YONEDA LEMMA: COMPLETE ENUNCIATION AND PROOF (INCLUDING THE OCCURRENCE OF THE EMPTY SET SITUATION

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Abstract. A central result in Category Theory is the Yoneda Lemma. The enunciation and proof of this lemma exhibited here are complete, including the situation (generally not taken into account in standard literature) when some of the occurrent sets are empty. An example illustrating the aforementioned situation is introduced.

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1. Introduction

The present paper is dedicated to a systematic and methodical discussion pertaining to the Yoneda Lemma (abbreviated in this Introduction YL). Namely our (limited) goal is to exhibit a rigorous enunciation and a rigorous proof (followed by an example) taking into account all possible situations, namely including the cases when sets which are empty appear (these cases are generally neglected in all standard texts).

YL is a fundamental result of Category Theory. It establishes a bijection (valid for any functor F and any object A between the set of natural transformations from the standard functor H_A (respectively h_A) to F and the image $F(A)$. YL constitutes the basis of the representability theory of functors and allows the embedding of any locally small category $\mathcal{C}$ into the category of contravariant functors from $\mathcal{C}$ to Ens. YL remains valid for preadditive categories.

The Japanese mathematician Nobuo Yoneda (28 March 1930 - 22 April 1996) stated YL in his paper [7]. Namely, the lemma is contained (and essentially proved) in a remark in the proof of some results concerning $R-$modules in the aforementioned paper (so, an explicit enunciation and an explicit proof do no appear in [7]). The real public history of YL originated in an interview of Nobuo Yoneda by Saunders Mac Lane (who proposed the name "Yoneda Lemma") at the Paris Gare du Nord in 1954 (see also [5]). It seems that YL was stated in print for the first time in [3].

A brief survey of the content of the present paper follows. In the first part following this Introduction, some necessary prerequisites appear (notations and basic notions) and, afterwards, we state and prove YL in a complete manner (see the beginning of the Introduction). The final part of the paper is dedicated to an example where we explicitly compute the functorial morphisms appearing in YL, putting into evidence the empty set situations. Reference texts for Category Theory used: the famous standard monograph [6] and the remarkable text [4] (unfortunately not translated).

2. The Yoneda Lemma (Enunciation and Proof)

As usual, in the sequel $\emptyset$ will be the empty set.

Let $\mathcal{C}$ be a category. The class of all objects in $\mathcal{C}$ is $Ob(\mathcal{C})$. If A, B are in $Ob(\mathcal{C})$, the class of all morphisms $f : A \to B$ is denoted by $Hom(A, B)$. The unit of A is $1_A \in Hom(A, A)$. In case $Hom(A, B)$ is a set for any A, B in $Ob(\mathcal{C})$ we say that the category $\mathcal{C}$ is *locally small*.

Assume that $\mathcal{C}$ and $\mathcal{C}'$ are given categories and let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C} \to \mathcal{C}'$ be two covariant (respectively contravariant) functors.

A *functorial morphism* (*natural transformation* –alternative name) $f : F \to G$ is a family $(f_Y)_{Y \in Ob(\mathcal{C})}$ of $\mathcal{C}'-$morphisms having the following property: for any Y, Z in $Ob(\mathcal{C})$ and any u in $Hom(Y, Z)$ (respectvely any u in $Hom(Z, Y)$), the following diagram is commutative:

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{f_Y} & G(Y) \\
F(u) \downarrow & & \downarrow G(u) \\
F(Z) & \xrightarrow{f_Z} & G(Z)
\end{array}
$$

(i.e. $G(u) \circ f_Y = f_Z \circ F(u)$).

Recall that Ens (many authors write Set instead of Ens) is the locally small category having all the sets as objects and all the functions as morphisms (i.e. for any sets A, B, one has $Hom(A, B) =$all the functions $f : A \to B$)

Continuing to speak about Ens, we shall recall some technical details about functions. In the spirit of the fundamental definitions (see e.g. [1]), we recall that, if A, B are sets, a function $f : A \to B$ is a set $f \subset A \times B$ (i.e. f is a *correspondence between A and B*) such that:

a) If $a \in A$ and $b_1 \in B$, $b_2 \in B$, one has the implication

$$
((a, b_1) \in f \ \text{ and } \ (a, b_2) \in f) \Rightarrow (b_1 = b_2)
$$

(i.e. *the correspondence is functional*).

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b) One has the equality

$$\{a \in A \mid \text{There exists } b \in B \text{ such that } (a,b) \in f\} = A$$

(i.e. *the domain of f is A*).

For $f : A \to B$, $x \in A$ and $y \in B$ such that $(x,y) \in f$ one usually writes $y = f(x)$, arriving at "normal" facts.

Let $f : A \to B$ be a function. We have the equivalence $(f = \emptyset) \Leftrightarrow (A = \emptyset)$ and the implications $(B = \emptyset) \Rightarrow (A = \emptyset) \Rightarrow (f = \emptyset)$. So, we can write $\emptyset : \emptyset \to \emptyset$, hence $\emptyset = 1_\emptyset$, consequently $Hom(\emptyset, \emptyset) = \{1_\emptyset\}$ (and $Hom(A, A) \neq \emptyset$ for any set A). If $f : A \to B$ and $g : B \to C$, one can compute $g \circ f : A \to C$ in Ens. We have the implications $(A = \emptyset) \Rightarrow (f = \emptyset) \Rightarrow (g \circ f = \emptyset)$ and also $(B = \emptyset) \Rightarrow (A = \emptyset)$, hence $(B = \emptyset) \Rightarrow (f = g = \emptyset)$. Finally notice that $(C = \emptyset) \Rightarrow (B = \emptyset)$ a. s. o. So, if one of the sets A, B, C is empty , one has $g \circ f = \emptyset$.

Now, let us consider a locally small category $\mathcal{C}$ and let X in $Ob(\mathcal{C})$. We shall recall the definition of the standard covariant functor $H_X : \mathcal{C} \to Ens$ (respectively contravariant functor $h_X : \mathcal{C} \to Ens$).

The action of H_X (respectively h_X) is described in the sequel. On objects: for any A in $Ob(\mathcal{C})$, $H_X(A) \stackrel{def}{=} Hom(X, A)$ (respectively $h_X(A) \stackrel{def}{=} Hom(A, X)$). On morphisms: take A, B in $Ob(\mathcal{C})$ such that $Hom(A, B) \neq \emptyset$, let $u \in Hom(A, B)$ and define

$$H_X(u) : H_X(A) = Hom(X, A) \to Hom(X, B) = H_X(B)$$

(respectively

$$h_X(u) : h_X(B) = Hom(B, X) \to Hom(A, X) = h_X(A))$$

as follows.

If $Hom(X, A) = \emptyset$ (respectively $Hom(B, X) = \emptyset$), define $H_X(u) = \emptyset$ (respectively $h_X(u) = \emptyset$). If $Hom(X, A) \neq \emptyset$ and $t \in Hom(X, A)$ (respectively $Hom(B, X) = \emptyset$ and $t \in Hom(B, X)$) define $H_X(u)(t) = u \circ t$, according to the schema $X \stackrel{t}{\to} A \stackrel{u}{\to} B$ (respectively $h_X(u)(t) = t \circ u$, according to the schema $A \stackrel{u}{\to} B \stackrel{t}{\to} X$) .

The present notations are not standard. Many authors write h^X or $Hom(X, -)$ instead of H_X (respectively h_X or $Hom(-, X)$ instead of h_X).

Now, let us consider a locally small category $\mathcal{C}$ and a covariant (respectively contravariant) functor $F : \mathcal{C} \to Ens$. We shall consider the class $Hom(H_X, F)$ (respectively $Hom(h_X, F)$ of all functorial morphism $f : H_X \to F$ (respectively $f : h_X \to F$). Such f has the form $f = (f_A)_{A \in Ob(\mathcal{C})}$, where, for any A in $Ob(\mathcal{C})$,

$f_A : H_X(A) \to F(A)$ (respectively $f_A : h_X(A) \to F(A)$ and the diagram

$$
\begin{array}{ccc}
Hom(X, A) & \xrightarrow{f_A} & F(A) \\
H_X(u) \downarrow & & \downarrow F(u) \\
Hom(X, B) & \xrightarrow{f_B} & F(B)
\end{array}
\tag{1}
$$

(respectively

$$
\begin{array}{ccc}
Hom(A, X) & \xrightarrow{f_A} & F(A) \\
h_X(u) \downarrow & & \downarrow F(u) \\
Hom(B, X) & \xrightarrow{f_B} & F(B)
\end{array}
\tag{1'})
$$

is commutative, for any A, B in $Ob(\mathcal{C})$ such that $Hom(A, B) \neq \emptyset$ (respectively $Hom(B, A) \neq \emptyset$) and any $u \in Hom(A, B)$ (respectively $u \in Hom(B, A)$).

The commutativity of the diagram (1) (respectively (1')) is guaranteed in case one of the sets $Hom(X, A)$, $Hom(X, B)$, $F(A)$, $F(B)$ (respectively $Hom(A, X)$, $Hom(B, X)$, $F(A)$, $F(B)$) is empty, according to the preceding facts. In case $Hom(X, A) \neq \emptyset$ (respectively $Hom(A, X) \neq \emptyset$), the commutativity of the schema (1) (respectively (1') means that, for any $X \xrightarrow{x} A \xrightarrow{u} B$ (respectively $B \xrightarrow{u} A \xrightarrow{x} X$ in $\mathcal{C}$ one has

$$
F(u)\,(f_A(x)) = f_B(u \circ x) \text{ (respectively } F(u)\,(f_A(x)) = f_B(x \circ u)). \tag{2}
$$

Lemma 1 (Yoneda Lemma). *Let $\mathcal{C}$ be a locally small category and assume that $F : \mathcal{C} \to Ens$ is a covariant (respectively contravariant) functor. Then, for any X in $Ob(\mathcal{C})$, either the sets $F(X)$ and $Hom\,(H_X, F)$ (respectively $F(X)$ and $Hom\,(h_X, F)$) are simultaneously empty, or the sets $F(X)$ and $Hom\,(H_X, F)$ (respectively $F(X)$ and $Hom\,(h_X, F)$) are simultaneously not empty.*

In the last case, there exists a bijection $a_X : Hom\,(H_X, F) \to F(X)$ (respectively $a_X : Hom\,(h_X, F) \to F(X)$) which acts as follows: for any $f = (f_A)_{A \in Ob(\mathcal{C})}$ in $Hom\,(H_X, F)$ (respectively $Hom\,(h_X, F)$), one has

$$
a_X(f) = f_X\,(1_X)\,.
$$

The inverse of a_X is the bijection $b_X : F(X) \to Hom\,(H_X, F)$, (respectively $b_X : F(X) \to Hom\,(h_X, F)$) and will be described in the sequel. Take an arbitrary $t \in F(X)$. Then $b_X(t) = (f_A)_{A \in Ob(\mathcal{C})}$ where, for any A in $Ob(\mathcal{C})$, one has $f_A : Hom(X, A) \to F(A)$ (respectively $f_A : Hom(A, X) \to F(A)$). In case $Hom(X, A) = \emptyset$ (respectively $Hom(A, X) = \emptyset$), one has $f_A = \emptyset$. In case $Hom(X, A) \neq \emptyset$ (respectively $Hom(A, X) \neq \emptyset$), f_A acts upon $x \in Hom(X, A)$ (respectively $x \in Hom(A, X)$), via

$$
f_A(x) = F(x)(t).
$$

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Proof. Take an arbitrary X in $Ob(\mathcal{C})$.

1. Assume first that $F(X)$ is empty. Then $Hom(H_X, F)$ (respectively $Hom(h_X, F)$) must be empty. Indeed, in the contrary case, take $f = (f_A)_{A \in Ob(\mathcal{C})}$ in $Hom\,(H_X, F)$ (respectively in $Hom\,(h_X, F)$). Then $f_X : Hom\,(X, X) \to f(X)$ and $f_X\,(1_X) \in F(X)$, contradiction. We proved the implication

$$(F(X) = \emptyset) \Rightarrow (Hom\,(H_X, F) = \emptyset \ \text{(respectively} \ Hom\,(h_X, F) = \emptyset)). \qquad (3)$$

2. Now, assume that $F(X)$ is not empty. For any $t \in F(X)$, let us perform the construction of $b_X(t)$ indicated in the enunciation. We must show that $b_X(t) = (f_A)_{A \in Ob(\mathcal{C})}$ is in $Hom\,(H_X, F)$ (respectively in $Hom\,(h_X, F)$). To this end, we take A, B in $Ob(\mathcal{C})$ such that $Hom(A, B) \neq \emptyset$ (respectively $Hom(B, A) \neq \emptyset$).

In case F is covariant, we must show (see (2)) that for any $u : A \to B$ in $\mathcal{C}$ and any $x : X \to A$ in $\mathcal{C}$ (hence we work for $Hom(X, A) \neq \emptyset$) one has $F(u)\,(f_A(x)) = f_B(u \circ x)$, which means $F(u)(F(x)(t)) = F(u \circ x)(t)$, i. e. $(F(u) \circ F(x))(t) = F(u \circ x)(t)$, obvious.

In case F is contravariant, we must show (see (2)) that for any $u : B \to A$ in $\mathcal{C}$ and any $x : A \to X$ in $\mathcal{C}$ (hence we work for $Hom(A, X) = \emptyset$), one has $F(u)\,(f_A(x)) = f_B(x \circ u)$, which means $F(u)(F(x)(t)) = F(x \circ u)(t)$, i.e. $(F(u) \circ F(x))(t) = f(x \circ u)(t)$, obvious.

We succeeded in defining the function $b_X : F(X) \to Hom\,(H_X, F)$ (respectively $b_X : F(X) \to Hom\,(h_X, F)$) in case F is covariant (respectively contravariant) acting via $t \mapsto b_X(t)$. This shows that $Hom\,(H_X, F)$ (respectively $Hom\,(h_X, F)$) is not empty, in case $F(X)$ is not empty. Consequently, we also proved the implication

$$(Hom\,(H_X, F) = \emptyset \ \text{(respectively} \ Hom\,(h_X, F) = \emptyset)) \Rightarrow (F(X) = \emptyset). \qquad (4)$$

The equivalence given by (3) and (4) shows that the sets $F(X)$ and $Hom\,(H_X, F)$ (respectively $F(X)$ and $Hom\,(h_X, F)$) are either simultaneously empty, or simultaneously not empty.

3. Assume that F is covariant and $Hom\,(H_X, F)$ is not empty (respectively F is contravariant and $Hom\,(h_X, F)$ is not empty), hence $F(X)$ is not empty. We can construct the function $a_X : Hom\,(H_X, F) \to F(X)$ (respectively $a_X : Hom\,(h_X, F) \to F(X)$) via

$$a_X(f) = f_X\,(1_X)$$

where $f = (f_A)_{A \in Ob(\mathcal{C})}$ is arbitrarily taken in $Hom\,(H_X, F)$ (respectively $Hom\,(h_X, F)$).

4. The final step of our proof will consist in showing that the functions a_X and b_X are mutually inverse. Hence, we must show that:

a) For any $t \in F(X)$ one has $a_X(b_X(t)) = t$.

b) One has $b_X(a_X(f)) = f$, for any $f \in Hom(H_X, F)$ in case F is covariant (respectively, for any $f \in Hom(h_X, F)$, in case F is contravariant).

Proof of a)

Take $t \in F(X)$ and construct $b_X(t) = f = (f_A)_{A \in Ob(\mathcal{C})}$ as in the enunciation.

In case F is covariant and $Hom(X, A) \neq \emptyset$, which is the case of $Hom(X, X)$ (respectively in case F is contravariant and $Hom(A, X) \neq \emptyset$, which is the case of $Hom(X, X)$), we have $f_A : Hom(X, A) \to F(A)$ (respectively $f_A : Hom(A, X) \to F(A)$), acting via $f_A(x) = F(x)(t)$, for any $x \in Hom(X, A)$ (respectively $x \in Hom(A, X)$). Hence $a_X(b_X(t)) = a_X(f) = f_X(1_X) = F(1_X)(t) = 1_{F(X)}(t) = t$.

Proof of b)

In case F is covariant (respectively contravariant) we take arbitrarily $f = (f_A)_{A \in Ob(\mathcal{C})}$ in $Hom(H_X, F)$ (respectively $Hom(h_X, F)$), then construct $t = a_X(f) = f_X(1_X)$ and finally we construct $b_X(a_X(f)) = b_X(t) = g = (g_A)_{A \in Ob(\mathcal{C})}$. We must prove that $g = f$, i.e. $f_A = g_A$ for any $A \in Ob(\mathcal{C})$. Notice that $f_A = g_A = \emptyset$ in case $Hom(X, A) = \emptyset$ (respectively in case $Hom(A, X) = \emptyset$).

So, in case F is covariant respectively contravariant), to prove that $f = g$ means to prove that, for any A in $Ob(\mathcal{C})$ such that $Hom(X, A) \neq \emptyset$ (respectively $Hom(A, X) \neq \emptyset$)and for any $x \in Hom(X, A)$ (respectively $x \in Hom(A, X)$), one has $g_A(x) = f_A(x)$. For such x one has $x : X \to A$ (respectively $x \in A \to X$) in C and one can consider the commutative diagram

$$
\begin{array}{ccc}
Hom(X, X) & \xrightarrow{f_X} & F(X) \\
H_X(x) \downarrow & & \downarrow F(x) \\
Hom(X, A) & \xrightarrow{f_A} & F(A)
\end{array}
\qquad (5)
$$

(respectively

$$
\begin{array}{ccc}
Hom(X, X) & \xrightarrow{f_X} & F(X) \\
h_X(x) \downarrow & & \downarrow F(x) \\
Hom(A, X) & \xrightarrow{f_A} & F(A)
\end{array}
\qquad (5'))
$$

Using (5) (respectively (5′) for $1_X \in Hom(X, X)$, we obtain

$$F(x)(f_X(1_X)) = f_A(H_X(x)(1_X)). \text{ i.e. } F(x)(t) = f_A(x \circ 1_X) = f_A(x)$$

(respectively

$$F(x)(f_X(1_X)) = f_A(h_X(x)(1_X)). \text{ i.e. } F(x)(t) = f_A(1_X \circ x) = f_A(x)).$$

But, according to the construction of $g = b_X(t)$, one has for $x \in Hom(X, A)$ (respectively $Hom(A, X)$) the equality $g_A(x) = F(x)(t)$. This shows that $g_A(x) = f_A(x)$.

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The proof is complete. □

Remark. Applying the Yoneda Lemma for the functor $F = H_Y$ (respectively $F = h_Y$) one obtains, for any two objects X, Y in $Ob(\mathcal{C})$, a canonical bijection between the sets $Hom(H_X, H_Y)$ and $Hom(Y, X)$ (respectively $Hom(h_X, h_Y)$ and $Hom(X, Y)$). This procedure was used in [2] in order to obtain, e. g., an abstract representation of the dual X' of a topological vector space X.

3. EXAMPLE (YONEDA LEMMA IN ACTION)

We exhibit an application of the Yoneda Lemma to an example where all kind of sets which are empty appear.

First we introduce the category $\mathcal{C}$ with $Ob(\mathcal{C}) = \{0, 1, 2\}$ and such that $Hom(0, 1) = \{u\}$, $Hom(1, 2) = \{v\}$, $Hom(0, 2) = \{w\}$, $Hom(i, j) = \emptyset$, whenever $i > j$ and $Hom(i, i) = \{1_i\}$, $i = 0, 1, 2$.

The basic composition rule is $w = v \circ u$.

See the following commutative diagram which illustrates $\mathcal{C}$

$$0 \xrightarrow{\ u\ } 1$$
$$\searrow w \quad \downarrow v$$
$$2$$

(we also have, of course the compulsory rules $u = u \circ 1_0 = 1_1 \circ u$, $v = v \circ 1_1 = 1_2 \circ v$, $w = w \circ 1_0 = 1_2 \circ w$). The reader can see that $\mathcal{C}$ is the category generated by the totally ordered set $\{0, 1, 2\}$.

Next, we construct a covariant functor $F : \mathcal{C} \to Ens$. To this end, consider two non empty sets A, B and an arbitrary function $\varphi : A \to B$. The functor $F : \mathcal{C} \to Ens$ acts as follows. First, on objects: $F(0) \overset{def}{=} \emptyset$, $F(1) \overset{def}{=} A$, $F(2) \overset{def}{=} B$. Next, on morphisms: $F(u) = \emptyset$, $F(v) = \varphi$, $F(w) = \emptyset$, $F(1_0) = \emptyset$, $F(1_1) = 1_A$, $F(1_2) = 1_B$.

Explanations (F is correctly defined as a covariant functor):

$u : 0 \to 1 \Rightarrow F(u) : F(0) = \emptyset \to F(1) = A$, hence $F(u) = \emptyset$,

$v : 1 \to 2 \Rightarrow F(v) : F(1) = A \to B = F(2)$ and we can take $F(v) = \varphi : A \to B$.

$w : 0 \to 2 \Rightarrow F(w) : F(0) = \emptyset \to F(2) = B$, hence $F(w) = \emptyset$.

$F(1_0) = 1_{F(0)} = 1_\emptyset = \emptyset$; $F(1_1) = 1_{F(1)} = 1_A$; $F(1_2) = 1_{F(2)} = 1_B$.

Notice that the relation $F(v \circ u) = F(v) \circ F(u)$ is true because $F(v \circ u) = F(w) = \emptyset$ and $F(v) \circ F(u) = \varphi \circ \emptyset = \emptyset$.

The assertion in the Yoneda Lemma will be checked via inspection, computing all $Hom(H_X, F)$, $X \in Ob(\mathcal{C})$.

For $X = 0$, we have $F(0) = \emptyset$ and we have seen that $Hom(H_0, F) = F(0) = \emptyset$.

Next, we shall obtain the bijections between $A = F(1)$ and $Hom\,(H_1, F)$ (respectively, between $B = F(2)$ and $Hom\,(H_2, F)$), stipulated in the Yoneda Lemma, computing $Hom\,(H_1, F)$ and $Hom\,(H_2, F)$.

Specifically, for $i = 1$ or $i = 2$, an element $f \in Hom\,(H_i, F)$ will be a family $f = \left(f_t^i\right)_{i \in I}$, where $I = \{0, 1, 2\}$ and $f_t^i : Hom\,(i, t) \to F(t)$, $t = 0, 1, 2$ are function such that the following diagram is commutative

$$
\begin{array}{ccc}
Hom(i, s) & \xrightarrow{f_s^i} & F(s) \\
H_i(x) \downarrow & & \downarrow F(x) \\
Hom(i, t) & \xrightarrow{f_t^i} & F(t)
\end{array}
\tag{6}
$$

whenever $x \in Hom(s, t)$, s, t being in $\{0, 1, 2\}$. So, the situations when $s > t$ (i.e.: $s = 1$, $t = 0$; $s = 2$, $t = 0$; $s = 2$, $t = 1$), implying $Hom(s, t) = \emptyset$, will not be taken into accont.

Case $i = 1$ ($f \in Hom(H_1, F)$ has the form $f = \left(f_0^1, f_1^1, f_2^1\right)$).

Using diagram (6) for $i = 1$, one sees that, if $s = 0$, one has $Hom(1, 0) = \emptyset$, hence $f_0^1 = \emptyset$ and (6) is commutative for any f_t^1, $t = 0, 1, 2$.

Taking $s = 1$, we have either $t = 1$ or $t = 2$. If $s = t = 1$, one has in (6) $x : 1 \to 1$, hence $x = 1_1$. Because $Hom(1, 1) = \{1_1\}$ one must have
$\left(F\left(1_1\right) \circ f_1^1\right)\left(1_1\right) = \left(f_1^1 \circ H_1\left(1_1\right)\right)\left(1_1\right)$, i.e.
$1_A\left(f_1^1\left(1_1\right)\right) = f_1^1\left(H_1\left(1_1\right)\left(1_1\right)\right) = f_1^1\left(1_1\right)$. This is true for any value $f_1^1\left(1_1\right) = a \in A$.

If $s = 1$, $t = 2$ one has in (6) $x : 1 \to 2$, hence $x = v$. Again $Hom(1, 1) = \{1_1\}$ and, because $F(v) = \varphi$, one must have
$\left(\varphi \circ f_1^1\right)\left(1_1\right) = \left(f_2^1 \circ H_1(v)\right)\left(1_1\right)$, i.e.
$\varphi\left(f_1^1\left(1_1\right)\right) = f_2^1\left(H_1(v)\left(1_1\right)\right) = f_2^1\left(v \circ 1_1\right) = f_2^1(v)$.

The remaining case is $s = 2$, $t = 2$, hence in (6) $x : 2 \to 2$, i.e. $x = 1_2$. Because $Hom(1, 2) = \{v\}$, one must have $\left(F\left(1_2\right) \circ f_2^1\right)(v) = \left(f_2^1 \circ H_1\left(1_2\right)\right)(v)$, i.e. $1_B\left(f_2^1(v)\right) = f_2^1\left(1_2 \circ v\right)$ which is true for any value $f_2^1(v) \in B$.

The conclusion for $i = 1$:

$$
Hom\,(H_1, F) = \left\{\left(\emptyset, f_1^1, f_2^1\right)\right\}
$$

where $f_1^1 : \{1_1\} \to A$ and $f_2^1 : \{v\} \to B$ are (of course) constant functions, connected via the relation $f_2^1(v) = \varphi\left(f_1^1\left(1_1\right)\right)$. Writing $f_1^1\left(1_1\right) = a \in A$, we have $f_2^1(v) = \varphi(a)$ and $Hom\,(H_1, F)$ is cardinally equivalent to the set $\{(\emptyset, a, \varphi(a)) \mid a \in A\}$.

Here, for $X = 1$, the Yoneda bijection $a_X : Hom\,(H_X, F) \to F(X)$, acting via $a_X(f) = f_X\left(1_X\right)$ is (taking into account that $f_X = f_1^1$):

$$
a_1 : Hom\,(H_1, F) \to F(1) = A, \quad a_1(f) = f_1^1\left(1_1\right).
$$

Case $i = 2$ ($f \in Hom(H_2, F)$ has the form $f = \left(f_0^2, f_1^2, f_2^2\right)$).

Studying the diagram (6), we see that $Hom(2, s) = \emptyset$ for $s = 0$ or $s = 1$, hence one must have $f_0^2 = f_1^2 = \emptyset$ and the diagram is automatically commutative for any f_t^2, $t = 0, 1, 2$.

It remains to study the case $s = t = 2$, Hence, in (6) one has $x : 2 \to 2$ consequently $x = 1_2$. Because $Hom(2, 2) = \{1_2\}$, one most have $\left(F\left(1_2\right) \circ f_2^2\right)\left(1_2\right) = \left(f_2^2 \circ H_2\left(1_2\right)\right)\left(1_2\right)$, i.e. $1_B\left(f_2^2\left(1_2\right)\right) = f_2^2\left(H_2\left(1_2\right)\left(1_2\right)\right)$, which means $f_2\left(1_2\right) = f_2^2\left(1_2\right)$. This is true for any value $f_2^2\left(1_2\right) = b \in B$.

The conclusion for $i = 2$:

$$Hom\left(H_2, F\right) = \left\{\left(\emptyset, \emptyset, f_2^2\right)\right\}$$

where $f_2^2 : \{1_2\} \to B$ is (of course) a constant function. Writing $f_2^2\left(1_2\right) = b \in B$, it follows that $Hom\left(H_2, F\right)$ is cardinally equivalent to the set $\{(\emptyset, \emptyset, b)) \mid b \in B\}$.

Here, for $X = 2$, the Yoneda bijection $a_X : Hom\left(H_X, F\right) \to F(X)$, acting via $a_X(f) = f_X\left(1_X\right)$ is (taking into account that $f_X = f_2^2$):

$$a_2 : Hom\left(H_2, F\right) \to F(2) = B, \quad a_2 f) = f_2^2\left(1_2\right).$$

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