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APPROXIMATE SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER BY USING ANALYTICAL TECHNIQUES

A.A. Hamoud, K.P. Ghadle

Abstract. In this paper, Adomian decomposition and modified Laplace Adomian decomposition methods are successfully applied to find the approximate solution of Volterra integro-differential equation of fractional order. The reliability of the methods and reduction in the size of the computational work give these methods a wider applicability. Also, the behavior of the solution can be formally determined by analytical approximate. Moreover, we proved the convergence of the solutions. Finally, an example is included to demonstrate the validity and applicability of the proposed techniques.

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1. Introduction

In this paper, we consider Caputo fractional Volterra integro-differential equation of the form:

$$^{c}D^{\alpha}y(x) = a(x)y(x) + g(x) + \int_0^x k(x,t)F(y(t))dt, \tag{1.1}$$

with the initial condition

$$y(0) = y_0, \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$ and $y : [0,1] \longrightarrow \mathbb{R}$, where $[0,1]$ is the continuous function which has to be determined, $g : [0,1] \longrightarrow \mathbb{R}$ and $k : [0,1] \times [0,1] \longrightarrow \mathbb{R}$, are continuous functions. $F : \mathbb{R} \longrightarrow \mathbb{R}$, is Lipschitz continuous function. An application of fractional derivatives was first given in 1823 by Abel [2] who applied the fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [4, 6, 7, 8, 9, 10, 20, 22, 23, 24]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can remember the following works. Al-Samadi and Gumah [5] applied the homotopy analysis method for fractional SEIR epidemic model, Zurigat et al. [26] applied HAM for system of fractional integro-differential equations, Yang and Hou [24] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [23] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [22] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [5, 12, 15, 16, 17, 19, 26]. The homotopy analysis method that was first proposed by Liao [21], is implemented to derive analytic approximate solutions of fractional integro-differential equations and convergence of HAM for this kind of equations is considered. Unlike all other analytical methods, HAM adjusts and controls the convergence region of the series solution via an auxiliary parameter $\hbar$.

The main objective of the present paper studies the behavior of the solution that can be formally determined by an analytical approximated methods as the Adomian decomposition and modified Laplace Adomian decomposition techniques. Moreover, we proved convergence of the solutions for Caputo fractional Volterra integro-differential equation.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [13, 14, 18, 25].

Definition 2.1. (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{2.1}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. (Caputo fractional derivative). The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$
\begin{aligned}
{}^cD_x^\alpha f(x) &= J^{m-\alpha}D^m f(x) \\
&= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}\frac{d^m f(t)}{dt^m}dt, & m-1<\alpha<m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases}
\end{aligned}
\tag{2.2}
$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

1. $J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v > 0.$
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$
3. $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0.$
4. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 2.3. (Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.3}$$

Theorem 1. *The Laplace transform of the Caputo derivative is defined as*

$$\mathcal{L}[{}^cD^\alpha f(x)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{2.4}$$

3. Adomian Decomposition Method

Consider the equation (1.1) where ${}^cD^\alpha$ is the operator defined as (2.3). Operating with J^α on both sides of the equation (1.1) we get [1, 3, 11, 23]

$$y(x) = y_0 + J^\alpha\left(g(x) + \int_0^x k(x,t)F(y(t))dt\right). \tag{3.1}$$

Adomian decomposition method defines the solution $y(x)$ by the series

$$y = \sum_{n=0}^{\infty} y_n, \tag{3.2}$$

and the nonlinear function F is decomposed as

$$F = \sum_{n=0}^{\infty} A_n, \tag{3.3}$$

where A_n is the Adomian polynomials given by

$$A_n = \frac{1}{n!}[\frac{d^n}{d\phi^n} F(\sum_{i=0}^{n} \phi^i y_i)]_{\phi=0}. \tag{3.4}$$

The Adomian polynomials were introduced in [1, 3, 10] as:

$$\begin{aligned} A_0 &= F(y_0), \\ A_1 &= y_1 F^{'}(y_0), \\ A_2 &= y_2 F^{'}(y_0) + \frac{1}{2} y_1^2 F^{''}(y_0), \\ A_3 &= y_3 F^{'}(y_0) + y_1 y_2 F^{''}(y_0) + \frac{1}{3} y_1^3 F^{'''}(y_0). \end{aligned}$$

The components $y_0, y_1, y_2, ...$ are determined recursively by

$$y_0(x) = y_0 + J^{\alpha} g(x), \tag{3.5}$$

$$y_{k+1} = J^{\alpha} \left(\int_0^x k(x,t) A_k dt \right). \tag{3.6}$$

Having defined the components $y_0, y_1, y_2, \cdots$, the solution y in a series form defined by (3.2) follows immediately. It is important to note that the decomposition method suggests that the 0^{th} component y_0 be defined by the initial conditions and the function $g(x)$ as described above. The other components namely $y_1, y_2, \cdots$, are derived recurrently.

4. Modified Laplace Adomian Decomposition Method

Secondly, we consider the fractional Volterra-Fredholm integro-differential equation [7, 9, 24]. We apply the Laplace transform to both sides of (1.1)

$$\mathcal{L}[{}^c D^{\alpha} y(x)] = \mathcal{L}[g(x)] + \mathcal{L}\Big[\int_0^x k(x,s) F(y(s)) ds \Big]. \tag{4.1}$$

Using the differentiation property of Laplace transform (2.4) we get

$$s^{\alpha} \mathcal{L}[y(x)] - y_0 = \mathcal{L}[g(x)] + \mathcal{L}\Big[\int_0^x k(x,s) F(y(s)) ds \Big], \tag{4.2}$$

Thus, the given equation is equivalent to

$$\mathcal{L}[y(x)] = \frac{y_0}{s^\alpha} + \frac{1}{s^\alpha}\mathcal{L}[g(x)] + \frac{1}{s^\alpha}\mathcal{L}\Big[\int_0^x k(x,s)F(y(s))ds\Big]. \tag{4.3}$$

Substituting (3.2), and (3.3) into (4.3), we will get

$$\mathcal{L}\left[\sum_{n=0}^{\infty} y_n\right] = \frac{y_0}{s^\alpha} + \frac{1}{s^\alpha}\mathcal{L}\Big[g(x)\Big] + \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x k(x,s)\sum_{n=0}^{\infty} A_n(s)ds\right] \tag{4.4}$$

Matching both sides of (4.4) yields the following iterative algorithm:

$$\begin{aligned}
\mathcal{L}[y_0] &= \frac{y_0}{s^\alpha} + \frac{1}{s^\alpha}\mathcal{L}\left[g(x)\right],\\
\mathcal{L}[y_1] &= \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x k(x,s)A_0(s)ds\right],\\
\mathcal{L}[y_2] &= \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x k(x,s)A_1(s)ds\right],\\
&.\\
&.\\
&.\\
\mathcal{L}[y_{n+1}] &= \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x k(x,s)A_n(s)ds\right].
\end{aligned}$$

The solution $y(x)$ defines by the series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{4.5}$$

5. Convergence Results

In this section, we shall give convergence results of Eq.(1.1), with the initial condition (1.2) and prove it.
Before starting and proving the results, we introduce the following hypotheses:
(H1) The two functions $a, g : [0,1] \longrightarrow \mathbb{R}$ are continuous.
(H2) There exists a constant $M > 0$, such that

$$|k(x,t)| \le M, \quad \forall \quad 0 \le x,t \le 1.$$

(H3) There exists a constant $L > 0$ such that, for any $y, y^* \in C([0,1],\mathbb{R})$

$$|F(y(t)) - F(y^*(t))| \le L|y - y^*|.$$

Theorem 2. *Suppose that (H1), (H2) and (H3) hold, and if the series solution* $y(x) = \sum_{i=0}^{\infty} y_i(x)$ *and* $\|y_1\| < \infty$ *obtained by the ADM is convergent, when* $0 < \delta = \frac{ML}{\Gamma(\alpha+1)} < 1$. *Then it converges to the exact solution of the fractional integro-differential equation* (1.1) – (1.2).

Proof. Denote as $(C[0,1], \|.\|)$ the Banach space of all continuous functions on $[0,1]$ with $|y_1(x)| \leq \infty$ for all x in $[0,1]$.

First we define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that $s_n = \sum_{i=0}^{n} y_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\|s_n - s_m\| &= \max_{\forall x \in J} |s_n - s_m| \\
&= \max_{\forall x \in J} \Big| \sum_{i=0}^{n} y_i(x) - \sum_{i=0}^{m} y_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} y_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} \Big(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big[\int_0^t k(t,s) A_i(s) ds\Big] dt\Big) \Big| \\
&= \max_{\forall x \in J} \Big| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big[\int_0^t k(t,s) \sum_{i=m}^{n-1} A_i(s) ds\Big] dt \Big|.
\end{aligned}$$

From (3.3), we have

$$\sum_{i=m}^{n-1} A_i = F(s_{n-1}) - F(s_{m-1}),$$

so,

$$\begin{aligned}
\|s_n - s_m\| &= \max_{\forall x \in J} \Big| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big[\int_0^t k(t,s)(F(s_{n-1}) - F(s_{m-1})) ds\Big] dt \Big| \\
&\leq \max_{\forall x \in J} \Big(\frac{1}{\Gamma(\alpha)} \int_0^x |x-t|^{\alpha-1} \Big[\int_0^t |k(t,s)||(F(s_{n-1}) - F(s_{m-1}))| ds\Big] dt\Big) \\
&\leq \frac{1}{\Gamma(\alpha+1)} [ML \|s_{n-1} - s_{m-1}\|] \\
&= \frac{LM}{\Gamma(\alpha+1)} \|s_{n-1} - s_{m-1}\| \\
&= \delta \|s_{n-1} - s_{m-1}\|.
\end{aligned}$$

Let $n = m + 1$, then

$$\|s_n - s_m\| \leq \delta\|s_m - s_{m-1}\| \leq \delta^2\|s_{m-1} - s_{m-2}\| \leq \cdots \leq \delta^m\|s_1 - s_0\|,$$

so,

$$\begin{aligned}\|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \\ &\leq [\delta^m + \delta^{m+1} + \cdots + \delta^{n-1}]\|s_1 - s_0\| \\ &\leq \delta^m[1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}]\|s_1 - s_0\| \\ &\leq \delta^m(\frac{1 - \delta^{n-m}}{1 - \delta})\|y_1\|.\end{aligned}$$

Since $0 < \delta < 1$, we have $(1 - \delta^{n-m}) < 1$, then

$$\|s_n - s_m\| \leq \frac{\delta^m}{1 - \delta}\|y_1\|.$$

But $|y_1(x)| < \infty$, so, as $m \longrightarrow \infty$, then $\|s_n - s_m\| \longrightarrow 0$. We conclude that s_n is a Cauchy sequence in $C[0, 1]$, therefore

$$y = \lim_{n \to \infty} y_n.$$

Then, the series is convergent and the proof is complete.

6. Illustrative Example

In this section, we present the analytical techniques based on the Adomian decomposition method and the modified Laplace Adomian decomposition method to solve Caputo fractional Volterra integro-differential equation.

Example 6.1 Consider the following Caputo fractional Volterra integro-differential equation.

$$^cD^{0.75}[y(t)] = \frac{6t^{2.25}}{\Gamma(3.25)} - \frac{t^2e^t}{5}y(t) + \int_0^t e^t sy(s)ds, \tag{6.1}$$

with the initial condition

$$y(0) = 0, \tag{6.2}$$

and the the exact solution is $y(t) = t^3$.

Firstly, we apply the Adomian decomposition method

Applying the operator $J^{0.75}$ to both sides of Eq.(6.1)

$$y(t) = y_0 + \frac{6}{\Gamma(3.25)} J^{0.75}\Big(t^{2.25}\Big) - \frac{1}{5} J^{0.75}\Big(t^2 e^t y(t)\Big) + J^{0.75}\Big(\int_0^t e^t s y(s) ds\Big)$$

Then,

$$\begin{aligned} y_0(t) &= y_0 + \frac{6}{\Gamma(3.25)} J^{0.75}\Big(t^{2.25}\Big) \\ &= 0 + \frac{6}{\Gamma(3.25)} \frac{\Gamma(9/4+1)}{\Gamma(9/4+3/4+1)} t^{(9/4+3/4)} \\ &= t^3, \\ y_1(t) &= -\frac{1}{5} J^{0.75}\Big(t^2 e^t y_0(t)\Big) + J^{0.75}\Big(\int_0^t e^t s y_0(s) ds\Big) \\ &= -\frac{1}{5} J^{0.75}\Big(t^2 e^t y_0(t)\Big) + J^{0.75}\Big(\int_0^t e^t s^4 ds\Big) \\ &= -\frac{1}{5} J^{0.75}(t^2 e^t y_0(t)) + J^{0.75}(\frac{1}{5} e^t t^5) \\ &= -\frac{1}{5} J^{0.75}\Big(t^2 e^t y_0(t)\Big) + \frac{1}{5} J^{0.75}\Big(e^t t^2 y_0(t)\Big) \\ &= 0, \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ y_n(t) &= 0. \end{aligned}$$

Therefore, the obtained solution is

$$y(t) = t^3.$$

Secondly, we apply the modified Laplace Adomian decomposition method

We apply the Laplace transform to both sides of (6.1)

$$\mathcal{L}\left[{}^c D^{0.75} y(t)\right] = \mathcal{L}\left[(-\frac{t^2 e^t}{5}) y(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] + \mathcal{L}\left[\int_0^t e^t s y(s) ds\right].$$

Using the property of Laplace transform and the initial condition (6.2), we get

$$s^{\frac{3}{4}}\mathcal{L}\left[y(t)\right] = \mathcal{L}\left[(-\frac{t^2e^t}{5})y(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] + \mathcal{L}\left[\int_0^t e^t sy(s)ds\right],$$

and

$$\mathcal{L}\left[y(t)\right] = \frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})y(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] + \mathcal{L}\left[\int_0^t e^t sy(s)ds\right]\right).$$

Substituting (3.2) and (3.3) into the above equation, we have

$$\mathcal{L}\left[\sum_{n=0}^{\infty} y_n(t)\right] = \frac{1}{s^{\frac{3}{4}}}\mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] + \frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})\sum_{n=0}^{\infty} y_n(t)\right] + \mathcal{L}\left[\int_0^t e^t s\sum_{n=0}^{\infty} A_n ds\right]\right).$$

Match both side of above equation, we have the following relation:

$$\begin{aligned}
\mathcal{L}\left[y_0(t)\right] &= \frac{1}{s^{\frac{3}{4}}}\mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right],\\
\mathcal{L}\left[y_1(t)\right] &= \frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})y_0(t)\right] + \mathcal{L}\left[\int_0^t e^t sA_0 ds\right]\right),\\
&\cdot\\
&\cdot\\
&\cdot\\
\mathcal{L}\left[y_{n+1}(t)\right] &= \frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})y_n(t)\right] + \mathcal{L}\left[\int_0^t e^t sA_n ds\right]\right).
\end{aligned}$$

Applying inverse Laplace transform to above equations we get

$$\begin{aligned}
y_0(t) &= t^3,\\
y_1(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})y_0(t)\right] + \frac{1}{s^{\frac{3}{4}}}\mathcal{L}\left[\int_0^t e^t s^4 ds\right]\right)\right),\\
&= 0,\\
&\cdot\\
&\cdot\\
&\cdot\\
y_n(t) &= 0.
\end{aligned}$$

Therefore, the obtained solution is

$$y(t) = t^3.$$

7. Conclusion

The Adomian decomposition and modified Laplace Adomian decomposition methods are successfully applied to find the approximate solution of Volterra integro-differential equation of fractional order. The reliability of the methods and reduction in the size of the computational work give these methods a wider applicability. Also, the behavior of the solution can be formally determined by analytical approximate. Moreover, we proved the convergence of the solutions. Finally, an example is included to demonstrate the validity and applicability of the proposed techniques.

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Ahmed A. Hamoud
[1]Department of Mathematics,
Taiz University, Taiz, Yemen.
[2]Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad, 431004, India.
email: *drahmedselwi985@gmail.com*

Kirtiwant P. Ghadle
Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad, 431004, India.
email: *drkp.ghadle@gmail.com*