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ON A CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and study two new subclasses of bi-univalent functions in the open unit disk $U = \{z : |z| < 1\}$ and obtain bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The result presented in this paper generalize the recent work of Srivastava et al. [9].

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k \, z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let S denote the subclass of A, which consist of functions of the form (1.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.

A function $f \in S$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \;\; z \in U$$

and convex of order $\alpha (0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \;\; z \in U.$$

Denote these classes respectively by $S^*(\alpha)$ and $K(\alpha)$.

It is well known by the Koebe one quarter theorem [4] that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}[f(z)] = z,\ (z \in U)$ and $f[f^{-1}(w)] = w,\ (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$.

The inverse of $f(z)$ has a series expansion in some disk about the origin of the form

$$f^{-1}(w) = w + A_2w^2 + A_3w^3 + A_4w^4 + \cdots \quad . \tag{1.2}$$

A function $f(z)$ univalent in a neighborhood of the origin and its inverse satisfy the condition

$$w = f[f^{-1}(w)]$$

Using (1.1), we have

$$w = f^{-1}(w) + a_2\ (f^{-1}(w))^2 + a_3\ (f^{-1}(w))^3 + a_4\ (f^{-1}(w))^4 + \cdots \quad . \tag{1.3}$$

Now using (1.2) we get the following result

$$f^{-1}(w) = w - a_2\ w^2 + (2a_2^2 - a_3)\ w^3 - (5a_2^3 - 5a_2a_3 + a_4)\ w^4 + \cdots \quad . \tag{1.4}$$

A function $f \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by (1.1). Examples of functions in the class Σ are

$$\frac{z}{1-z}\ ,\quad -log(1-z)\ ,\quad \frac{1}{2}log\left(\frac{1+z}{1-z}\right)\ ,$$

and so on. However, the familiar Koebe function is not bi-univalent. Also functions in S such as

$$z - \frac{z^2}{2}\quad \text{and}\quad \frac{z}{1-z^2}$$

are also not members of Σ (see [9]).

Lewin [6] first investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [7], on the other hand, showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. The coefficient problem i.e. bound of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem.

Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of the univalent function class S. Thus following Brannan and Taha [3], a function $f \in A$ of form (1.1) is in the class $S^*_\Sigma(\alpha)$ $(0 < \alpha \leq 1)$ of strongly bi-starlike functions of order α if it satisfies the following conditions:

$$f \in \Sigma \quad \text{and} \quad \left| arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U; 0 < \alpha \leq 1)$$
$$\text{and} \quad \left| arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in U; 0 < \alpha \leq 1)\ ,$$

where g is the extension of f^{-1} to U. The classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding(respectively) to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ (for details see [3]).

Recently, several authors introduced and investigated the various subclasses of bi-univalent functions and obtained bounds for the initial coefficients $|a_2|$ and $|a_3|$ (see, for example, [1, 5, 9, 10].

Srivastava et al. [9] introduced two new subclasses of analytic and bi-univalent functions as follows:

Definition 1. *A function $f(z)$ given by* (1.1) *is said to be in the class $\mathcal{H}^\alpha_\Sigma$ if the following conditions are satisfied:*

$$f \in \Sigma$$
$$\textit{and} \quad \left|arg\left(f'(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U; 0 < \alpha \leq 1)$$
$$\textit{and} \quad \left|arg\left(g'(w)\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in U; 0 < \alpha \leq 1)\ ,$$

where the function g is extension of f^{-1} to U, and is given by

$$g(w) = w - a_2\ w^2 + (2a_2^2 - a_3)\ w^3 - (5a_2^3 - 5a_2a_3 + a_4)\ w^4 + \cdots \quad .$$

Definition 2. *A function $f(z)$ given by* (1.1) *is said to be in the class $\mathcal{H}_\Sigma(\beta)$ if the following conditions are satisfied:*

$$f \in \Sigma$$
$$\textit{and} \quad Re\left(f'(z)\right) > \beta \qquad (z \in U;\ 0 \leq \beta < 1)$$
$$\textit{and} \quad Re\left(g'(w)\right) > \beta \qquad (w \in U;\ 0 \leq \beta < 1)\ ,$$

where the function g is extension of f^{-1} to U, and is given by

$$g(w) = w - a_2\ w^2 + (2a_2^2 - a_3)\ w^3 - (5a_2^3 - 5a_2a_3 + a_4)\ w^4 + \cdots\quad .$$

The object of present investigation is to generalize above two subclasses $\mathcal{H}_\Sigma^\alpha$ and $\mathcal{H}_\Sigma(\beta)$ of analytic bi-univalent function class Σ and find estimates on the initial coefficients $|a_2|$ and $|a_3|$. The techniques used are same as of Srivastava et al. [9].

We begin by setting

$$F_\mu(z) = (1-\mu)f(z) + \mu\ zf'(z)\ , \quad 0 \le \mu \le 1, \quad f \in S, \tag{1.5}$$

so that

$$F_\mu(z) = z + \sum_{n=2}^{\infty}[1 + \mu(n-1)]a_n\ z^n. \tag{1.6}$$

Clearly $F_\mu \in S$ and has an inverse F_μ^{-1}, defined by $F_\mu^{-1}[F_\mu(z)] = z,\ (z \in U)$ and $F_\mu^{-1}[F_\mu(w)] = w,\ (|w| < r_0(f);\ r_0(f) \ge \frac{1}{4})$. In fact, the inverse function F_μ^{-1} is given by

$$F_\mu^{-1}(w) = w - a_2(1+\mu)\ w^2 + [2a_2^2(1+\mu)^2 - a_3(1+2\mu)]\ w^3 + \cdots\quad .$$

In order to derive our main results, we have to recall here the following lemma.

Lemma 1. [8]. Let $h \in P$ the family of all functions h analytic in U for which $Re\{h(z)\} > 0$ and have the form

$$h(z) = 1 + p_1\ z + p_2\ z^2 + p_3\ z^3 + \cdots$$

for $z \in U$. Then $|p_n| \le 2$, for each n.

2. Main results

2.1. Coefficient bounds for the function class $\mathcal{GH}_\Sigma^{\alpha,\mu}$

Definition 3. *A function $F_\mu(z)$ given by* (1.6) *is said to be in the class $\mathcal{GH}_\Sigma^{\alpha,\mu}$ if the following conditions are satisfied:*

$$F_\mu \in \Sigma$$

$$\text{and} \quad \left|arg\left(F'_\mu(z)\right)\right| < \frac{\alpha\pi}{2} \quad (z \in U; 0 < \alpha \le 1) \tag{2.1}$$

$$\text{and} \quad \left|arg\left(G'_\mu(w)\right)\right| < \frac{\alpha\pi}{2} \quad (w \in U; 0 < \alpha \le 1)\ , \tag{2.2}$$

where the function G_μ is extension of F_μ^{-1} to U, and is given by

$$G_\mu(w) = w - a_2(1+\mu)\ w^2 + [2a_2^2(1+\mu)^2 - a_3(1+2\mu)]\ w^3 + \cdots \quad .$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{GH}_\Sigma^{\alpha,\mu}$.

Theorem 1. . *Let $F_\mu(z)$ given by (1.6) be in the class $\mathcal{GH}_\Sigma^{\alpha,\mu}$ Then*

$$|a_2| \le \frac{\alpha}{(1+\mu)}\sqrt{\frac{2}{\alpha+2}} \tag{2.3}$$

$$|a_3| \le \frac{\alpha(3\alpha+2)}{3(1+2\mu)} \tag{2.4}$$

Proof. Clearly, conditions (2.1) and (2.2) can be written as

$$F'_\mu(z) = [p(z)]^\alpha \tag{2.5}$$

and

$$G'_\mu(w) = [q(w)]^\alpha \tag{2.6}$$

respectively.

Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

$$\text{and} \quad q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \quad .$$

Clearly,

$$[p(z)]^\alpha = 1 + \alpha\ p_1\ z + \left(\alpha\ p_2 + \frac{\alpha(\alpha-1)}{2}\ p_1^2\right)\ z^2 + \cdots$$

$$\text{and} \quad [q(w)]^\alpha = 1 + \alpha\ q_1 w + \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2\right) w^2 + \cdots \quad .$$

Also

$$F'_\mu(z) = 1 + (1+\mu)2a_2\ z + (1+2\mu)3a_3\ z^2 + \cdots$$

$$\text{and} \quad G'_\mu(w) = 1 - a_2(1+\mu)\ 2w + [2a_2^2(1+\mu)^2 - a_3(1+2\mu)]\ 3w^2 + \cdots \quad .$$

Now, equating the coefficients in (2.5) and (2.6), we get

$$(1+\mu)\ 2a_2 = \alpha p_1 \ , \tag{2.7}$$

$$(1+2\mu)\ 3a_3 = \alpha\ p_2 + \frac{\alpha(\alpha-1)}{2}\ p_1^2 \ , \tag{2.8}$$

$$-(1+\mu)\ 2a_2 = \alpha q_1 \ , \tag{2.9}$$

$$[2a_2^2(1+\mu)^2 - a_3(1+2\mu)]3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2 \ . \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \tag{2.11}$$

$$\text{and} \quad 8\ a_2^2(1+\mu)^2 = \alpha^2\left(p_1^2+q_1^2\right) \tag{2.12}$$

Now by adding equation (2.10) and equation (2.8) , we get

$$6\ a_2^2(1+\mu)^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right) \ ,$$

by using (2.12), we get

$$6\ a_2^2(1+\mu)^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\left(\frac{8\ a_2^2(1+\mu)^2}{\alpha^2}\right) \ ,$$

$$\Rightarrow \quad a_2^2 = \frac{\alpha^2(p_2+q_2)}{2(\alpha+2)(1+\mu)^2} \ .$$

Applying Lemma 1 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \leq \frac{\alpha}{(1+\mu)}\sqrt{\frac{2}{\alpha+2}} \ .$$

This gives the bound on $|a_2|$ as asserted in (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting equation (2.10) from equation (2.8), we get

$$6\ a_3\ (1+2\mu) - 6a_2^2(1+\mu)^2 = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right) \ .$$

From (2.11) we get $p_1^2 = q_1^2$ and also using (2.12), we have

$$6\ a_3 - 6\left(\frac{\alpha^2(p_1^2+q_1^2)}{8(1+\mu)^2}\right) = \alpha(p_2-q_2)$$

$$6\ a_3 - 6\left(\frac{\alpha^2(2\ p_1^2)}{8(1+\mu)^2}\right) = \alpha(p_2-q_2) \quad (\text{by using } p_1^2 = q_1^2)$$

$$\Rightarrow \quad a_3 = \frac{\alpha^2\ p_1^2}{4(1+2\mu)} + \frac{\alpha(p_2-q_2)}{6(1+2\mu)} \ .$$

Applying Lemma 1 once again for the coefficients p_1, q_1, p_2 and q_2 , we get

$$|a_3| \leq \frac{\alpha^2\ (4)}{4(1+2\mu)} + \frac{\alpha\ (4)}{6(1+2\mu)} \quad .$$

Which yields

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3(1+2\mu)} \quad .$$

This completes the proof of Theorem 1.

3. Coefficient bounds for the function class $\mathcal{GH}_\Sigma(\beta,\mu)$

Definition 4. *A function $F_\mu(z)$ given by (1.6) is said to be in the class $\mathcal{GH}_\Sigma(\beta,\mu)$ if the following conditions are satisfied:*

$$F_\mu \in \Sigma$$

$$\text{and} \quad Re\left(F'_\mu(z)\right) > \beta \quad (z \in U;\ 0 \leq \beta < 1) \tag{3.1}$$

$$\text{and} \quad Re\left(G'_\mu(w)\right) > \beta \quad (w \in U;\ 0 \leq \beta < 1) \tag{3.2}$$

where the function G_μ is extension of F_μ^{-1} to U, and is given by

$$G_\mu(w) = w - a_2(1+\mu)\ w^2 + [2a_2^2(1+\mu)^2 - a_3(1+2\mu)]\ w^3 + \cdots \quad .$$

For functions in the class $\mathcal{GH}_\Sigma(\beta,\mu)$, the following coefficient estimates hold.

Theorem 2. . *Let $F_\mu(z)$ given by (1.6) be in the class $\mathcal{GH}_\Sigma(\beta,\mu)$ Then*

$$|a_2| \leq \frac{1}{(1+\mu)}\sqrt{\frac{2(1-\beta)}{3}} \tag{3.3}$$

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3(1+2\mu)} \tag{3.4}$$

Proof. Clearly, conditions (3.1) and (3.2) can be written as

$$F'_\mu(z) = \beta + (1-\beta)\ p(z) \tag{3.5}$$

and

$$G'_\mu(w) = \beta + (1-\beta)\ q(w) \tag{3.6}$$

respectively.

Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

$$\text{and} \quad q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \quad .$$

Clearly,

$$\beta + (1-\beta)\ p(z) = 1 + (1-\beta)\ p_1\ z + (1-\beta)\ p_2\ z^2 + \cdots$$

$$\text{and} \quad \beta + (1-\beta)\ q(w) = 1 + (1-\beta)\ q_1\ w + (1-\beta)\ q_2\ w^2 + \cdots \quad .$$

Also

$$F'_\mu(z) = 1 + (1+\mu)2a_2\ z + (1+2\mu)3a_3\ z^2 + \cdots$$

$$\text{and} \quad G'_\mu(w) = 1 - a_2(1+\mu)\ 2w + [2a_2^2(1+\mu)^2 - a_3(1+2\mu)]\ 3w^2 + \cdots \quad .$$

Now, equating the coefficients in (3.5) and (3.6), we get

$$(1+\mu)2a_2 = (1-\beta)\ p_1 \ , \tag{3.7}$$

$$(1+2\mu)3a_3 = (1-\beta)\ p_2 \ , \tag{3.8}$$

$$-(1+\mu)2a_2 = (1-\beta)\ q_1 \ , \tag{3.9}$$

$$[2a_2^2(1+\mu)^2 - a_3(1+2\mu)]\ 3 = (1-\beta)\ q_2 \ . \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

$$\text{and} \quad (1+\mu)^2\ 8\ a_2^2 = (1-\beta)^2\left(p_1^2 + q_1^2\right) \tag{3.12}$$

Now by adding equation (3.10) and equation (3.8) , we get

$$6a_2^2(1+\mu)^2 = (1-\beta)(p_2 + q_2)$$

$$a_2^2 = \frac{(1-\beta)}{6(1+\mu)^2}(p_2 + q_2)$$

Thus, we have

$$|a_2^2| \leq \frac{(1-\beta)}{6(1+\mu)^2}(|p_2| + |q_2|)$$

Applying Lemma 1 for the coefficients p_2 and q_2 , we have

$$|a_2| \leq \frac{1}{(1+\mu)}\sqrt{\frac{2(1-\beta)}{3}} .$$

Which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting equation (3.10) from equation (3.8), we get

$$\begin{aligned} 6\ a_3(1+2\mu) - 6\ a_2^2(1+\mu)^2 &= (1-\beta)(p_2-q_2) \\ 6\ a_3(1+2\mu) &= 6\ a_2^2(1+\mu)^2 + (1-\beta)(p_2-q_2) . \end{aligned}$$

From (3.11) we get $p_1^2 = q_1^2$ and also using (3.12), we have

$$\begin{aligned} 6\ a_3(1+2\mu) &= 6\ \frac{1}{8}(1-\beta)^2\left(p_1^2+q_1^2\right) + (1-\beta)(p_2-q_2) \\ 6\ a_3(1+2\mu) &= \frac{3}{4}(1-\beta)^2\left(2p_1^2\right) + (1-\beta)(p_2-q_2) \quad \text{(by using } p_1^2=q_1^2\text{)} \\ a_3 &= \frac{(1-\beta)^2 p_1^2}{4(1+2\mu)} + \frac{(1-\beta)(p_2-q_2)}{6(1+2\mu)} \end{aligned}$$

Applying Lemma 1 for the coefficients p_1, q_1, p_2 and q_2 , we get

$$\begin{aligned} |a_3| &\leq \frac{(1-\beta)^2 4}{4(1+2\mu)} + \frac{(1-\beta)(4)}{6(1+2\mu)} \\ \Rightarrow \quad |a_3| &\leq \frac{(1-\beta)(5-3\beta)}{3(1+2\mu)} . \end{aligned}$$

This completes the proof of Theorem 2.

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