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A NOTE ON POISSON TYPE INTEGRALS IN PSEUDOCONVEX AND CONVEX DOMAINS OF FINITE TYPE AND SOME RELATED RESULTS

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Abstract. In this short note we extend some well-known results on Hardy type spaces using Poisson type integrals(or representation) in higher dimensions in convex and bounded pseudoconvex domains with smooth boundary.

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1. On some new estimates for Poisson type integrals in pseudoconvex and convex domains.

The goal of this work to study questions related to so-called expanded integral operators acting between (analytic) spaces of various dimensions in context of Poisson type integral operators. Similar study for Bergman typpe integral operators plays very important role in solution of so -called Trace problem in various spaces of analytic functions in various type of domains (see, for example, [2], [4], [11], [14-16]) and various references there also. The following result can be seen in [6-7]. Let D be bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary with the defining ρ function.

Then there exist a pseudoconvex domain $\tilde{D} \supset D$ and functions $K(\xi, z)$ and $\Phi(\xi, z)$ defined for $\xi \in \partial D$, $z \in \tilde{D}$, such that $\tilde{K}(\xi, z)$, $\Phi(\xi, z)$ are holomorphic in $z \in \tilde{D}$, and continuous in $\xi \in \partial D$ for any holomorphic functions $f \in D$, so that for $f \in C\left(\bar{D}\right)$ and $z \in D$ integral formula (analogue of Poisson integral representation)

$$f(z) = \int\limits_{\partial D} (f(\xi)) \frac{\left(\tilde{K}(\xi, z)\right)}{[\Phi(\xi, z)]^n} d\sigma(\xi) \tag{1}$$

holds, where $d\sigma$ is the $(2n-1)$ dimensional measure on ∂D (see [6], [7]). In this note we will use (1) and related formulas to extend some classical results on Hardy type spaces. (other applications of (1) can be seen in [7]).

For same type functions, but on product domains we have using (1) by each variable separately that if $f \in C\left(\bar{D}^n\right)$ and $f \in H\left(D^n\right)$, then

$$f\left(z_1,...,z_n\right) = \int\limits_{\partial D} ... \int\limits_{\partial D} \left[f\left(\xi_1,...,\xi_n\right)\right]\left(\prod_{j=1}^{n} \frac{\tilde{K}\left(\xi_j,z_j\right)}{\left(\Phi\left(\xi_j,z_j\right)\right)^n}\right) d\sigma(\xi_j), \tag{2}$$

$z_j \in D, j = 1,...,n$. .For more detail on Φ and $\tilde{K}$ functions we refer to [6], [7]. (In this paper we assume $\tilde{K} \in C(\bar{D}^n)$).

Our goal first to use (2) for study of some estimates related to so-called trace operator $(T_R) f = f\left(z,...,z\right)$, $z \in D \subset \mathbb{C}^n$ acting in Hardy type spaces in bounded strictly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$. We refer to [4], [2], [11], [14-16] for results on trace operator (diagonal map). We will need the following lemma also taken from [6]. We denote various positive constants by C, c or by C with lower indexes.

Lemma A(see [6]). Let $k < n$, then

$$\int \left|\Phi^{-k}\left(\xi,z\right)\right| d\sigma\left(\xi\right) \le c_1, z \in D.$$

$$\int \left|\Phi^{-n}\left(\xi,z\right)\right| d\sigma\left(\xi\right) \le c_2 \left|\log\left(\rho\left(z\right)\right)\right|, z \in D.$$

If $k > n$

$$\int \left|\Phi^{-k}\left(\xi,z\right)\right| d\sigma\left(\xi\right) \le c \left|\rho\left(z\right)\right|^{n-k}, z \in D.$$

where ρ is a defining functions of our D domain.

The same two facts (analogues of these results) can be seen also in convex domains of finite type in [5] (see also various references there). We refer also to [17]. We now provide a new unit disk result and it is simple proof then extend it repeating same arguments used in proofs of these results into convex and bounded pseudoconvex domains in higher dimension (see [4] for the unit disk case and for $n = 1$ case).

Let U be the unit disk, $H\left(U\right)$ be the class of all analytic functions in U,

$$M_p^p\left(f,r\right) = \int_T \left|f\left(r\xi\right)\right|^p d\zeta, r \in \left(0,1\right), 0 < p \le \infty, T = \partial U.$$

Let $T^n = T \times ... \times T$,

$$\left(M_p^p\right)\left(f,\vec{r}\right) = \int_{T^n} \left|f\left(\vec{r}\xi\right)\right|^p d\xi, r_j \in \left(0,1\right), j = 1,...,n, f \in L^p\left(T^n\right), 1 \le p \le \infty.$$

If $\sup\limits_{r_j<1} M_p(f,\vec{r}) < \infty$ for analytic function in polydisk $U^n = U \times ... \times U$, $1 \le p \le \infty$ then we say $f \in H^p(U^n)$.

Various estimates are known in the unit ball, disk, polydisk for $M_p(f,r)$ function (see, for example, [1] and [2] – [4] and various references). We are interested to estimates related with product domains and Trace operators where $M_p(f,r)$ is involved. We have the following result in the unit polydisk, which extend known classical result from the case of the unit disk. We below provide a complete proof for simple case of the unit disk, and polydisk then based on this result we repeat same arguments to get same type results in higher dimensions also in other more complicated domains (theorem 1 and 2 below).

Theorem A. Let $1 \le p < q < \infty$. Then

$$[M_q(Df,r)](1-r)^{\frac{n}{p}-\frac{n}{q}} \le c(M_p)(f,r), n>1,\ r \in (0,1)$$

where $Df = f(z,...,z)$, $z \in U$, $f \in H^p(U^n)$.

Proof of theorem A.

We put $n=2$, since the proof of general case is the same.

Let $s>1$, $1-\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$, $c = \left(\frac{p}{p-1}\right)$, $b = \left(\frac{s}{s-1}\right)$, $a = q$.

Then we have obviously $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$. Let $\tilde{\Phi}(\xi,z) = \left(\prod\limits_{k=1}^{2} \frac{1}{\|1-(\xi_k,z_k)\|^2}\right)$, $\xi_k \in T$, $z_k \in U$, $k=1,2$.

By Holder"s inequality using Poisson integral representation (see [1]) we have

$$|f(z,z)| = c_1 \left| \int\limits_{T^2} \frac{f(\xi_1,\xi_2)\left(1-|z|^2\right)^2}{|1-\langle\xi_1,z_1\rangle|^2 |1-\langle\xi_2,z_2\rangle|^2} d\xi_1 d\xi_2 \right|, z \in U.$$

Hence we have using elementar estimate of Φ (see [1]).

$$\frac{|f(z,z)|}{(1-|z|)^2} \le c_2 \left(\int\limits_{T^2} |\Phi(\xi,z)|^s |f(\xi)|^p d\vec{\xi} \right)^{\frac{1}{s}} \times$$

$$\times \left(\int\limits_{T^2} |f(\xi)|^p d\tilde{\sigma}_1(\xi) \right)^{\frac{1}{b}} \times \left(\int\limits_{T^2} |\Phi(\xi,z)|^s d\vec{\xi} \right)^{\frac{1}{c}} \le c_3 A(f) B(f) \left(\frac{1}{(1-|z|)^\tau} \right),$$

where $z = |z|\varphi$, $z \in U$, and $\tau = \frac{2(2s-1)}{c}$ and $d\tilde{\sigma}_1(\vec{\xi}) = (d\xi_1 d\xi_2) = d\vec{\xi}$.

Then

$$\int_T |f_R(z,z)|^q (1-|z|)^{\frac{2q}{c}(2s-1)-2q} d\varphi \le c_4 \left(\int_T \int_{T^2} |\Phi(\xi,z)|^s |f_R(\xi)|^p d\xi d\phi \right) \|f_R\|_{H^p(T^2)}^{p\frac{q}{b}}.$$

Finnally we have for $z = r\varphi$, $z \in U$

$$\left(\int_T |f_R(z,z)|^q d\varphi \right)^{\frac{1}{q}} (1-R)^{\alpha} \le c_5 \|f_R\|_{H^p(T^2)}, R \in (0,1),$$

where $\alpha = \frac{4s-2-2q+\frac{2q}{c}(2s-1)}{q}$, $s \in [1,q)$, $\alpha > 0$. Hence we have that

$$\|Df_R\|_{L^q} (1-R)^{\frac{2}{p}-\frac{2}{q}} \le c_6 \|f_R\|_{H^p}, \; R \in (0,1).$$

Theorem A is proved.

Remark. For $n = 1$ (in the unit disk) we have that

$$\left(\int_T |f(r\xi)|^q d\xi \right)^{\frac{1}{q}} \le c_7 \left(1-r^2\right)^{\frac{1}{q}-\frac{1}{p}} M_p(f,r), r \in (0,1), f \in H(U).$$

This is known (see [1]), for $1 \le p < q$,and our goal to find versions of this results in context of various domains in $\mathbb{C}^n$. Note our proof is based on two tools only (Poisson formula and on estimate of Poisson type kernel).

We add some facts from [5] concerning convex domains of finite type and analytic functions on them first.

Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary. We denote by $d\sigma$ the normalized surface measure on ∂D, we denote by $H(D)$ the class of all analytic functions on D.

Let $\vec{N}$ be a real vector field in a neighborhood of ∂D which agree with the outward unit normal vector field on ∂D. For $z \in \partial D$ and $t > 0$ sufficiently small say $0 < t < \delta_0$ the curve of $\vec{N}$ has a unique intersection point with the hypersurface $\delta = t$.

We call this intersection point z_t. For any function f on ∂D let $f_t(z) = f(z_t)$ for $z \in \partial D$ and we define means of f $M_p(f,t) = \left(\int_{\partial D} |f_t|^p d\sigma \right)^{\frac{1}{p}}$, $0 < p < \infty$. We say $f \in H^p(D)$ iff $\left(\sup_{0<t<\delta_0} \right) M_p(f,t) \le \infty$.

H^p Hardy class usually identified with a subspace of $(L^p)(\partial D, d\sigma)$, $1 < p < \infty$. We need basic facts on reproducing kernels of smoothly bounded, convex domains of finite type m defined by a real-valued function ρ with convex intralevel sets (see [5]), $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$. Diederich-Fornaless (see [5]) constructed a good C^∞ family $S(z,\xi)$ of support functions on $D \times D$, holomorphic in $z \in \bar{D}$ and C^∞ in ξ chosen in a suitable neighborhood U of ∂D. Then (see [6] and [5]) there are C^∞ functions $(Q_j(z,\xi))$, $j = 1, ..., n$ holomorphic in Z were defined, such that $S(z,\xi) = \sum_{j=1}^{n} (Q_j(z,\xi))(z_j - \xi_j)$, Henkin proved the integral representation

$$f(z) = c\left(\int_{\xi\in\partial D} \frac{(f(\xi))\, Q\Lambda\left(\bar{\partial}^T Q\right)^{n-1}}{(S(z,\xi)^n)}\right), z \in D, f \in L^1(\partial D) \cap H(D), \qquad (3)$$

for some constant c where $\left(\bar{\partial}^T Q\right)$ means tangential components of $\bar{\partial} Q$. We define $[K(z,\xi)]$, so that

$$f(z) = c\int_{\partial D} f(\xi)\, K(z,\xi),$$

where c is a special constant (see [5]) (analogue of (1) for these convex domains).

Let $K_t(z,\xi)$ be the kernel on $\partial D \times \partial D$ defined by $K_t(z,\xi) = K(z_t,\xi)$. Then as analogue of Lemma A we have the following estimates (see [5]).

Lemma B (see [5]).

$$\int_{\xi\in\partial D} |K_t(z,\xi)|^s \, d\sigma(\xi) \le c_1\left(t^{-n(s-1)}\right) \qquad (C)$$

uniformly in $z \in \partial D$,

$$\int_{\xi\in\partial D} |K_t(z,\xi)|^s \, d\sigma(z) \le c_2\left(t^{-n(s-1)}\right) \qquad (D)$$

uniformly in $\xi \in \partial D$.

These two assertions are serving as a base of our proof on domains of convex type in $\mathbb{C}^n$ as in the unit disk. We refer to [5] - [6] for various results on analytic spaces on such type domains.

Let D be a domain in $\mathbb{C}^n$. Then $D^m = D \times ... \times D$ is a product domain $H(D^m)$ is a space of all analytic functions on D^m, $m \in \mathbb{N}$.

The simple and carefull analyse of the proof of the unit disk case (theorem A) shows that using assertions we provided above in bounded strictly pseudoconvex domains with smooth boundary and in convex domains of finite type the following assertions are valid (complete analogues of theorem A which we provided in case of unit disk above) for such type of domains in $\mathbb{C}^n$. We formulate theorems 1,2 repeating rather simple and short arguments of the unit disk case (see the formulation and the proof of our first theorem above).

Theorem 1. Let D be bounded strongly pseudoconvex domain with smooth boundary in C^n. Let $1 \leq p < q < \infty$. Then

$$\left(\int\limits_{\partial D} \left|\left(\tilde{D} f_r\right)(\xi)\right|^q d\sigma(\xi)\right)^{\frac{1}{q}} \left(r^{\frac{nm}{p}-\frac{nm}{q}}\right) \leq [cM_p(f,r)] =$$

$$= \tilde{c}\left(\int\limits_{\partial D} ... \int\limits_{\partial D} \left|f\left(r\vec{\xi}\right)\right|^p d\tilde{\sigma}\left(\vec{\xi}\right)\right)^{\frac{1}{p}}, n > 1,$$

where $\left(\tilde{D} f_r\right)(z) = f(rz,...,rz)$, $r \in (0, r_0)$, for some fixed positive r_0, $z \in D$ and where $(d\tilde{\sigma})\left(\vec{\xi}\right)$ is the Lebeques measure (normalized) on $\underbrace{\partial D \times ... \times \partial D}_{m}$, $m \in N$ for a positive constant C, $f \in C\left(\bar{D}^m\right)$, $f \in H\left(D^m\right)$.

Theorem 2. Let D be the convex domain of finite type $\tilde{m}$ in $\mathbb{C}^n$, $m \in \mathbb{N}$
Let $1 \leq p < q < \infty$. Then let $D^m = D \times ... \times D$, $f \in H\left(D^m\right) \cap L^1\left(D^m\right)$. Then

$$\left(\int\limits_{\partial D} \left|\tilde{D} f_t\right|^q dv\right)^{\frac{1}{q}} \left(t^{\frac{nm}{p}-\frac{mn}{q}}\right) \leq c\left(\int\limits_{\partial D} ... \int_{\partial D} |f_t(\xi)|^p d\tilde{v}(\xi)\right)^{\frac{1}{p}}, n > 1,$$

where $\left(\tilde{D} f_t\right)(z) = f\left(z_t^1, ..., z_t^m\right)$, $t \in (0, r_0)$,$r_0 > 0$, $z_t^j \in D$, $j = 1, ..., m$ and where $d\tilde{v}$ is the normalized Lebeques measure on $\partial D \times ... \times \partial D$, $z_t^j = z_t^{j+1}$.

We provided lemmas needed for the proof of these results following the proof of the unit disk case. We provide with this the full sketch of proofs of these theorems at the same time leaving small details to readers.

First note (1), (2), (3) and Lemma B serve as substitution of Poisson representation in the unit disk used in the proof of theorem A then we only need to use lemma A and repeat arguments of proof of theorem A to get the proof of theorem 1.

To prove theorem 2 we need other substitution of these two results in context of convex domains provided by us above and again we should repeat stepp by step the proof of theorem A.

2. Final remarks

In this section we add some vital final remarks.

Similar results are valid in context of tube domain over symmetric cones (we refer to [12] for comppplete analogue of Poisson formula and Lemma A.)

Such a result can be seen as an extension ($m = 1$ case) of a recent result of obtained in [12]. The proof is the same as proof of theorem A based on same tools and same simple arguments.

The key ingredient of all our estimates of extended Poisson integral representation are L^p estimates of kernel in such type integral representation. Same type results can be provided in Lie ball matrix domains and also in bounded symmetric domains. We refer the reader for some known for related with Poisson formula and estimates of Szego kernel needed for our proof in context of other domains to [8]-[11], and [12].

These type estimates can be used in solution of various results on traces of H^p functions (see, for example, [2, 4] for the unit disk and the unit polydisk case).

We mention that "diagonal" Poisson integral kernel in the unit disk was the key ingredient of proof of an sharp embedding theorem of type

$$\left(\int_U |f(z,...,z)|^p \, d\mu(z)\right)^{\frac{1}{p}} \leq c \, \|f\|_{h^p(U^n)},$$

where (see [11]), μ is positive Borel measure on U, $1 < p < \infty$, U^n is the unit polydisk and h^p is the space of all n-harmonic functions in the unit polydisk U^n. Probably same ideas can be used where U is acting as pseudoconvex domain or as a tubular domain over symmetric cones.

We add in addition some historical final remarks. Expanded Poisson integral probably for the first time appeared in [11] in the unit disk, then more general version of it expanded Bergman projection in [14-16] in relation with trace problem in Hardy type and Bergman type spaces on product domains. The expanded Bergman projection was studied in solutions of similar type problems in bounded pseudoconvex domains,in the unit polyball and in tubular domains over symmetric cones. The related expanded Poisson integral in C^n is less studied (see [11] for the unit disk case) and this paper is probably the first one that shed some light on this very interesting topic in higher dimension. Such type integral operators may have many applications in various problems in complex function theory also. We note finnaly we plan to continue to study these type operators in analytic function spaces in our subsequent papers.

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