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A NOTE ON HAMEL BASIS OF HILBERT SPACES

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Abstract. It is well known that every vector space has a Hamel basis. In this short note we give a novel proof to show that for an infinite-dimensional Hilbert space, a basis is never a Hamel basis.

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1. Introduction and Preliminaries

The formal idea of using and studying bases in normed linear spaces (or normed linear spaces with a basis) really began with the work, early in 1927, of a young Polish mathematician Juliusz Schauder (1927). He essentially took the idea of a basis from linear spaces and defined it as follows:

> a basis in a normed linear space X is a sequence $\{x_n\}$ such that for each $x \in X$, there is a unique sequence $\{\alpha_n\}$ of scalars so that $x = \sum_n \alpha_n x_n$.

Another strong reason to study bases comes from the fact that basis theory can be used as a tool or technique to help solve a variety of other problems from analysis. A final reason to study basis theory is that a number of classic problems in functional analysis can be rephrased so as to become a problem involving bases in Banach spaces.

A Hilbert space is the abstraction of the finite-dimensional Euclidean spaces of geometry. Its properties are very regular and contain few surprises, though the presence of an infinity of dimensions guarantees a certain amount of surprises. There are two reasons that accounts for the importance of Hilbert spaces. First, they arise as the natural infinite-dimensional generalization of Euclidean spaces, and as such, they enjoy the familiar properties of orthogonality, complemented by the important feature of completeness. Second, the theory of Hilbert spaces serves both as a conceptual framework and as a language that formulates some basic arguments in analysis in a more abstract setting.

It is well known that, as in Euclidean spaces, each Hilbert space can be coordinatized and the vehicle for introducing the coordinates is an orthonormal basis.

Definition 1. *[3] A set B is a Hamel basis for a vector space X if B is a maximal linearly independent subset of X. Alternatively, B is a Hamel basis if every $x \in X$ has a unique representation as a finite linear combination of elements of B.*

Definition 2. *[2] An orthonormal subset of a Hilbert space $\mathcal{H}$ is a subset B having the properties:*

1) for $e \in \mathcal{H}, ||e|| = 1$,

2) if $e_1, e_2 \in \mathcal{H}$ and $e_1 \neq e_2$, then $e_1 \perp e_2$.

A basis for $\mathcal{H}$ is a maximal orthonormal set.

Proposition 1. *[1],[2] If B is an orthonormal set in $\mathcal{H}$, then there is a basis for $\mathcal{H}$ that contains B.*

2. The Main Result

It is well known that every vector space has a Hamel basis. The term basis for a Hilbert space is defined as previous section and it relates to the inner product on $\mathcal{H}$. In this section we give an interesting proof to show that for an infinite-dimensional Hilbert space, a basis is never a Hamel basis.

Theorem 1. *For an infinite-dimensional Hilbert space, a basis is never a Hamel basis.*

Proof. Let $B = \{e_i \in \mathcal{H} : i \in I\}$ be a Hamel basis for the infinite-dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{F}$ be all finite subsets of I and define

$$h_F := \sum_{a_i \in F} \frac{1}{\sqrt{2^{i-1}}} e_{a_i}, \ F \in \mathcal{F}.$$

For $\epsilon > 0$, there is an N such that $\sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} < \frac{\epsilon^2}{2}$. Let $F_0 = \{b_1, \cdots, b_N\}$ and let $F, G \in \mathcal{F}$ such that both contain F_0. Then

$$\begin{aligned}
||h_F - h_G||^2 &= ||\sum_{a_i \in F} \frac{1}{\sqrt{2^{i-1}}} e_{a_i} - \sum_{c_i \in G} \frac{1}{\sqrt{2^{i-1}}} e_{c_i}||^2 \\
&= ||\sum_{a_i \in F \setminus G} \frac{1}{\sqrt{2^{i-1}}} e_{a_i} - \sum_{c_i \in G \setminus F} \frac{1}{\sqrt{2^{i-1}}} e_{c_i}||^2 \\
&= \sum_{a_i \in F \setminus G} \frac{1}{2^{i-1}} + \sum_{c_i \in G \setminus F} \frac{1}{2^{i-1}} \\
&\leq \sum_{a_i \in F \setminus F_0} \frac{1}{2^{i-1}} + \sum_{c_i \in G \setminus F_0} \frac{1}{2^{i-1}} \\
&\leq \sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} + \sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} \\
&< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.
\end{aligned}$$

Hence, $\{h_F : F \in \mathcal{F}\}$ is a Cauchy net in $\mathcal{H}$.

We now show that this net is not convergent in $\mathcal{H}$. Let $h \in \mathcal{H}$. Then h can be represented as following,

$$h = \sum_{i=1}^{N_1} \alpha_i e_{d_i}$$

for some N_1 and α_i. For $0 < \epsilon < 1$, there is an N_2 such that for $n \geq N_2$ we have, $\frac{1}{2^{n-N_1}} < \epsilon$. Define $N_0 := \max\{N_1, N_2\}$ and $E := \{d_1, \cdots, d_{N_0}\}$. If $F \in \mathcal{F}$ contains E and $|F| = n$ (the cardinal number of F), then $n \geq N_0$ and

$$\begin{aligned}
||h_F - h||^2 &= ||\sum_{a_i \in F} \frac{1}{\sqrt{2^{i-1}}} e_{a_i} - \sum_{i=1}^{N_1} \alpha_i e_{d_i}||^2 \\
&= \sum_{i=1}^{N_1} (\frac{1}{\sqrt{2^{i-1}}} - \alpha_i)^2 + \sum_{i=N_1+1}^{n} \frac{1}{2^{i-1}} \\
&\geq \sum_{i=N_1+1}^{n} \frac{1}{2^{i-1}} \\
&= \frac{1}{2^{N_1-1}} (1 - \frac{1}{2^{n-N_1}}) \\
&> \frac{1}{2^{N_1-1}} (1 - \epsilon).
\end{aligned}$$

Therefore, $\{h_F : F \in \mathcal{F}\}$ is not convergent in $\mathcal{H}$ and so $\mathcal{H}$ is not complete.

References

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