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# ON CO-FILTERS OF IMPLICATIVE SEMIGROUPS WITH APARTNESS

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Abstract. The setting of this research is the Bishop's constructive mathematics - a mathematics based on the Intuitionistic Logic and principled-philosophical orientation of Bishop's mathematics. Implicative semigroups with apartness introduced and analyzed by this author in his two recently published articles. In this paper, as a continuation of the research [26, 27], a description of co-filters was made in an implicative semigroup with apartness using one class of special subsets of such semi-group.

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## 1. Introduction

The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [9]. For the first generalization of implicative semilattice see Nemitz [20] and Blyth [7]. Moreover, there exists a close relationship between implicative semigroups and other domains. For example, there is a lot of implications in mathematical logic and set theory (see Birkhoff [6]). For the general development of implicative semilattice theory, the ordered filters play an important role. It has been shown by Nemitz [20]. Motivated by this, Chan and Shum [9] established some elementary properties and constructed quotient structure of implicative semigroups via ordered filters. Jun [13, 14], Jun, Meng and Xin [15], Jun and Kim [16] and Lee, Shum and Wu [17, 18] discussed ordered filters of implicative semigroups. Bang and So [1] analyzed some special substructures in implicative semigroups.

In paper [26], in setting of Bishop's constructive mathematics, following the ideas of Chan and Shum and other authors mentioned above, the author introduced the notion of implicative semigroups with (tight) apartness and gave some fundamental characterization of these semigroups. In [26] and in this article, using sets with apartness and co-order relation introduced by the author, instead of partial order. See for example [10, 22, 23, 24, 25, 30]. In this case, it is an excise relation, researched by Barony [2], Greenleaf [12], Negri [21] and von Plato [32]. So, in this research, the author studied side effects induced by existence of apartness and co-order. Additionally, in [26] the author introduced the notion of co-filter in an implicative semigroup and described its connections with filter. Further, in [27] he analyzed a connection between co-filters and strongly extensional homomorphisms of implicative semigroup with apartness.

In this article, as a continuation of his mentioned articles [26, 27], the author discuss about some forms of ordered co-filters of such semigroups.

## 2. Preliminaries

### 2.1. Bishop's constructive orientation

This investigation is in Bishop's constructive algebra in a sense of papers [10, 11, 22, 23, 24, 23, 28, 29, 30] and books [3, 4, 5, 8], [31](Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). The diversity relation "$\neq$" ([4]) is a binary relation on $S$, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z\ .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z)),$$

then, it is called apartness (A. Heyting). In this paper, is assume that the basic apartness is tight, i.e. that it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

For subset $X$ of $S$, we say that it is a strongly extensional subset of $S$ if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$ holds. For subsets $X$ and $Y$ of $S$, it is said that the subset $X$ is set-set apart from the subset $Y$, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. It's labeled like this $x \lhd Y$, instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^{\lhd} = \{x \in S : x \lhd X\}$ is denoted strong complement of $X$. A subset $G$ of set $(S, =, \neq)$ is a detachable subset of $S$ if $(\forall x \in S)(x \in G \vee x \lhd G)$ holds.

For a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ is says that it is strongly extensional if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b).$$

For relation $\alpha \subseteq S \times S$, it is says that it is an co-order relation on semigroup $S$, if it is consistent, cotransitive and linear

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1},$$

where $\alpha$ has to be compatible with the semigroup operation in the following way

$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \Longrightarrow (x, y) \in \alpha).$$

Speaking the language of the classical algebra, the relation $\alpha$ is left and right cancellative. Here, $*$ is the filed product between relations defined by the following way: If $\alpha$ and $\beta$ are relations on set $S$, then filed product $\beta * \alpha$ of relation $\alpha$ and $\beta$ is the relation given by $\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in \alpha \vee (y, z) \in \beta)\}$. In some earlier published texts, the alternative name used for this relation was 'anti-order relation'. Both names will be used throughout this text.

For undefined notions and notations and used in this article, the reader can look in some of the following articles [10, 11, 22, 23, 24, 25, 26, 27, 28, 29, 30].

## 2.2. Implicative semigroup with apartness

In this subsection, some definitions and the necessary results will be repeated. When it comes to a *negatively anti-ordered* semigroup (briefly, n.a-o. semigroup) ([26, 27]), then it is meant a set $S$ with a co-order $\alpha$ and a binary internal operation '$\cdot$' (it writed $xy$ instead of $x \cdot y$) such that for all $x, y, z \in S$ the following holds:

(1) $(xy)z = x(yz)$,

(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and

(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In that case for co-order (anti-order) $\alpha$ we will say that it is a *negative anti-order relation* on semigroup. The operation '$\cdot$' is extensional and strongly extensional function from $S \times S$ into $S$, i.e. it has to be

$$(x, y) = (x', y') \Longrightarrow xy = x'y'$$
$$(xy \neq x'y \vee yx \neq yx') \Longrightarrow x \neq x'$$

for any elements $x, x', y, y'$ of $S$.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$ for any elements $x, y, z$ of $S$.

In addition, let us recall that the internal binary operation '$\otimes$' must satisfy the following implications:

$$(a, b) = (u, v) \Longrightarrow a \otimes b = u \otimes v,$$
$$a \otimes b \neq u \otimes v \Longrightarrow (a, b) \neq (u, v).$$

The operation $\otimes$ is called *implication*. From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup*.

An implicative semigroup is to be *commutative* if it satisfies the following condition

$$(\forall x \in S)(\forall y \in S)(x \cdot y = y \cdot x).$$

Let $\alpha$ be a relation on $S$. For an element $a$ of $S$ we put $a\alpha = \{x \in S : (a, x) \in \alpha\}$ and $\alpha a = \{x \in S : (x, a) \in \alpha\}$. In the following proposition we give some properties of negative anti-order relation on semigroup.

**Theorem 1.** ([26], Theorem 3.1) *If $\alpha \subseteq S \times S$ is an anti-order relation on a semigroup $S$, then the following statements are equivalent:*
(i) *$\alpha$ is a negative co-order relation;*
(ii) *$\alpha b$ for any $b$ in $S$ has the following properties:*
$xy \in \alpha b \Longrightarrow x \in \alpha b \wedge y \in \alpha b,$
$x \in \alpha b \Longrightarrow (x, y) \in \alpha \vee y \in \alpha b;$
(iii) $(\forall a, b \in S)(\alpha a \cup \alpha b \subseteq \alpha(ab));$
(iv) *$a\alpha$ is an ideal of $S$ for any $a$ in $S$;*
(v) $(\forall a, b \in S)((ab)\alpha \subseteq a\alpha \cap b\alpha).$

In any implicative semigroup $S$ there exist a special element of $S$, the biggest element in $(S, \alpha^{\triangleleft})$, which is the left neutral element in $(S, \cdot)$.

Some elementary properties of semigroup with apartness are given in the following proposition ([26], Theorem 3.3, Theorem 3.4, Corollary 3.2 and Corollary 3.3).

**Theorem 2.** (a) $(\forall x \in S)(x \otimes x = 1)$;
(b) $(\forall x \in S)(\forall y \in S)((x, y) \in \alpha \Longleftrightarrow 1 \neq x \otimes y)$;
(c) $(\forall x \in S)(1 = x \otimes 1)$ *and* $(\forall x \in S)(x = 1 \otimes x)$.

### 2.3. Ordered co-filters

In this subsection the author reminds readers on the concept of co-filters of an implicative semigroup with apartness ([26, 27]): A subset $G$ of $S$ is called *ordered co-filter* if the following holds:
(G1) $xy \in G \Longrightarrow x \in G \vee y \in G$, that is, $G$ is a cosubsemigroup of $S$ and
(G2) $y \in G \Longrightarrow (y, x) \in \alpha^{-1} \vee x \in G$.

It is easy to check that anti-filter is a strongly extensional subset of $S$. Moreover, strong compliment $G^{\triangleleft}$ of an co-filter $G$ is a filter in $S$.

The following theorem gives equivalent conditions of ordered co-filters.

**Theorem 3.** ([26], Theorem 3.7) *An inhabited proper subset $G$ of an implicative semigroup $S$ is an ordered co-filter of $S$ if and only if it satisfies the following conditions:*
(G3) $1 \lhd G$;
(G4) $(\forall x \in S)(\forall y \in S)(y \in G \Longrightarrow x \otimes y \in G \vee x \in G)$.

It is easily checked that the union of each family of ordered co-filters of an implicative semi-group with apartness is also an ordered co-filter of that semigroup.

The subset $\{1\}^{\lhd}$ is the maximal ordered anti-filter of implicative semigroup $S$. Indeed. Obviously $1 \bowtie \{1\}^{\lhd}$. Let $y \in \{1\}^{\lhd}$. Thus, from $y \neq 1$ follows $y \neq x \otimes y$ or $x \otimes y \neq 1$. The second option gives $x \otimes y \in \{1\}^{\lhd}$. From the first option $1 \otimes y = y \neq x \otimes y$ follows $x \in \{1\}^{\lhd}$. So, the subset $\{1\}^{\lhd}$ is an ordered anti-filter of $S$ by Theorem 3.

## 3. The Main Results

Firstly, the author introduces (Definition 1) a class of special subsets of the implicit semigroup that naturally occurs (Theorem 4) in such semigroups. The class of these subsets allows to some determination of ordered co-filters by means of its elements.

**Definition 1.** *Let $S$ be an implicative semigroup ordered under an anti-order $\alpha$ and let $a_1, ..., a_n \in S$ and $n \in \mathbb{N}$. We define*

$$L_1(a_1) = \{y \in S : a_1 \otimes y \neq 1\} = \{y \in S : (a_1, y) \in \alpha\} = a_1\alpha,$$
$$L_n(a_n, ..., a_1) = \{y \in S : a_n \otimes (... \otimes (a_1 \otimes y)...) \neq 1\}.$$

By the following two propositions, gives a partly description of these subsets.

**Proposition 1.** $1 \lhd L_1(a_1)$, *and* $a_1 \lhd L_1(a_1)$ *for any* $a_1 \in L$.

*Proof.* Indeed, if $y \in L_1(a_1)$, then $a_1 \otimes y \neq 1$. Thus, from $a_1 \otimes y \neq 1 = a_1 \otimes 1$ follows $y \neq 1$. Therefore, $1 \lhd L_1(a_1)$. Further on, from $a_1 \otimes y \neq 1 = a_1 \otimes a_1$ follows $y \neq a_1$. So, $a_1 \lhd L_1(a_1)$.

**Proposition 2.** $L_n(a_n, a_{n-1}..., a_1) \subseteq L_{n-1}(a_{n-1}, ..., a_1) \subseteq ... \subseteq L_1(a_1)$.

*Proof.* Frstly, we have $L_2(a_2, a_1) \subseteq L_1(a_1)$. Suppose $y \in L_2(a_2, a_1)$. Then $a_2 \otimes (a_1 \otimes y) \neq 1 = a_2 \otimes 1$. Thus, $a_1 \otimes y \neq 1$. So, $y \in L_1(a_1)$. Finaly, we have

$$L_n(a_n, a_{n-1}..., a_1) \subseteq L_{n-1}(a_{n-1}, ..., a_1) \subseteq ... \subseteq L_2(a_2, a_1) \subseteq L_1(a_1).$$

by induction.

By Proposition 1 and Proposition 2 subsets $L_1(a_1), ..., L_n(a_n, ..., a_n)$ are the proper subsets of the semigroup $S$.

On the other hand, for each subset $A$ of semigroup $S$ there is the maximal ordered co-filter contained in $A$. Indeed, if $\mathfrak{F}$ is the family of all ordered co-filters contained in $A$, then $\bigcup \mathfrak{F}$ is an co-filter of $S$ also contained in $A$. So, for any natural $n$ and any subset $L_n(a_n, ..., a_1)$, generated by elements $a_n, ..., a_1 \in S$, there is the maximal ordered co-filter included in $L_n(a_n, ..., a_1)$. The validity of this assertion is established by simple direct checking of conditions (G3) and (G4). It is quite natural to analyze the question: When will this co-filter be equal to this set or, in other words, when this set will be a co-filter?

The first important result in this article is given by the following theorem.

**Theorem 4.** *For any ordered co-filter $G$ of a implicative semigroup $S$ with apartness, holds*

$$(\forall n \in \mathbb{N})(\forall a_1, ..., a_n \in S)(a_1 \lhd G, ..., a_n \lhd G)(G \subseteq L_n(a_n, ..., a_1)).$$

*Proof.* Let $G$ be an ordered co-filter of an implicative semigroup with apartness ordered under an co-order $\alpha$ an $a_1$ be an arbitrary element of $S$ such that $a_1 \lhd G$. Then, by definition of co-filter, the implication $y \in G \implies a_1 \otimes y \in G$ is valid. Also, $a_1 \otimes y \neq 1$ holds. So, $G \subseteq L_1(a_1)$ holds, also. Suppose that the inclusion $G \subseteq L_{n-1}(a_{n-1}, ..., a_1)$ is valid. Also, suppose that $a_{n-1} \otimes (... \otimes (a_1 \otimes y)...) \in G$ holds. Thus, again by determination of the co-filter, for the element $a_n \in S$ such that $a_n \lhd G$, finally is valid $a_n \otimes (... \otimes (a_1 \otimes y)...) \neq 1$. Therefore, $G \subseteq L_n(a_n, ..., a_1)$.

In what following, results expressed in two next lemmas will be required.

**Lemma 5.** *If $S$ is a commutative implicative semigroup, then for any bijection $\varphi : \{1, ..., n\} \longrightarrow \{1, ..., n\}$ holds $L_n(a_n, ..., a_1) = L_n(a_{\varphi(n)}, ..., a_{\varphi(1)})$.*

*Proof.* The proof immediately follows from the fact

$$y \in L_n(a_n, ..., a_1) \iff (a_n \cdot ... \cdot a_1, y) \in \alpha$$

and commutativity of the semigroup $S$

**Lemma 6.** *If $S$ is commutative, then $a_i \lhd L(a_n, ..., a_1)$ for all $i \in \{1, ..., n\}$.*

*Proof.* It is valid $a_1 \lhd L_1(a_1) \supseteq L_n(a_n, ..., a_1)$ by Proposition 1 and Proposition 2. Let $y \in L(b, a) = L_2(b, a)$, i.e. let $b \otimes (a \otimes y)) \neq 1$. It means $(b, a \otimes y) \in \alpha$. This is equivalent with $(ba, y) \in \alpha$ and $(ab, y) \in \alpha$ because $S$ is commutative. Hence, from $a \otimes (b \otimes y) \neq 1 = a \otimes 1$ follows $b \otimes y \neq 1 = b \otimes b$. Finally, $y \neq b$. Therefore, $b \lhd L(b, a)$.

The proof of this lemma is obtained by induction.

From Proposition 1 is immediately obtained $\bigcup_{a_2 \in S} L_2(a_2, a_1) \subseteq L_1(a_1)$. It can prove that the reverse inclusion also holds if the semigroup $S$ is commutative.

**Proposition 3.** *If $S$ is commutative and $a \in S$, then $L_1(a) = \bigcup_{b \in S} L_2(b, a)$.*

*Proof.* Let $t \in L_1(a)$ be an arbitrary element and let $b \in S$. Then, from $a \otimes t \neq 1$, by cotransitivity, holds

$$a \otimes t \neq b \otimes (a \otimes t) \ \vee \ b \otimes (a \otimes t) \neq 1.$$

Suppose $a \otimes t \neq b \otimes (a \otimes t)$. Thus $a \otimes t \neq a \otimes (b \otimes t)$ by commutativity of $S$ and $t \neq b \otimes t$. The last inequality means $(t, b \otimes t) \in \alpha$, i.e. $(t \cdot b, t) \in \alpha$. This is in contradiction with (3). Therefore, it has to be $t \in L_2(b, a)$.

The sets $L_n(a_n, ..., a_1)$ gives an description (as the second important result in this article) of an ordered co-filter of commutative implicative semigroup $S$ in a special case when the $G$ is a detechable subset of $S$.

**Theorem 7.** *Let $G$ be a detachable subset of a commutative implicative semigroup $S$. Then $G$ is an ordered co-filter of $S$ if and only if $G \subseteq L_n(a_n, ..., a_1)$ for any $a_n, ..., a_1 \in G^{\lhd}$ where $n \in \mathbb{N}$.*

*Proof.* First, if $G$ is an ordered co-filter of $S$, theb holds $G \subseteq L_n(a_n, ..., a_1)$ for any $a_n, ..., a_1 \in G^{\lhd}$ by Theorem 1.

Conversely, suppose that $G \subseteq L_n(a_n, ..., a_1)$ holds for any $a_n, ..., a_1 \in G^{\lhd}$. It is holds $1 \lhd G$ by Proposition 1 and Proposition 2. Let $y \in G$ and $x$ be an arbitrary element of $S$. Then $x \in G$ or $x \lhd G$ because $G$ is a detachable subset of $S$. In the case $x \lhd G$, from $y \in G \subseteq L_n(1, ..., 1, x)$ we have $(x, y) \in \alpha$. Therefore, the detachable subset $G$ is an ordered co-filter of $S$ since the condition (G4) is satisfied.

An affirmative answer to the above-mentioned question can be obtained if certain specific implicit semigroups are observed. An implicative semigroup $S$ is said to be left self-distributive ([13]) if

$$x \otimes (y \otimes z) = (x \otimes y) \otimes (x \otimes z)$$

for any $x, y, z \in S$. An example of such semigroup one can find in article [13].

**Theorem 8.** *Let $S$ be an implicative semigroup with apartness satisfies the left self-distributive law under the operation $'\otimes'$. Then the set $L_n(a_n, ..., a_1)$ is an ordered co-filter of $S$ for all $a_n, ..., a_1 \in S$.*

*Proof.* Put $L(b,a) = L_2(b,a)$. It is clear that $1 \lhd L(b,a)$. Let $y \in L(b,a)$ and let $x$ be an arbitrary element of $S$. Then, $b \otimes (a \otimes y) \neq 1$. Thus, $b \otimes (a \otimes y) \neq b \otimes (a \otimes (x \otimes y))$ or $b \otimes (a \otimes (x \otimes y)) \neq 1$. The second part means $x \otimes y \in L(b,a)$. Now,

$$1 \otimes (b \otimes (a \otimes y)) \neq (b \otimes (a \otimes x)) \otimes (b \otimes (a \otimes y))$$

by left self distributivity and $1 \neq b \otimes (a \otimes x)$. So, $x \in L(b,a)$. Therefore, $L(b,a)$ is an ordered co-filter of $S$. So, the set $L_n(a_n, ..., a_1)$ is an ordered co-filter of $S$ for all $a_n, ..., a_1 \in S$ by induction.

## 4. Final Observation

Bishop's constructive mathematics includes the following two aspects:

(1) The Intuitionistic logic and

(2) The principled-philosophical orientations of constructivism.

Intuitionistic logic does not accept the TND principle as an axiom. In addition, Intuitionistic logic does not accept the validity of the 'double negation' principle. This makes it possible to have a difference relation in sets which is not a negation of equality relation. Therefore, we accept that in Bishop's constructive mathematics we consider set $A$ as one relational system $(A, =, \neq)$. In Bishop's constructive algebra we always encounter the following two problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined;

(b) Since every predicate has at least one of its duals, how to construct a dual of the algebraic concept defined with a given predicate(s).

Constructive algebra is not only a different way of observing algebraic structures and their interactions. The specificity of Intuitionistic logics make it possible to identify, understand and analyze the observed types of algebraic structures as specific relational systems. In the texts [10, 11, 22, 23, 25], the author alone or in collaboration with his associates analyzes semigroups with apartness. In such semigroups, among other things, the object of research is also the relations of co-order and co-quasiorder as logical dual(s) of order (and quasi-order relations) in semigroups. The existence of these special order relations in semigroups with apartness and their compatibility with internal binary operation in such semigroups provides some new substructures in such semigroups that do not have an analogy in the classical semigroup theory. The text [30] provides a retrospective of some algebraic structures ordered under co-quasiorder relation. Particulary, this author is researched semilattice-ordered semigroups with apartness in his papers [24, 28]. The object of the author's research in the article [29] is semilattice-ordered semiring with apartnes. In all the mentioned algebraic structures, the author observed the specific concepts of the co-ideals and co-filters substructures whose analogue does not exist in the classical algebra. These logical possibilities strongly increase the complexity and the number of substructures of any algebraic structure.

In this article, as a continuation of his research started in articles [26, 27], the author tries to describe in more detail the concept of co-filters in implicative semigroups with apartness. To that end, the author uses a special class of subsets that naturally appear in such semi-groups. The results are obtained by using of some special features of the classical and constructive algebras and have no parallel in the classical semigroup theory.

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