sigma\nu+\sigma+\frac{1}{2},2\sigma\right),\left(-\omega+\lambda+\sigma\nu+\sigma+\frac{1}{2},2\sigma\right),(1,1)\,; \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right),(-\tau+\lambda+\sigma\nu+\sigma+1,2\sigma)\,; \end{array} \frac{-x}{4\alpha^{2\sigma}}\right].$$
(24)

Corollary 9. *If we set $c = -1$ and $k = 1$, then the formula Eq. (23) reduces in the following form*

$$\int_0^\infty t^{\lambda-1} e^{-\frac{\alpha t}{2}}\, \mathcal{W}_{\tau,\omega}(\alpha t)\, L_\nu\left(x^{1/2}t^\sigma\right) dt = \alpha^{-(\lambda+\sigma(\nu+1))} \left(\frac{\sqrt{x}}{2}\right)^{\nu+1}$$

$$\times {}_3\psi_3 \left[\begin{array}{c} \left(\omega+\lambda+\sigma\nu+\sigma+\frac{1}{2},2\sigma\right),\left(-\omega+\lambda+\sigma\nu+\sigma+\frac{1}{2},2\sigma\right),(1,1)\,; \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right),(-\tau+\lambda+\sigma\nu+\sigma+1,2\sigma)\,; \end{array} \frac{x}{4\alpha^{2\sigma}}\right].$$
(25)

Theorem 10. *Assume that $k \in \Re$, ν, c, λ are complex numbers with $\nu > \frac{3}{2}k$, $\Re(\omega) > 0$ and $|\Re(\lambda \pm \tau)| > 0$, then the following integral holds true*

$$\int_0^\infty t^{\lambda-1} \mathcal{K}_\tau(\omega t)\, S_{\nu,c}^k\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} 2^{\frac{\sigma\nu}{k}+\lambda+\sigma-2}\, \omega^{-\left(\frac{\sigma\nu}{k}+\lambda+\sigma\right)} k^{-\frac{\nu}{k}-\frac{1}{2}}$$

$$\times \; _3\psi_2 \left[\begin{array}{c} \left(\frac{\tau+\lambda+\sigma\left(\frac{\nu}{k}+1\right)}{2},\sigma\right), \left(\frac{\lambda-\tau+\sigma\left(\frac{\nu}{k}+1\right)}{2},\sigma\right), (1,1) \; ; \\ \left(\frac{\nu}{k}+\frac{3}{2},1\right), \left(\frac{3}{2},1\right) \; ; \end{array} \; \frac{-cx}{4^{1-\sigma}k\,\omega^{2\sigma}} \right]. \quad (26)$$

Proof. The left-hand side of Theorem 10 denoted by I_4. Using the definition of k-Struve function Eq. (5), we have

$$I_4 = \int_0^\infty t^{\lambda-1}\mathcal{K}_\tau\left(\omega t\right) \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{x^{1/2}t^\sigma}{4}\right)^{2n+\frac{\nu}{k}+1} dt,$$

By applying an important simplification, this equation becomes

$$= \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^\infty \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n \int_0^\infty t^{\lambda+2\sigma n+\frac{\nu}{k}+\sigma-1}\mathcal{K}_\tau\left(\omega t\right) dt,$$

Then applying formula Eq. (13), we arrive at

$$I_4 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} 2^{\lambda+\sigma\left(\frac{\nu}{k}+1\right)-2}\omega^{-\left(\lambda+\sigma\left(\frac{\nu}{k}+1\right)\right)}$$

$$\times \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\lambda+\sigma\left(\frac{\nu}{k}+1\right)+\tau+2\sigma n}{2}\right)\Gamma\left(\frac{\lambda+\sigma\left(\frac{\nu}{k}+1\right)-\tau+2\sigma n}{2}\right)\Gamma\left(n+1\right)}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)n!} \left(\frac{-cx}{4^{1-\sigma}\omega^{2\sigma}}\right)^n,$$

In view of Eq. (4) and Eq. (15), we arrive at the desired result (26).

Corollary 11. *Taking $k=c=1$, then the formula Eq. (26) reduces in the following form*

$$\int_0^\infty t^{\lambda-1}\mathcal{K}_\tau\left(\omega t\right)H_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} 2^{\lambda+\sigma(\nu+1)-2}\omega^{-(\lambda+\sigma(\nu+1))}$$

$$\times \; _3\psi_2 \left[\begin{array}{c} \left(\frac{\lambda+\sigma(\nu+1)+\tau}{2},\sigma\right), \left(\frac{\lambda+\sigma(\nu+1)-\tau}{2},\sigma\right), (1,1) \; ; \\ \left(\nu+\frac{3}{2},1\right), \left(\frac{3}{2},1\right) \; ; \end{array} \; \frac{-x}{4^{1-\sigma}\omega^{2\sigma}} \right]. \quad (27)$$

Corollary 12. *Taking $c=-1$ and $k=1$, then the formula Eq. (26) reduces in the following form*

$$\int_0^\infty t^{\lambda-1}\mathcal{K}_\tau\left(\omega t\right)L\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} 2^{\lambda+\sigma(\nu+1)-2}\omega^{-(\lambda+\sigma(\nu+1))}$$

$$\times \; _3\psi_2 \left[\begin{array}{c} \left(\frac{\lambda+\sigma(\nu+1)+\tau}{2},\sigma\right), \left(\frac{\lambda+\sigma(\nu+1)-\tau}{2},\sigma\right), (1,1) \; ; \\ \left(\nu+\frac{3}{2},1\right), \left(\frac{3}{2},1\right) \; ; \end{array} \; \frac{x}{4^{1-\sigma}\omega^{2\sigma}} \right]. \quad (28)$$

Theorem 13. *Suppose that $k \in \Re$, ν, c are complex numbers with $\nu > \frac{3}{2}k$, then the following integral holds true*

$$\int_R e^{i\omega^{\frac{1}{\beta}}t} S_{\nu,c}^k \left(x^{1/2} t^\sigma\right) dt = k^{-\frac{\nu}{k}-\frac{1}{2}} \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{(-1)^{2\sigma n + \frac{\sigma\nu}{k}+\sigma}}{i^{2\sigma n + \frac{\sigma\nu}{k}+\sigma+1} \omega^{\left(2\sigma n + \frac{\sigma\nu}{k}+\sigma+1\right)/\beta}}$$

$$\times\, {}_2\psi_2 \left[\begin{array}{c} \left(\sigma\left(\frac{\nu}{k}+1\right)+1, 2\sigma\right), (1,1)\, ; \\ \left(\frac{\nu}{k}+\frac{3}{2}, 1\right), \left(\frac{3}{2}, 1\right)\, ; \end{array} \frac{-cx}{4k} \right]. \tag{29}$$

Proof. The left-hand side of Theorem 13 denoted by I_5. Using the definition of k-Struve function Eq. (5), we have

$$I_5 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n \int_R e^{i\omega^{\frac{1}{\beta}}t} t^{2\sigma n + \frac{\sigma\nu}{k}+\sigma}\, dt,$$

By substituting $i\omega^{\frac{1}{\beta}}t = -\mu$, then above equation can be written as

$$I_5 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-cx}{4}\right)^n$$

$$\times \int_{-\infty}^{0} e^{-\mu} \left(\frac{-\mu}{i\omega^{\frac{1}{\beta}}}\right)^{2\sigma n + \frac{\sigma\nu}{k}+\sigma} \left(\frac{-1}{i\omega^{\frac{1}{\beta}}}\right) d\mu,$$

By applying important simplification on the above expression, we arrive at

$$I_5 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{(-1)^{2\sigma n + \frac{\sigma\nu}{k}+\sigma}}{i^{2\sigma n + \frac{\sigma\nu}{k}+\sigma+1} \omega^{\left(2\sigma n + \frac{\sigma\nu}{k}+\sigma+1\right)/\beta} \Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)}$$

$$\times \left(\frac{-cx}{4}\right)^n \int_0^{-\infty} e^{-\mu} \mu^{2\sigma n + \frac{\sigma\nu}{k}+\sigma}\, d\mu$$

Then apply the definition of Laplace Transform on the integral form, we get

$$I_5 = \left(\frac{\sqrt{x}}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{(-1)^{2\sigma n + \frac{\sigma\nu}{k}+\sigma}}{i^{2\sigma n + \frac{\sigma\nu}{k}+\sigma+1} \omega^{\left(2\sigma n + \frac{\sigma\nu}{k}+\sigma+1\right)/\beta}}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma\left(2\sigma n + \frac{\sigma\nu}{k}+\sigma+1\right)\Gamma\left(n+1\right)}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)n!} \left(\frac{-cx}{4}\right)^n$$

Finally, interpreting this equation by virtue of Eq. (4) and Eq. (15), we arrive at the desired result Eq. (29).

Corollary 14. *If we set $k = c = 1$, then the formula Eq. (29) reduces in the following form*

$$\int_R e^{i\omega^{\frac{1}{\beta}}t} H_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} \sum_{n=0}^\infty \frac{(-1)^{2\sigma n+\sigma\nu+\sigma}}{i^{2\sigma n+\sigma\nu+\sigma+1}\omega^{(2\sigma n+\sigma\nu+\sigma+1)/\beta}}$$

$$\times\ {}_2\psi_2\left[\begin{array}{c} \left(\sigma\left(\nu+1\right)+1,2\sigma\right),\left(1,1\right);\ \frac{-x}{4} \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right);\quad \end{array}\right]. \tag{30}$$

Corollary 15. *If we set $c = -1$ and $k = 1$, then the formula Eq. (29) reduces in the following form*

$$\int_R e^{i\omega^{\frac{1}{\beta}}t} L_\nu\left(x^{1/2}t^\sigma\right) dt = \left(\frac{\sqrt{x}}{2}\right)^{\nu+1} \sum_{n=0}^\infty \frac{(-1)^{2\sigma n+\sigma\nu+\sigma}}{i^{2\sigma n+\sigma\nu+\sigma+1}\omega^{(2\sigma n+\sigma\nu+\sigma+1)/\beta}}$$

$$\times\ {}_2\psi_2\left[\begin{array}{c} \left(\sigma\left(\nu+1\right)+1,2\sigma\right),\left(1,1\right);\ \frac{x}{4} \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right);\quad \end{array}\right]. \tag{31}$$

Theorem 16. *Suppose that $k \in \Re$, ν, c are complex numbers with $\nu > \frac{3}{2}k$ and $0 < \alpha \leq 1$, then the following integral holds true*

$$\Im_\alpha\left[S_{\nu,c}^k(t)\right](\omega) = \left(\frac{1}{2}\right)^{\frac{\nu}{k}+1} k^{-\frac{\nu}{k}-\frac{1}{2}} \sum_{k=0}^\infty \frac{(-1)^{2n+\frac{\nu}{k}+1}}{i^{2n+\frac{\nu}{k}+2}\omega^{\left(2n+\frac{\nu}{k}+2\right)/\alpha}}$$

$$\times\ {}_2\Psi_2\left[\begin{array}{c} \left(\frac{\nu}{k}+2,2\right),\left(1,1\right);\ \frac{-c}{4k} \\ \left(\frac{\nu}{k}+\frac{3}{2},1\right),\left(\frac{3}{2},1\right);\quad \end{array}\right]. \tag{32}$$

Proof. Using Eqs. (5) and (14), in the left hand side of Eq. (32) gives

$$I_6 = \int_R e^{i\omega^{(1/\alpha)}t} \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{t}{2}\right)^{2n+\frac{\nu}{k}+1} dt$$

$$= \left(\frac{1}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^\infty \frac{1}{\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-c}{4}\right)^n \int_R e^{i\omega^{(1/\alpha)}t} t^{2n+\frac{\nu}{k}+1} dt,$$

Setting $i\omega^{1/\alpha}t = -\rho$, then becomes

$$= \left(\frac{1}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^\infty \frac{(-1)^{2n+\frac{\nu}{k}+1}}{i^{2n+\frac{\nu}{k}+2}\omega^{\left(2n+\frac{\nu}{k}+2\right)/\alpha}\Gamma_k\left(nk+\nu+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{-c}{4}\right)^n$$

86

$$\times \int_R e^{-\rho} \rho^{2n+\frac{\nu}{k}+1} d\rho,$$

Now, applying Eq. (4), we obtain

$$= \left(\frac{1}{2}\right)^{\frac{\nu}{k}+1} \sum_{n=0}^{\infty} \frac{(-1)^{2n+\frac{\nu}{k}+1} k^{-\frac{\nu}{k}-\frac{1}{2}}}{i^{2n+\frac{\nu}{k}+2} \omega^{(2n+\frac{\nu}{k}+2)/\alpha}} \sum_{n=0}^{\infty} \frac{\Gamma\left(2n+\frac{\nu}{k}+2\right)\Gamma(n+1)}{\Gamma\left(n+(\nu/k)+\frac{3}{2}k\right)\Gamma\left(n+\frac{3}{2}\right) n!} \left(\frac{-c}{4k}\right)^n,$$

Finally, interpreting this equation by virtue of Eq. (15), we arrive at the desired result Eq. (32).

Corollary 17. *If we take $k = c = 1$, then the formula Eq. (32) reduces in the following form*

$$\Im_\alpha \left[H_\nu(t)\right](\omega) = \left(\frac{1}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^{2n+\nu+1}}{i^{2n+\nu+2} \omega^{(2n+\nu+2)/\alpha}} {}_2\Psi_2 \left[\begin{array}{c} (\nu+2,2),(1,1); \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right); \end{array} \frac{-1}{4}\right].$$

$$(33)$$

Corollary 18. *If we take $c = -1$ and $k = 1$, then the formula Eq. (32) reduces in the following form*

$$\Im_\alpha \left[L_\nu(t)\right](\omega) = \left(\frac{1}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^{2n+\nu+1}}{i^{2n+\nu+2} \omega^{(2n+\nu+2)/\alpha}} {}_2\Psi_2 \left[\begin{array}{c} (\nu+2,2),(1,1); \\ \left(\nu+\frac{3}{2},1\right),\left(\frac{3}{2},1\right); \end{array} \frac{1}{4}\right].$$

$$(34)$$

Lastly, we conclude this paper by remarking that, the integral transform formulas deduced in this paper, for k-Struve functions, are significant and can lead to yield numerous transforms for variety of Struve functions. The transforms established here are general in nature and are likely to find useful in applied problem of sciences, engineering and technology.

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