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STARLIKENESS CONDITIONS FOR NORMALIZED ANALYTIC FUNCTIONS INCLUDING RUSCHEWEYH OPERATOR

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Abstract. In the present paper, we introduce special subclass of analytic functions using Ruscheweyh operator. By making use of the notion of differential subordination, we find conditions on the parameters M, α, δ and μ for which

$$\left|\left(1-\alpha+\alpha(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}\right)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu}-\alpha(\lambda+1)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu+1}-1\right|<M,$$

implies that $f \in S_n^*(\delta)$, where $n \in \mathbb{N}$. The results obtained here generalize some previously results given in the literature.

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1. Introduction

Let $\mathcal{A}_n$ denote the class of all analytic functions $f(z)$ in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ which are in the form

$$f(z) = z + a_{n+1}z^{n+1} + \dots, \tag{1}$$

with $\mathcal{A} = \mathcal{A}_1$.

Let $0 \le \delta < 1$. The class of (normalized) starlike functions of order δ, $S_n^*(\delta)$, is defined by

$$S_n^*(\delta) = \left\{ f \in \mathcal{A}_n : \Re\frac{zf'(z)}{f(z)} > \delta,\ z \in \mathbb{D} \right\},$$

with $S^*(\delta) = S_1^*(\delta)$. It is well known that $S^*(0) = S^*$, where S^* is the class of (normalized) starlike functions in $\mathbb{D}$, (see [3]). Simillary, we denote by $K_n(\delta)$ the class of (normalized) convex functions of order δ and define by

$$K_n(\delta) = \left\{ f \in \mathcal{A}_n : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta,\ z \in \mathbb{D} \right\}.$$

It is well known that $f \in K_n(\delta)$, if and only if $zf'(z) \in S_n^*(\delta)$, (see [3]).

Let f, g be analytic in $\mathbb{D}$. We say that f is subordinate to g (or g is superordinate to f) and written as $f \prec g$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ such that

$$w(0) = 0,\ |w(z)| < 1 \text{ and } f(z) = g(w(z)).$$

Let $f, g \in \mathcal{A}$ be given by Teylor series expansions of the forms

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathbb{D}).$$

The Hadamard product (or convolution) of f and g, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{2}$$

Suppose that $f \in \mathcal{A}$. The Ruscheweyh derivative operator [4], $R^\lambda : \mathcal{A} \to \mathcal{A}$, is defined as follows

$$R^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad (\lambda \geq -1, z \in \mathbb{D}). \tag{3}$$

By an easy calculation we find that

$$R^0 f(z) = f(z),\ R^1 f(z) = zf'(z) \text{ and } R^2 f(z) = \frac{z}{2}(2f'(z) + zf''(z)),$$

and so on. Using (3) and strightforward calculations we deduce that for each $\lambda \geq -1$ and $z \in \mathbb{D}$

$$z(R^\lambda f)'(z) = (\lambda+1) R^{\lambda+1} f(z) - \lambda R^\lambda f(z). \tag{4}$$

In [6] some conditions on M, α, δ and μ were determined so that

$$\left| (1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1 \right| < M$$

implies $f \in S_n^*(\delta)$.

Motivated by the recent work of Zhu [6], in the present paper we see that the results remain true for the functions $f \in \mathcal{A}_n$ that satisfy the following condition:

$$\left| \left(1 - \alpha + \alpha(\lambda+2)\frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)}\right)\left(\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right)^{\mu} - \alpha(\lambda+1)\left(\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right)^{\mu+1} - 1 \right| < M. \tag{5}$$

For special choices of α and λ, (5) reduces to the interesting cases that will be given in the corollaries. For the similar results see [1, 2, 5].

To prove our main results we shall use the following lemmas.

Lemma 1. ([6]) Let $B(z), C(z)$ and $D(z)$ be complex functions in $\mathbb{D}$ and let n be a positive integer. Suppose that $D(0) = 0, B(z) \neq 0$ and $\Re\frac{C(z)}{B(z)} \geq -n$ for all $z \in \mathbb{D}$. If $p(z) = p_n z^n + \ldots$ is analytic in $\mathbb{D}$ and satisfies

$$|B(z)zp'(z) + C(z)p(z) + D(z)| < M,$$

for all $z \in \mathbb{D}$, then $|p(z)| < N$ in $\mathbb{D}$, where

$$N = \sup\left\{\frac{M + |D(z)|}{|nB(z) + C(z)|} : z \in \mathbb{D}\right\}.$$

Lemma 2. ([6]) Let $\alpha > 0, \mu > 0$ and

$$M_n(\alpha, \delta, \mu) = \begin{cases} \frac{(\mu+n\alpha)(1-\delta)}{n+\mu(1-\delta)} & ; \quad \alpha \geq \alpha_2 \\ \frac{(\mu+n\alpha)\sqrt{2\alpha(1-\delta)-1}}{\sqrt{n^2\alpha^2+2(n\mu+(1-\delta)\mu^2)\alpha}} & ; \quad \alpha_1 \leq \alpha \leq \alpha_2 \\ \frac{\alpha(\mu+n\alpha)(1-\delta)}{2\mu+(n-\mu+\mu\delta)\alpha} & ; \quad 0 < \alpha < \alpha_1 \end{cases}$$

where $\alpha_2 = \frac{n+\mu(1-\delta)}{n(1-\delta)}$ and

$$\alpha_1 = \frac{\sqrt{9\mu^2 + 2n\mu + n^2 - (18\mu^2 + 2n\mu)\delta + 9\mu^2\delta^2} - 3\mu + n + 3\mu\delta}{2n(1-\delta)}.$$

If $p(z)$ and $q(z)$ are analytic in $\mathbb{D}$ with $p(z) = 1 + p_n z^n + \ldots$, and $q(z) = 1 + q_n z^n + \ldots$, and satisfy $q(z) \prec 1 + \frac{\mu M z}{n\alpha+\mu}$ also $q(z)(1 - \alpha + \alpha p(z)) \prec 1 + Mz$ with $0 < M \leq M_n(\alpha, \delta, \mu)$, then $\Re(p(z)) > \delta$ for all $z \in \mathbb{D}$.

2. Main Results

Using Lemmas 1 and 2, we state and prove the following results.

Theorem 3. Suppose that α, μ, δ, M and $M_n(\alpha, \delta, \mu)$ be defined as in Lemma 2. If $f \in \mathcal{A}_n$ satisfies

$$\left(1 - \alpha + \alpha(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}\right)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu} - \alpha(\lambda+1)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu+1} \prec 1 + Mz,$$

then

$$\Re\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right) > \delta.$$

Proof. Let $q(z) = \left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu}$. Using (4), after an easy computation, we obtain

$$\frac{1}{\mu}\frac{zq'(z)}{q(z)} = (\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} - 1.$$

This gives that

$$q(z) + \frac{\alpha}{\mu}zq'(z)$$
$$= \left(1-\alpha+\alpha(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}\right)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu} - \alpha(\lambda+1)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu+1}.$$

By the assumption of the theorem we have $q(z)+\frac{\alpha}{\mu}zq'(z) \prec 1+Mz$, or equivalently $\left|\frac{\alpha}{\mu}zq'(z)+q(z)-1\right| < M$. From this we see that all conditions of Lemma 1 are satisfied. So, we obtain $|q(z)-1| < N = \frac{\mu M}{\mu+n\alpha}$, which is equivalent to $q(z) \prec 1+\frac{\mu M}{\mu+n\alpha}z$. Let,

$$p(z) = (\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}.$$

The assumption of the theorem shows that

$$q(z)(1-\alpha+\alpha p(z)) \prec 1+Mz.$$

Applying Lemma 2, we see that $\Re p(z) > \delta$. This completes the proof.

Taking $\lambda = -1$ in Theorem 3 we obtain [[6], Theorem 2]:

Corollary 4. Let α, μ, δ, M and $M_n(\alpha,\delta,\mu)$ be defined as in Lemma 2. If $f \in \mathcal{A}_n$ satisfies

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1+Mz$$

then $f \in S_n^*(\delta)$.

Taking $\lambda = \delta = 0$ and $\mu = 1$ in Theorem 3 we obtain the following result:

Corollary 5. Let $\alpha > 0$ and

$$M_n(\alpha) = \begin{cases} \frac{(1+n\alpha)}{n+1} & ; \quad \alpha \geq \frac{n+1}{n} \\ \frac{(1+n\alpha)\sqrt{2\alpha-1}}{\sqrt{n^2\alpha^2+2(n+1)\alpha}} & ; \quad \frac{\sqrt{9+2n+n^2}-3+n}{2n} \leq \alpha < \frac{n+1}{n} \\ \frac{\alpha(1+n\alpha)}{2+(n-1)\alpha} & ; \quad 0 < \alpha < \frac{\sqrt{9+2n+n^2}-3+n}{2n}. \end{cases}$$

If $f \in \mathcal{A}_n$ satisfies

$$\left(1+\alpha+\alpha\frac{zf''(z)}{f'(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)-\alpha\left(\frac{zf'(z)}{f(z)}\right)^2 \prec 1+Mz,$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \Re\left(\frac{zf'(z)}{f(z)}\right)-1.$$

Theorem 6. Let $\mu > 0$ and $0 < \beta \leq \frac{\mu+n}{\sqrt{\mu^2+(\mu+n)^2}}$. If $f \in \mathcal{A}_n$ satisfies

$$\left|\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu}\left[(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right]-1\right| < \beta,$$

then

$$\Re\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right) > \delta$$

where,

$$\delta = \begin{cases} \frac{(\mu+n)(1-\beta)}{\mu+n+\mu\beta} & ; \quad 0 < \beta < \frac{\mu+n}{2\mu+n} \\ \frac{(\mu+n)^2(1-\beta^2)-\mu^2\beta^2}{2(\mu+n)^2-2\mu^2\beta^2} & ; \quad \frac{\mu+n}{2\mu+n} \leq \beta \leq \frac{\mu+n}{\sqrt{\mu^2+(\mu+n)^2}}. \end{cases} \tag{6}$$

Proof. From (6) we have

$$\beta = \begin{cases} \frac{(\mu+n)\sqrt{1-2\delta}}{\sqrt{n^2+2(\mu n+(1-\delta)\mu^2)}} & ; \quad 0 \leq \delta \leq \frac{\mu}{3\mu+n} \\ \frac{(\mu+n)(1-\delta)}{n+\mu+\mu\delta} & ; \quad \frac{\mu}{3\mu+n} < \delta < 1. \end{cases}$$

It is easy to show that the inequality

$$\frac{\sqrt{9\mu^2+2n\mu+n^2-(18\mu^2+2n\mu)\delta+9\mu^2\delta^2}-3\mu+n+3\mu\delta}{2n(1-\delta)} \leq 1$$

is equivalent to $\delta \leq \frac{\mu}{3\mu+n}$. Hence, it is seen that all conditions of Theorem 3 are satisfied with $\beta = M_n(1,\delta,\mu)$ and we obtain $\Re(p(z)) > \delta$, where δ is given by (6) and

$$p(z) = (\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}.$$

Taking $\lambda = -1$ in Theorem 6 we obtain [[6], Theorem 3]:

Corollary 7. Let $\mu > 0$ and $0 < \beta \leq \frac{\mu+n}{\sqrt{\mu^2+(\mu+n)^2}}$. If $f \in \mathcal{A}_n$ satisfies

$$\left| f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right| < \beta; \quad (z \in \mathbb{D}),$$

then $f \in S_n^*(\delta)$, where δ is given by (6).

Finally, taking $\lambda = -1, \mu = 1$ and $zf'(z)$ instead of $f(z)$ in Theorem 3 we obtain [[6], Theorem 4]:

Corollary 8. Let $0 \leq \delta < 1, \alpha > 0$ and

$$M_n(\alpha,\delta) = \begin{cases} \frac{(1+n\alpha)(1-\delta)}{n+1-\delta} & ; \quad \alpha \geq \alpha_2 \\ \frac{(1+n\alpha)\sqrt{2\alpha(1-\delta)-1}}{\sqrt{n^2\alpha^2+2(n+1-\delta)\alpha}} & ; \quad \alpha_1 \leq \alpha \leq \alpha_2 \\ \frac{\alpha(1+n\alpha)(1-\delta)}{2+(n-1+\delta)\alpha} & ; \quad 0 < \alpha < \alpha_1 \end{cases}$$

where $\alpha_2 = \frac{n+1-\delta}{n(1-\delta)}$ and

$$\alpha_1 = \frac{\sqrt{9+2n+n^2-(18+2n)\delta+9\delta^2} - 3 + n + 3\delta}{2n(1-\delta)}.$$

If $f \in \mathcal{A}_n$ satisfies

$$|f'(z) + \alpha z f''(z) - 1| < M; \; (z \in \mathbb{D}),$$

with $0 < M \leq M_n(\alpha,\delta)$, then $zf' \in S_n^*(\delta)$, i.e., f is convex-univalent function of order δ.

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