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Some strong and △-convergence theorems for multi-valued mappings in hyperbolic spaces

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Abstract: We introduce an iteration process for three multi-valued mappings in hyperbolic spaces and establish the strong and Δ-convergence theorems using the new iteration process. The results presented in this paper extend, unify and generalize some previous works from the current existing literature.

Keywords: Hyperbolic space, multi-valued mappings, common fixed point, △-convergence, strong convergence.

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1 Introduction

Let K be a nonempty subset of a metric space (X, d). The set K is called *proximinal* if for any $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$, where $d(x, K) = \inf \{d(x, y) : y \in K\}$. We shall denote $CB(K)$ and $P(K)$ be the family of nonempty closed bounded all subsets and nonempty proximinal bounded all subsets of K, respectively. *The Hausdorff metric* on $CB(X)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} \quad \text{for all } A, B \in CB(X).$$

Let $T : K \to CB(K)$ be a multi-valued mapping. An element $p \in K$ is a fixed point of T if $p \in Tp$. Denote by $F(T)$ the set of all fixed points of T and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. It follows from the definition of P_T that $d(x, Tx) \leq d(x, P_T(x))$ for any $x \in K$. The mapping T is said to be

(i) *nonexpansive* if $H(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(ii) *quasi-nonexpansive* [17] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$;

(iii) *Lipschitzian* if there exists a constant $L > 0$ such that $H(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in K$;

(iv) *Lipschitzian quasi-nonexpansive* if both (ii) and (iii) hold.

It is clear that each multi-valued nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist the multi-valued quasi-nonexpansive mappings that are not nonexpansive (see [16, 17]). Moreover, each multi-valued nonexpansive mapping is Lipschitzian with $L = 1$.

Agarwal, O'Regan and Sahu [1] introduced the following iteration process, which is independent of both Mann [13] and Ishikawa [7] iterations, for a single-valued nonexpansive mapping in a Banach space:

$$\begin{cases} x_1 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. They showed that the rate convergence of this iteration process is similar to the Picard iteration and faster than the Mann iteration for contraction mappings.

Recently, Khan and Abbas [8] studied the two multi-valued mappings version of the iteration process (1) in a hyperbolic space.

Motivated by these results, we now modify the iteration process (1) for three multi-valued mappings in a hyperbolic space as follows:

Let K be a nonempty convex subset of a hyperbolic space X and $Q, S, T : K \to P(K)$ be three multi-valued mappings. Then the sequence $\{x_n\}$ is generated as

$$\begin{cases} x_0 \in K, \\ y_n = W\left(t_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \\ x_{n+1} = W(u_n, v_n, \alpha_n), \quad \forall n \geq 0, \end{cases} \tag{2}$$

where $t_n \in P_Q(x_n)$, $v_n \in P_S(x_n)$, $u_n \in P_T(y_n) = P_T(W(t_n, x_n, \frac{\beta_n}{1-\alpha_n}))$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ such that $\alpha_n + \beta_n < 1$.

In this paper, we prove some convergence theorems of the iteration process (2) for approximating a common fixed point of three multi-valued Lipschitzian quasi-nonexpansive mappings in a hyperbolic space. Our results generalize some recent results given in [8, 15].

2 Preliminaries and lemmas

We consider the concept of hyperbolic space introduced by Kohlenbach [10] which is more restrictive than the hyperbolic type introduced in Goebel and Kirk [4] and more general than the concept of hyperbolic space in Reich and Shafrir [14].

A hyperbolic space [10] is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is a function satisfying

(W1) $d(z, W(x, y, \lambda)) \leq (1-\lambda)d(z, x) + \lambda d(z, y)$,

(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| \, d(x, y)$,

(W3) $W(x, y, \lambda) = W(y, x, (1-\lambda))$,

(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1-\lambda)d(x, y) + \lambda d(z, w)$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$.

If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi [20]. A subset K of a hyperbolic space X is *convex* if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$. CAT(0) space in the sense of Gromov (see [2]) and Banach space are the examples of hyperbolic space. The class of hyperbolic space also contains Hadamard manifolds (see [3]), the Hilbert balls equipped with the hyperbolic metric (see [5]), Cartesian products of Hilbert balls and $\mathbb{R}$-trees, as special cases.

A hyperbolic space (X, d, W) is said to be *uniformly convex* [18] if for all $u, x, y \in X, r > 0$ and $\varepsilon \in (0,2]$, there exists a constant $\delta \in (0,1]$ such that $d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r$ whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ is called *the modulus of uniform convexity* if $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$. The function η is *monotone* if it decreases with r (for a fixed ε).

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, define a continuous functional $r(.,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r_K(\{x_n\})$ of $\{x_n\}$ with respect to a subset K of X is given by

$$r_K(\{x_n\}) = \inf \ \{r(x,\{x_n\}) : x \in K\}.$$

The asymptotic center $A_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r_K(\{x_n\})\}.$$

$r(\{x_n\})$ and $A(\{x_n\})$ will denote the asymptotic radius and the asymptotic center of $\{x_n\}$ with respect to X, respectively. In general, the set $A_K(\{x_n\})$ may be empty or may even contain infinitely many points. It has been shown in Proposition 3.3 of [11] that every bounded sequences have unique asymptotic center with respect to nonempty closed convex subsets in a complete uniformly convex hyperbolic space with the monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in X is said to be $\triangle$-*convergent* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ (see [12]). In this case, we write $\triangle$-$\lim_{n\to\infty} x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

In the sequel, we shall need the following results.

Lemma 1 *(see [9, Lemma 2.5]) Let (X,d,W) be a uniformly convex hyperbolic space with the monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x) \le r$, $\limsup_{n\to\infty} d(y_n,x) \le r$ and $\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = r$ for some $r \ge 0$, then*

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

Lemma 2 *(see [9, Lemma 2.6]) Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X and $\{x_n\}$ be a bounded sequence in K with $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m,\{x_n\}) = r(y,\{x_n\})$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 3 *(see [19, Lemma 1]) Let K be a nonempty subset of a metric space (X,d) and $T : K \to P(K)$ be a multi-valued mapping. Then the followings are equivalent:*

(1) $x \in F(T)$, that is, $x \in Tx$;

(2) $P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$;

(3) $x \in F(P_T)$, that is, $x \in P_T(x)$.

Further, $F(T) = F(P_T)$.

3 Main results

From now on for three multi-valued mappings Q, S and T, we set $F = F(Q) \cap F(S) \cap F(T) \neq \emptyset$.

We start with proving key lemmas for later use.

Lemma 4 *Let K be a nonempty closed convex subset of a hyperbolic space X and $Q,S,T : K \to P(K)$ be three multi-valued mappings such that P_Q, P_S and P_T are quasi-nonexpansive. Then for the sequence $\{x_n\}$ defined by (2), $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. Then by Lemma 3, $p \in P_Q(p) = \{p\} = P_S(p) = P_T(p)$. From (2), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(u_n, v_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(u_n, p) + \alpha_n d(v_n, p) \\ &= (1-\alpha_n) d(u_n, P_T(p)) + \alpha_n d(v_n, P_S(p)) \\ &\leq (1-\alpha_n) H(P_T(y_n), P_T(p)) + \alpha_n H(P_S(x_n), P_S(p)) \\ &\leq (1-\alpha_n) d(y_n, p) + \alpha_n d(x_n, p) \end{aligned} \tag{3}$$

and

$$\begin{aligned} d(y_n, p) &= d\left(W\left(t_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(t_n, p) + \frac{\beta_n}{1-\alpha_n} d(x_n, p) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) H(P_Q(x_n), P_Q(p)) + \frac{\beta_n}{1-\alpha_n} d(x_n, p) \\ &\leq \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) + \frac{\beta_n}{1-\alpha_n} d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{4}$$

Combining (3) and (4), we get

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. ■

Lemma 5 *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with the monotone modulus of uniform convexity η and $Q, S, T : K \to P(K)$ be three multi-valued mappings such that P_Q, P_S and P_T are Lipschitzian quasi-nonexpansive with $d(x_n, v_n) \leq d(u_n, v_n)$. Let $\{x_n\}$ be the sequence defined by (2) with $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then*

$$\lim_{n\to\infty} d(x_n, P_Q(x_n)) = \lim_{n\to\infty} d(x_n, P_S(x_n)) = \lim_{n\to\infty} d(x_n, P_T(x_n)) = 0.$$

Proof. By Lemma 4, $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F$. We assume that

$$\lim_{n\to\infty} d(x_n, p) = r \quad \text{for some } r \geq 0. \tag{5}$$

The case $r = 0$ is trivial. Next, we deal with the case $r > 0$. Now (3) can be rewritten as

$$(1-\alpha_n) d(x_{n+1}, p) \leq (1-\alpha_n) d(y_n, p) + \alpha_n d(x_n, p) - \alpha_n d(x_{n+1}, p).$$

This implies that

$$\begin{aligned} d(x_{n+1}, p) &\leq d(y_n, p) + \frac{\alpha_n}{1-\alpha_n}[d(x_n, p) - d(x_{n+1}, p)] \\ &\leq d(y_n, p) + \frac{b}{1-b}[d(x_n, p) - d(x_{n+1}, p)] \end{aligned}$$

and so $r \leq \liminf_{n\to\infty} d(y_n, p)$. Taking limit superior on both sides in the inequality (4), we get $\limsup_{n\to\infty} d(y_n, p) \leq r$. Hence

$$\lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d\left(W\left(t_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) = r. \tag{6}$$

Since

$$d(t_n, p) \leq H(P_Q(x_n), P_Q(p)) \leq d(x_n, p),$$

then we have

$$\limsup_{n\to\infty} d(t_n, p) \leq r. \tag{7}$$

From (5)-(7) and Lemma 1, we obtain

$$\lim_{n\to\infty} d(t_n, x_n) = 0. \tag{8}$$

Since $d(x, P_Q(x)) = \inf_{z\in P_Q(x)} d(x, z)$, therefore

$$d(x_n, P_Q(x_n)) \leq d(x_n, t_n) \to 0 \quad \text{as } n \to \infty.$$

By (4) and the quasi-nonexpansiveness of P_T, we have

$$d(u_n, p) \leq H(P_T(y_n), P_T(p)) \leq d(y_n, p) \leq d(x_n, p).$$

Hence

$$\limsup_{n\to\infty} d(u_n, p) \leq r. \tag{9}$$

Since

$$d(v_n, p) \leq H(P_S(x_n), P_S(p)) \leq d(x_n, p),$$

then we have

$$\limsup_{n\to\infty} d(v_n, p) \leq r. \tag{10}$$

In addition,

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(u_n, v_n, \alpha_n), p) = r. \tag{11}$$

From (9)-(11) and Lemma 1, we obtain

$$\lim_{n\to\infty} d(u_n, v_n) = 0.$$

Hence, from the hypothesis $d(x_n, v_n) \leq d(u_n, v_n)$, we have

$$d(x_n, P_S(x_n)) \leq d(x_n, v_n) \leq d(u_n, v_n) \to 0 \quad \text{as } n \to \infty.$$

Since

$$d(x_n, u_n) \leq d(x_n, v_n) + d(v_n, u_n) \leq 2d(u_n, v_n) \to 0 \quad \text{as } n \to \infty, \tag{12}$$

we conclude that

$$d(x_n, P_T(y_n)) \leq d(x_n, u_n) \to \infty \quad \text{as } n \to \infty.$$

In addition, by (8) and (12), we get

$$\begin{aligned} d(x_n, P_T(x_n)) &\leq d(x_n, u_n) + d(u_n, P_T(x_n)) \\ &\leq d(x_n, u_n) + H(P_T(y_n), P_T(x_n)) \\ &\leq d(x_n, u_n) + Ld(y_n, x_n) \\ &\leq d(x_n, u_n) + L\left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(t_n, x_n) \\ &\leq d(x_n, u_n) + L\left(1 - \frac{a}{1-a}\right) d(t_n, x_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This completes the proof. ∎

We now give our △-convergence theorem.

Theorem 6 *Let X, K and $\{x_n\}$ satisfy the hypotheses of Lemma 5 and $Q, S, T : K \to P(K)$ be three multi-valued mappings such that P_Q, P_S and P_T are nonexpansive. If X is complete, then the sequence $\{x_n\}$ is $\triangle$-convergent to a point in F.*

Proof. It follows from Lemma 4 that the sequence $\{x_n\}$ is bounded. Then $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$. Let $\{z_n\}$ be any subsequence of $\{x_n\}$ with $A_K(\{z_n\}) = \{z\}$. By Lemma 5, we have

$$\lim_{n\to\infty} d(z_n, P_Q(z_n)) = \lim_{n\to\infty} d(z_n, P_S(z_n)) = \lim_{n\to\infty} d(z_n, P_T(z_n)) = 0.$$

Now, we claim that z is a common fixed point of P_Q, P_S and P_T. For this, we define a sequence $\{w_m\}$ in $P_T(z)$. So, we calculate

$$\begin{aligned} d(w_m, z_n) &\leq d(w_m, P_T(z_n)) + d(P_T(z_n), z_n) \\ &\leq H(P_T(z), P_T(z_n)) + d(P_T(z_n), z_n) \\ &\leq d(z, z_n) + d(P_T(z_n), z_n). \end{aligned}$$

Then

$$r(w_m, \{z_n\}) = \limsup_{n\to\infty} d(w_m, z_n) \leq \limsup_{n\to\infty} d(z, z_n) = r(z, \{z_n\}).$$

This implies that $|r(w_m, \{z_n\}) - r(z, \{z_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 2 that $\lim_{m\to\infty} w_m = z$. Note that $Tz \in P(K)$ being proximinal is closed, hence $P_T(z)$ is closed. Consequently $\lim_{m\to\infty} w_m = z \in P_T(z)$ and so $z \in F(P_T)$. Similarly, $z \in F(P_S)$ and $z \in F(P_Q)$. Hence $z \in F$. By the uniqueness of asymptotic center, we can get $x = z$. It implies that the sequence $\{x_n\}$ is $\triangle$-convergent to $x \in F$. The proof is completed. ■

Remark 1 *If we take $Q = S$ in Theorem 6, we get the $\triangle$-convergence theorem in [8].*

Theorem 7 *Let X, K, Q, S, T and $\{x_n\}$ be the same as in Lemma 5. Then*

(i) $\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0$ if $\{x_n\}$ converges strongly to a common fixed point in F.

(ii) $\{x_n\}$ converges strongly to a common fixed point in F if X is complete and either $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$.

Proof. (i) Let $p \in F$. Since $\{x_n\}$ converges strongly to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\epsilon > 0$, there exists $n_0 \in N$ such that $d(x_n, p) < \epsilon$ for all $n \geq n_0$. Taking infimum over $p \in F$, we get

$$d(x_n, F) < \epsilon \quad \text{for all } n \geq n_0.$$

This means $\lim_{n\to\infty} d(x_n, F) = 0$ so that

$$\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0.$$

(ii) Suppose that X is complete and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. It follows from Lemma 4 that $\lim_{n\to\infty} d(x_n, F)$ exists. Then, we get

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

The proof of the remaining part follows the proof of Theorem 2.5 in [8]. ■

Recall that a multi-valued mapping $T : K \to P(K)$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

Gu and He [6] defined the concept of condition $(A^{'})$ for N multi-valued mappings. We can define this concept for three multi-valued mappings as follows.

The mappings Q, S and T are said to satisfy *condition* $(A^{'})$ if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \ \leq \frac{1}{3}\left[d(x, Qx) + d(x, Sx) + d(x, Tx)\right] \quad \text{for all} \quad x \in K.$$

By using the above definitions, we can easily prove the following strong convergence result.

Theorem 8 *Let X, K, Q, S, T and $\{x_n\}$ be satisfy the hypotheses of Lemma 5 and X be a complete. If one of the mappings P_Q, P_S and P_T is semi-compact or P_Q, P_S and P_T satisfy condition $(A^{'})$, then the sequence $\{x_n\}$ is convergent strongly to a point in F.*

Remark 2 *(i) Theorems 7, 8 contain the corresponding results of Khan and Abbas [8] when S, T are two multi-valued mappings such that P_S and P_T are nonexpansive and $Q = S$.*

(ii) Our results generalize the corresponding results of Şahin and Başarır [15] from three nonexpansive self mappings to three multi-valued Lipschitzian quasi-nonexpansive mappings.

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