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THE UNIVALENCE CONDITIONS FOR A GENERAL INTEGRAL OPERATOR

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Abstract. In this paper, we obtain univalence conditions for a new general integral operator defined on the space of normalized analytic function in the open unit disk U. Some corollaries follow as special cases.

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1. Introduction

Let $\mathcal{A}$ be the class of the functions f which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} :\mid z \mid< 1\}$ and $f(0) = f^{'}(0) - 1 = 0$.

We denote by S the subclass of A consisting of functions $f \in A$, which are univalent in $\mathcal{U}$.

Let $\mathcal{P}$ denote the class of functions p which are analytic in $\mathcal{U}$, $p(0) = 1$ and $\mathrm{Re}p(z) > 0$, for all $z \in \mathcal{U}$.

We consider the integral operator

$$\mathcal{M}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i-1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left({g_i}'(t) \right)^{\beta_i} \cdot \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}}, \qquad (1)$$

for $f_i, g_i \in \mathcal{A}$ and the complex numbers $\delta, \alpha_i, \beta_i, \gamma_i$, with $\delta \neq 0$, $i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$.

2. Preliminary results

We need the following lemmas.

Lemma 1. [6] *Let γ, δ be complex numbers, $Re\gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function F_δ defined by

$$F_\delta(z) = \left[\delta \int_0^z t^{\delta-1} f'(t)\,dt\right]^{\frac{1}{\delta}},$$

is regular and univalent in $\mathcal{U}$.

Lemma 2. [5] *Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} z^m,$$

the equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m} z^m,$$

where θ is constant.

3. Main results

Theorem 3. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$, $i = \overline{1,n}$, M_i, N_i, P_i real positive numbers, $i = \overline{1,n}$, and $f_i, g_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i,$$

$$\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \leq N_i,$$

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq P_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n}\left[|\alpha_i - 1|\, M_i + |\beta_i|\, P_i + |\gamma_i|\, N_i\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \qquad (2)$$

then for all δ complex numbers, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{M}_n$, given by (1) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i{}'(t)\right)^{\beta_i} \cdot \left(\frac{g_i(t)}{t}\right)^{\gamma_i} \mathrm{dt},$$

for $f_i, g_i \in \mathcal{A}$, $i = \overline{1, n}$.

The function H_n is regular in $\mathcal{U}$ and satisfy the following usual normalization conditions $H_n(0) = H_n'(0) - 1 = 0$.

Now, calculating the derivatives of H_n of the first and second orders, we readily obtain

$$H_n{}'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \cdot \left(g_i{}'(z)\right)^{\beta_i} \cdot \left(\frac{g_i(z)}{z}\right)^{\gamma_i},$$

$$H_n{}''(z) =$$

$$\sum_{i=1}^{n} \left[(\alpha_i - 1)\left(\frac{f_i(z)}{z}\right)^{\alpha_i - 2} \cdot \left(g_i{}'(z)\right)^{\beta_i} \cdot \left(\frac{g_i(z)}{z}\right)^{\gamma_i} \cdot \left(\frac{z f_i{}'(z) - f_i(z)}{z^2}\right)\right] \cdot$$

$$\cdot \prod_{\substack{k=1 \\ k \neq i}}^{n} \left[\left(\frac{f_k(z)}{z}\right)^{\alpha_k - 1} \cdot \left(g_k{}'(z)\right)^{\beta_k} \cdot \left(\frac{g_k(z)}{z}\right)^{\gamma_k}\right] +$$

$$\sum_{i=1}^{n} \left[\beta_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \cdot \left(g_i{}'(z)\right)^{\beta_i - 1} \cdot g_i{}''(z) \cdot \left(\frac{g_i(z)}{z}\right)^{\gamma_i}\right] \cdot$$

$$\cdot \prod_{\substack{k=1 \\ k \neq i}}^{n} \left[\left(\frac{f_k(z)}{z}\right)^{\alpha_k - 1} \cdot \left(g_k{}'(z)\right)^{\beta_k} \cdot \left(\frac{g_k(z)}{z}\right)^{\gamma_k}\right] +$$

$$+ \sum_{i=1}^{n} \left[\gamma_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \cdot \left(g_i{}'(z)\right)^{\beta_i} \cdot \left(\frac{g_i(z)}{z}\right)^{\gamma_i - 1} \cdot \left(\frac{z g_i{}'(z) - g_i(z)}{z^2}\right)\right] \cdot$$

$$\cdot \prod_{\substack{k=1 \\ k \neq i}}^{n} \left[\left(\frac{f_k(z)}{z}\right)^{\alpha_k - 1} \cdot \left(g_k{}'(z)\right)^{\beta_k} \cdot \left(\frac{g_k(z)}{z}\right)^{\gamma_k}\right],$$

for all $z \in \mathcal{U}$.

We have

$$\frac{zH_n''(z)}{H_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} + \gamma_i \left(\frac{zg_i'(z)}{g_i(z)} - 1 \right) \right],$$

for all $z \in \mathcal{U}$.

Therefore

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq$$

$$\leq \frac{1-|z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| + |\gamma_i| \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \right], \quad (3)$$

for all $z \in \mathcal{U}$.

By applying the General Schwarz Lemma to the functions f_i, g_i, $i = \overline{1,n}$ we obtain

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i |z|,$$

$$\left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \leq N_i |z|,$$

$$\left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq P_i |z|,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$.

Using these inequalities from (3) we have

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{1-|z|^{2c}}{c} |z| \sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| P_i + |\gamma_i| N_i \right], \quad (4)$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\leq 1} \frac{\left(1-|z|^{2c}\right)|z|}{c} = \frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

from (4) we obtain

$$\frac{1-|z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \leq \frac{2}{(2c+1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| P_i + |\gamma_i| N_i \right],$$

and hence, by (2) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\cdot\frac{(2c+1)^{\frac{2c+1}{2c}}}{2}=1,$$

for all $z\in\mathcal{U}$.

So,

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le 1. \tag{5}$$

and using (5), by Lemma 1, it results that the integral operator $\mathcal{M}_n$, given by (1) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 3.1, obtain the next corollary:

Corollary 4. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$, $c = Re\gamma$, $i=\overline{1,n}$, M_i, N_i, P_i real positive numbers, $i=\overline{1,n}$, and $f_i, g_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i=\overline{1,n}$*

If

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right| \le M_i,$$

$$\left|\frac{zg_i'(z)}{g_i(z)}-1\right| \le N_i,$$

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \le P_i,$$

for all $z\in\mathcal{U}$, $i=\overline{1,n}$ and

$$\sum_{i=1}^{n}\left[|\alpha_i-1|\,M_i+|\beta_i|\,P_i+|\gamma_i|\,N_i\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{N}_n$ defined by

$$\mathcal{N}_n(z)=\int_0^z \prod_{i-1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left(g_i{}'(t)\right)^{\beta_i}\cdot\left(\frac{g_i(t))}{t}\right)^{\gamma_i}\right]dt \tag{6}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1=\gamma_2=...=\gamma_n=0$ in Theorem 3.1, obtain the next corollary:

Corollary 5. *Let $\gamma, \alpha_i, \beta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, M_i, P_i real positive numbers, $i = \overline{1,n}$, and $f_i, g_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i,$$

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq P_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| P_i\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{F}_n$ defined by

$$\mathcal{F}_n(z) = \int_0^z \prod_{i-1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left({g_i}'(t)\right)^{\beta_i}\right] dt \qquad (7)$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = ... = \beta_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 6. *Let $\gamma, \alpha_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, M_i, N_i real positive numbers, $i = \overline{1,n}$, and $f_i, g_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1,n}$*

If

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i,$$

$$\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| \leq N_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\gamma_i| N_i\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{G}_n$ *defined by*

$$\mathcal{G}_n(z) = \int_0^z \prod_{i-1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{8}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 7. *Let* $\gamma, \beta_i, \gamma_i$ *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, N_i, P_i *real positive numbers,* $i = \overline{1,n}$, *and* $g_i \in \mathcal{A}$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1,n}$

If

$$\left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \leq N_i,$$

$$\left| \frac{zg_i''(z)}{g_i'(z)} \right| \leq P_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\beta_i| \, P_i + |\gamma_i| \, N_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$ *defined by*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i-1}^{n} \left[\left(g_i{}'(t) \right)^{\beta_i} \cdot \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{9}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$, $\beta_1 = \beta_2 = ... = \beta_n = 0$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 8. *Let* γ, α_i*be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1,n}$, M_i *real positive numbers,* $i = \overline{1,n}$, *and* $f_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $i = \overline{1,n}$

If

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq M_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \, M_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$ *defined b*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i-1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} dt \tag{10}$$

is in the class $\mathcal{S}$*.*

If we consider $\delta = 1$, $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 9. *Let* γ, β_i *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1, n}$, P_i *real positive numbers,* $i = \overline{1, n}$, *and* $g_i \in \mathcal{A}$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1, n}$

If

$$\left| \frac{z g_i''(z)}{g_i'(z)} \right| \leq P_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1, n}$ *and*

$$\sum_{i=1}^{n} \left[|\beta_i| \, P_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$ *defined by*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i-1}^{n} \left({g_i}'(t) \right)^{\beta_i} dt \tag{11}$$

is in the class $\mathcal{S}$*.*

If we consider $\delta = 1$, $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ and $\beta_1 = \beta_2 = ... = \beta_n = 0$ in Theorem 3.1, obtain the next corollary:

Corollary 10. *Let* γ, γ_i *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, $i = \overline{1, n}$, N_i *real positive numbers,* $i = \overline{1, n}$, *and* $g_i \in \mathcal{A}$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1, n}$

If

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| \leq N_i,$$

for all $z \in \mathcal{U}$, $i = \overline{1, n}$ *and*

$$\sum_{i=1}^{n}\left[|\gamma_i|\,N_i\right]\leq\frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{I}_n$ *defined by*

$$\mathcal{I}_n(z)=\int_0^z\prod_{i-1}^{n}\left(\frac{g_i(t))}{t}\right)^{\gamma_i}dt \tag{12}$$

is in the class $\mathcal{S}$.

If we consider $n=1$, $\delta=\gamma=\alpha$ and $\alpha_i-1=\beta_i=\gamma_i$ in Theorem 3.1, obtain the next corollary:

Corollary 11. *Let* α *be complex numbers,* $Re\alpha>0$, M,N,P *real positive numbers, and* $f,g\in\mathcal{A}$, $f(z)=z+a_2z^2+a_3z^3+....$, $g(z)=z+b_2z^2+b_3z^3+....$,

If

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq M,$$

$$\left|\frac{zg'(z)}{g(z)}-1\right|\leq N,$$

$$\left|\frac{zg''(z)}{g'(z)}\right|\leq P,$$

for all $z\in\mathcal{U}$, *and*

$$|\alpha-1|\,(M+N+P)\leq\frac{(2Re\alpha+1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{M}$ *defined by*

$$\mathcal{M}(z)=\left\{\alpha\int_0^z\left[f(t)\cdot g'(t)\cdot\frac{g(t))}{t}\right]^{\alpha-1}dt\right\}^{\frac{1}{\alpha}}, \tag{13}$$

is in the class $\mathcal{S}$.

Theorem 12. *Let* $\gamma,\alpha_i,\beta_i,\gamma_i$ *be complex numbers,* $i=\overline{1,n}$, $c=Re\gamma>0$ *and* $f_i,g_i\in\mathcal{S}$, $g_i'\in\mathcal{P}$, $f_i(z)=z+a_{2i}z^2+a_{3i}z^3+....$, $g_i(z)=z+b_{2i}z^2+b_{3i}z^3+....$, $i=\overline{1,n}$

If

$$\sum_{i=1}^{n}|\alpha_i-1|+\sum_{i=1}^{n}|\beta_i|+2\sum_{i=1}^{n}|\gamma_i|\leq\frac{c}{2},\quad for\quad 0<c<1 \tag{14}$$

or

$$2\sum_{i=1}^{n}|\alpha_i-1|+\sum_{i=1}^{n}|\beta_i|+2\sum_{i=1}^{n}|\gamma_i|\leq\frac{1}{2},\quad for\quad c\geq 1 \tag{15}$$

then for any complex numbers δ, $Re\delta\geq c$, the integral operator $\mathcal{M}_n$ defined by (1) is in the class $\mathcal{S}$.

Proof. We consider the function

$$H_n(z)=\int_0^z\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left({g_i}'(t)\right)^{\beta_i}\cdot\left(\frac{g_i(t)}{t}\right)^{\gamma_i}\mathrm{d}t,$$

for $f_i,g_i\in\mathcal{S}$, $g_i'\in\mathcal{P}$, $i=\overline{1.n}$.

The function H_n is regular in $\mathcal{U}$ and satisfy the following usual normalization conditions $H(0)=H'(0)-1=0$.

We obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq$$

$$\leq\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|+|\gamma_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right],$$

for all $z\in\mathcal{U}$.

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq$$

$$\leq\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|+|\gamma_i|\left(\left|\frac{zg_i'(z)}{g_i(z)}\right|+1\right)\right],$$

for all $z\in\mathcal{U}$. Since $f_i,g_i\in\mathcal{S}$ we have

$$\left|\frac{zf_i'(z)}{f_i(z)}\right|\leq\frac{1+|z|}{1-|z|},$$

$$\left|\frac{zg_i'(z)}{g_i(z)}\right|\leq\frac{1+|z|}{1-|z|},$$

for all $z\in\mathcal{U}$, $i=\overline{1,n}$.

For $g_i'\in\mathcal{P}$ we have

$$\left| \frac{z g_i''(z)}{g_i'(z)} \right| \leq \frac{2\,|z|}{1-|z|^2},$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$.

Using these relations we get

$$\frac{1-|z|^{2c}}{c} \left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq$$

$$\leq \frac{1-|z|^{2c}}{c} \left[\left(\frac{1+|z|}{1-|z|} + 1 \right) \sum_{i=1}^{n} |\alpha_i - 1| \right] + \frac{1-|z|^{2c}}{c} \cdot \frac{2\,|z|}{1-|z|^2} \sum_{i=1}^{n} |\beta_i| +$$

$$+ \frac{1-|z|^{2c}}{c} \left[\left(\frac{1+|z|}{1-|z|} + 1 \right) \sum_{i=1}^{n} |\gamma_i| \right] \leq \frac{1-|z|^{2c}}{c} \cdot \frac{2}{1-|z|} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{1-|z|^{2c}}{c} \cdot$$

$$\cdot \frac{2\,|z|}{1-|z|^2} \sum_{i=1}^{n} |\beta_i| + \frac{1-|z|^{2c}}{c} \cdot \frac{2}{1-|z|} \sum_{i=1}^{n} |\gamma_i|, \tag{16}$$

for all $z \in \mathcal{U}$.

For $0 < c < 1$, we have $1-|z|^{2c} \leq 1-|z|^2$, $z \in \mathcal{U}$ and by (16) we obtain

$$\frac{1-|z|^{2c}}{c} \left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq \frac{4}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2}{c} \sum_{i=1}^{n} |\beta_i| + \frac{4}{c} \sum_{i=1}^{n} |\gamma_i|, \tag{17}$$

for all $z \in \mathcal{U}$.

From (14) and (17) we have

$$\frac{1-|z|^{2c}}{c} \left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq 1. \tag{18}$$

for all $z \in \mathcal{U}$ and $0 < c < 1$.

For $c \geq 1$ we have $\frac{1-|z|^{2c}}{c} \leq 1-|z|^2$, $z \in \mathcal{U}$ and by (16) we obtain

$$\frac{1-|z|^{2c}}{c} \left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq 4 \sum_{i=1}^{n} |\alpha_i - 1| + 2 \sum_{i=1}^{n} |\beta_i| + 4 \sum_{i=1}^{n} |\gamma_i|, \tag{19}$$

for all $z \in \mathcal{U}$ and $c \geq 1$.

From (15) and (19) we obtain

$$\frac{1-|z|^{2c}}{c} \left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq 1. \tag{20}$$

for all $z \in \mathcal{U}$ and $c \geq 1$. and by (18), (20) and Lemma 1 it results that the integral operator $\mathcal{M}_n$ defined by (1) is in the class $\mathcal{S}$.

If we consider $\delta = 1$ in Theorem 3.2, we obtain the next corollary:

Corollary 13. *Let* $\gamma, \alpha_i, \beta_i, \gamma_i$ *be complex numbers,* $i = \overline{1,n}$, $0 < Re\gamma \leq 1$ *and* $f_i, g_i \in \mathcal{S}$, $g_i' \in \mathcal{P}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1,n}$

If

$$2\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + 2\sum_{i=1}^{n}|\gamma_i| \leq \frac{Re\gamma}{2},$$

then the integral operator $\mathcal{N}_n$ *defined by (6) belongs to the class* $\mathcal{S}$.

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = ... = \beta_n = 0$ in Theorem 3.2, we obtain the next corollary:

Corollary 14. *Let* $\gamma, \alpha_i, \gamma_i$ *be complex numbers,* $i = \overline{1,n}$, $0 < Re\gamma \leq 1$ *and* $f_i, g_i \in \mathcal{S}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $i = \overline{1,n}$

If

$$\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\gamma_i| \leq \frac{Re\gamma}{4},$$

then the integral operator $\mathcal{F}_n$ *given by (7) is in the class* $\mathcal{S}$.

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