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SHARP FEKETE-SZEGŐ COEFFICIENTS FUNCTIONAL FOR CERTAIN P-VALENT CLOSE-TO-CONVEX FUNCTIONS

S. Kant and P. P. Vyas

Abstract. In the present paper certain subclass $\mathcal{K}_{p,s}(\phi)$ of p-valent close-to-convex functions in the unit disc are defined by means of subordination. Sharp estimates for the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_{p,s}(\phi)$ is obtained. Sharp distortion theorem and growth theorem are also obtained.

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1. Introduction and definitions

Let $\mathcal{A}_p$ be the class of all p-valent analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in N)\,, \tag{1}$$

in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we write $\mathcal{A}_1 = \mathcal{A}$.

Let f and g be analytic in $\mathbb{U}$, then f is subordinate to g in $\mathbb{U}$, written as $f \prec g$ if there is an analytic function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, then f is subordinate to g iff $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let ϕ be an analytic function with positive real part in $\mathbb{U}$ with $\phi(0) = 1, \phi'(0) > 0$ and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $\mathcal{S}_p^*(\phi)$ be the class of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad (z \in \mathbb{U}) \tag{2}$$

and $\mathcal{C}_p(\phi)$ be the class of functions $f \in \mathcal{A}_p$ satisfing

$$\frac{1}{p}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) \prec \phi(z) \qquad (z \in \mathbb{U}). \tag{3}$$

These classes are recently studied by Ali et. *al* [1] and they obtained sharp distortion, growth, covering and rotation theorems for these classes. The classes $\mathcal{S}_p^*(\phi)$ and $\mathcal{C}_p(\phi)$ include several well known subclasses of p-valent starlike and p-valent convex function as special cases. In particular, for an analytic function $\phi(z) = \frac{1+(1-\frac{2\gamma}{p})z}{1-z}$ in $\mathbb{U}$, (2) gives

$$\frac{zf'(z)}{f(z)} \prec \frac{p+(p-2\gamma)z}{1-z} \qquad (0 \le \gamma < p, z \in \mathbb{U}) \tag{4}$$

or equivalently

$$Re\Big(\frac{zf'(z)}{f(z)}\Big) > \gamma \qquad (0 \le \gamma < p, z \in \mathbb{U}). \tag{5}$$

Any function satisfying (4) or (5) belongs to class of p-valent starlike function of order γ denoted by $\mathcal{S}_p^*(\gamma)$. For $p = 1$ the classes $\mathcal{S}_p^*(\phi)$ and $\mathcal{C}_p(\phi)$ are introduced and studied by Ma and Minda (see[8]). We denote that

$$\mathcal{S}_p^*(0) = \mathcal{S}_p^*, \quad \mathcal{S}_1^*(\gamma) = \mathcal{S}^*(\gamma) \quad and \quad \mathcal{S}_1^*(0) = \mathcal{S}^*.$$

Let $\mathcal{K}_p(\phi)$ be the class of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\frac{zf'(z)}{g(z)} \prec \phi(z) \qquad (z \in \mathbb{U}) \tag{6}$$

for some $g \in \mathcal{S}_p^*$.
The class $\mathcal{K}_p(\phi)$ include several well known subclasses of p-valent close-to-convex function as special cases. Very recently by means of subordination one of the author Kant [6] discussed the following subclass $\mathcal{K}_p^s(\phi)$ of the class $\mathcal{K}_p(\phi)$:

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p^s(\phi)$, if it satisfies the subordination relation

$$\frac{1}{p}\Big(\frac{(-1)^p z^{p+1} f'(z)}{g(z)g(-z)}\Big) \prec \phi(z) \qquad (z \in \mathbb{U})$$

for some function $g \in \mathcal{S}_p^*(p/2)$.
Here the assumption of g is a starlike function of oder $\frac{p}{2}$ makes the function $\frac{g(z)g(-z)}{(-z)^p}$ p-valent starlike function in $\mathbb{U}$. So instead of $\frac{g(z)g(-z)}{(-z)^p}$ with $g \in \mathcal{S}_p^*(p/2)$, we can consider $\frac{g(z)-g(-z)}{2}$ with $g(z) \in \mathcal{S}_p^*$, which motivates us to define a new subclass $\mathcal{K}_{p,s}(\phi)$ of close-to-convex function as follows:

Definition 1. *Let ϕ be an analytic univalent function with positive real part in $\mathbb{U}$ with $\phi(0) = 1$. The class $\mathcal{K}_{p,s}(\phi)$ consists of functions $f \in \mathcal{A}_p$ satisfying*

$$\frac{1}{p}\left(\frac{zf'(z)}{\frac{g(z)-g(-z)}{2}}\right) \prec \phi(z) \qquad (z \in \mathbb{U}) \tag{7}$$

for some function $g \in \mathcal{S}_p^*$.

For some recent investigations on the class of close-to-convex functions, one can find in [[2], [3], [4], [7], [9], [11], [12], [13], [14], [15]] and the references cited therein. In the present investigation, we obtain a sharp estimates for the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_{p,s}(\phi)$. Also distortion, growth and covering theorems are derived.

2. Fekete-Szegő inequality

In this section we assume that ϕ is an analytic function with positive real part in $\mathbb{U}$ with $\phi(0) = 1, \phi'(0) > 0$ and which maps the open unit disc $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. In such case the function ϕ has an expansion of the form

$$\phi(z) = 1 + A_1 z + A_2 z^2 + \cdots \qquad (A_1 > 0, z \in \mathbb{U}). \tag{8}$$

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{9}$$

satisfying the condition $|w(z)| \leq 1$ in $\mathbb{D}$. We need the following lemmas to prove our results:

Lemma 1. *[5] If $w \in \Omega$, then for any complex number ν:*

$$|w_1| \leq 1, \qquad |w_2 - \nu w_1^2| \leq \max\{1, |\nu|\}. \tag{10}$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

Theorem 2. *Let $f \in \mathcal{A}_p$ of the form (1) belonging to the class $\mathcal{K}_{p,s}(\phi)$, Then*

$$|a_{p+1}| \leq \frac{p}{p+1} A_1 \tag{11}$$

and

$$\left|a_{p+2} - \nu a_{p+1}^2\right| \leq \frac{p^2}{p+2} + \frac{pA_1}{p+2}. \max\{1, |\frac{A_2}{A_1} - \frac{\nu p(p+2)}{(p+1)^2} A_1|\} \quad (\nu \in \mathbb{C}). \tag{12}$$

The results are sharp.

Proof. Let $f \in \mathcal{K}_{p,s}(\phi)$. In view of Definition 1, there exists a Schwarz function w such that

$$\frac{1}{p}\Big(\frac{zf'(z)}{\frac{g(z)-g(-z)}{2}}\Big) = \phi(w(z)) \qquad (z \in \mathbb{U}), \tag{13}$$

for some function $g \in \mathcal{S}_p^*$. Let

$$g(z) = z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots .$$

Then by a simple calculation, we have

$$\frac{g(z)-g(-z)}{2} = z^p + b_{p+2}z^{p+2} + b_{p+4}z^{p+4} + \cdots . \tag{14}$$

Series expansion (14) and Taylor expansion (1) for f, give

$$\frac{1}{p}\Big(\frac{zf'(z)}{\frac{g(z)-g(-z)}{2}}\Big) = 1 + \frac{p+1}{p}a_{p+1}z + \big(\frac{p+2}{p}a_{p+2} - b_{p+2}\big)z^2 + \cdots . \tag{15}$$

Also,

$$\phi(w(z)) = 1 + A_1w_1z + (A_1w_2 + A_2w_1^2)z^2 + \cdots . \tag{16}$$

Making use of (15), (16) in (13) and then equating the coefficients of z and z^2 in the resulting equation, we get

$$a_{p+1} = \frac{p}{p+1}A_1w_1 \tag{17}$$

and

$$a_{p+2} = \frac{p}{p+2}\big(b_{p+2} + A_1w_2 + A_2w_1^2\big). \tag{18}$$

Thus for a complex number ν, we have

$$\begin{aligned} a_{p+2} - \nu a_{p+1}^2 &= \frac{p}{p+2}\big(b_{p+2} + A_1w_2 + A_2w_1^2\big) - \nu\big(\frac{p}{p+1}A_1w_1\big)^2 \\ |a_{p+2} - \nu a_{p+1}^2| &= \frac{p}{p+2}\big|b_{p+2} + A_1\big\{w_2 - \big(\frac{\nu p(p+2)A_1}{(p+1)^2} - \frac{A_2}{A_1}\big)w_1^2\big\}\big| \\ &\leq \frac{p}{p+2}\big|b_{p+2}\big| + \frac{pA_1}{p+2}|w_2 - \big(\frac{\nu p(p+2)A_1}{(p+1)^2} - \frac{A_2}{A_1}\big)w_1^2|. \end{aligned} \tag{19}$$

By virtue of Lemma 1 and a result [[10], Theorem 4, p.66], we have

$$|a_{p+2} - \nu a_{p+1}^2| \leq \frac{p^2}{p+2} + \frac{pA_1}{p+2}.\max\{1, |\frac{\nu p(p+2)}{(p+1)^2}A_1 - \frac{A_2}{A_1}|\}.$$

This complete the required assertions (11) and (12).

For sharpness consider the function f_1 by

$$f_1(z) = p\int_0^z \frac{t^{p-1}}{(1-t^2)^p}\phi(t)dt.$$

The function f_1 clearly belongs to the class $\mathcal{K}_{p,s}(\phi)$ with $g(z) = \frac{z^p}{(1-z^2)^p} \in \mathcal{S}_p^*$. Since

$$\frac{pz^{p-1}}{(1-z^2)^p}\phi(z) = p\{z^{p-1} + A_1 z^p + (A_2+p)z^{p+1} + \cdots\},$$

we have

$$\begin{aligned} f_1(z) &= p\int_0^z \{t^{p-1} + A_1 t^p + (A_2+p)t^{p+1} + \cdots\}dt \\ &= z^p + \frac{pA_1}{p+1}z^{p+1} + \frac{p(A_2+p)}{p+2}z^{p+2} + \cdots . \end{aligned}$$

Next, we consider

$$f_2(z) = p\int_0^z \frac{t^{p-1}}{(1-t^2)^p}\phi(t^2)dt.$$

Then, we obtain

$$f_2(z) = z^p + \frac{p(A_1+p)}{p+2}z^{p+2} + \cdots .$$

Functions f_1 and f_2 show that the results (11) and (12) are sharp.

3. Distortion and Growth Theorems

Theorem 3. *Let ϕ be an analytic univalent function with positive real part and*

$$\phi(-r) = \min_{|z|=r<1}|\phi(z)| \quad , \quad \phi(r) = \max_{|z|=r<1}|\phi(z)|.$$

If p is an odd number and f belongs to the class $\mathcal{K}_{p,s}(\phi)$, then

$$\frac{\phi(-r)r^{p-1}}{(1+r^2)^p} \leq |f'(z)| \leq \frac{\phi(r)r^{p-1}}{(1-r^2)^p} \qquad (|z| = r < 1) \tag{20}$$

and

$$\int_0^r \frac{\phi(-l)l^{p-1}}{(1+l^2)^p}dl \leq |f(z)| \leq \int_0^r \frac{\phi(l)l^{p-1}}{(1-l^2)^p}dl \qquad (|z| = r < 1). \tag{21}$$

The results are sharp.

Proof. Suppose $f \in \mathcal{K}_{p,s}(\phi)$. By (7), we have

$$\frac{zf'(z)}{pG(z)} \prec \phi(z) \tag{22}$$

where

$$G(z) = \frac{g(z) - g(-z)}{2}$$

is an odd p-valent starlike function, which has the inequalities

$$\frac{r^p}{(1+r^2)^p} \leq |G(z)| \leq \frac{r^p}{(1-r^2)^p} \qquad (|z| = r < 1). \tag{23}$$

From (22) for a Schwarz funcion w, we have

$$\begin{aligned} |f'(z)| &= \frac{p|G(z)|}{|z|}|\phi(w(z))| \\ &\leq \frac{pr^{p-1}}{(1-r^2)^p}.\max_{|z|=r}|\phi(z)| \qquad (|z| = r < 1) \\ &\leq \frac{pr^{p-1}}{(1-r^2)^p}\phi(r) \qquad (|z| = r < 1). \end{aligned} \tag{24}$$

Similarly

$$|f'(z)| \geq \frac{pr^{p-1}}{(1+r^2)^p}\phi(-r) \qquad (|z| = r < 1). \tag{25}$$

To prove the sharpness of our results, we consider the function

$$f_1(z) = p\int_0^z \frac{t^{p-1}}{(1-t^2)^p}\phi(t)\, dt$$

and

$$f_2(z) = p\int_0^z \frac{t^{p-1}}{(1+t^2)^p}\phi(t)\, dt.$$

Clearly $f_1(z)$ and $f_2(z)$ are of p-valent close-to-convex functions with $g_1(z) = \frac{z^p}{(1-z^2)^p}$ and $g_2(z) = \frac{z^p}{(1+z^2)^p}$ respectively in $\mathbb{U}$. Functions g_1 and g_2 are of p-valent starlike. Thus the functions f_1 and f_2 are members of the class $\mathcal{K}_{p,s}(\phi)$. The sharpness of upper estimates for $|f'|$ and $|f|$ are given by the function f_1 while the sharpness for lower estimates are provided by f_2.

Conflicts of Interest

The author declare that there is no conflict of interest regarding the publication of this paper.

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Shashi Kant
Department of Mathematics,
Government Dungar College,
Bikaner-334001, INDIA
email: *drskant.2007@yahoo.com*

Prem Pratap Vyas
Department of Mathematics,
Government Dungar College,
Bikaner-334001, INDIA
email: *prempratapvyas@gmail.com*