Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 40/2014
pp. 291-303
doi: 10.17114/j.aua.2014.40.24

QUASI-CONFORMAL CURVATURE TENSOR ON GENERALIZED (κ, μ)-CONTACT METRIC MANIFOLDS

U.C. De, S. Samui

Abstract. The object of the present paper is to characterize 3-dimensional generalized (κ, μ)-contact metric manifolds satisfying certain curvature conditions on quasi-conformal curvature tensor.

2000 *Mathematics Subject Classification*: 53C15, 53C25.

Keywords: Quasi-conformal curvature tensor, (κ, μ)-contact metric manifold, 3-dimensional generalized (κ, μ)-contact metric manifolds, N(k)-contact metric manifolds, η-Einstein manifolds.

1. Introduction

In 1995, Blair, Koufogiorgos and Papantoniou [9] introduced the notion of (κ, μ)-contact metric manifolds where κ, μ are real constants. Assuming κ, μ smooth functions, Koufogiorgos and Tsichlias [18] introduced the notion of generalized (κ, μ)-contact metric manifolds and gave several examples. Again they also show that such manifold does not exist in dimension greater than three. In a recent paper [2], Yildiz, De and Cetinkaya study concircular curvature tensor in 3-dimensional generalized (κ, μ)-contact metric manifolds. Generalized (κ, μ)-contact metric manifolds have been studied by several authors ([17], [11], [19], [1]) and many others.
In [6], the authors studied extended pseudo projective curvature tensor on contact metric manifolds. Quasi-conformal curvature tensor on Sasakian manifolds has been studied by De, Jun and Gazi [23]. After the Reimannian curvature tensor, Weyl conformal curvature tensor plays an important role in differential geometry as well as in theory of relativity. In [16], Yano and Sawaki defined the notion of the quasi-conformal curvature tensor which is extended form of conformal curvature tensor. According to them a quasi-conformal curvature is defined by

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - \\ &\quad g(X,Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \qquad (1)\end{aligned}$$

for all $X, Y \in$ TM, where a and b are constants, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the n-dimensional manifold $M^n (n \geq 3)$. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1) takes the form

$$\begin{aligned} \widetilde{C}(X,Y)Z &= R(X,Y)Z \\ &-\frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+g(Y,Z)QX - g(X,Z)QY] \\ &+\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z, \end{aligned} \tag{2}$$

where C is conformal curvature tensor [15]. Thus C is a particular case of the tensor $\widetilde{C}$. In a recent paper [25], De and Matsuyama studied quasi-conformally flat manifolds satisfying certain curvature condition on the Ricci tensor. They proved that a quasi-conformally flat manifold satisfying

$$S(X,Y) = rT(X)T(Y), \tag{3}$$

where S is the Ricci tensor, r is the scalar curvature and T is a nonzero 1-form defined by $T(X) = g(X, \rho)$, ρ is a unit vector field, can be expressed as a locally wraped product $I \times_{e^q} M^*$, where M^* is an Einstein manifold. From this result, it easily follows that a quasi-conformal flat space time satisfying (3) is a Robertson-Walker space time [4].

Let M be an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). Since at each point $p \in M$ the tangent space T_pM can be decomposed into direct sum $T_pM = \varphi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$, the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \varphi(T_P) \oplus \{\xi_p\} \;\; p \in M$$

. It may be natural to consider to consider the following particular cases: (1) the projection of the image of C in $\varphi(T_pM)$ is zero; (2) the projection of the image of C in $\{\xi_p\}$ is zero; (3) the projection of image of $C|_{\varphi(T_pM)\times\varphi(T_pM)\times\varphi(T_pM)}$ in $\varphi(T_pM)$ is zero. An almost contact metric manifold satisfying the case (1), (2) and (3) is said to be conformally symmetric [12], ξ-conformally flat [13] and φ-conformally flat [14] respectively. In an analogas way, we define ξ-quasi-conformally flat generalized (κ, μ)-contact metric manifolds.
In [24], the authors studied ξ-conformally flat $N(\kappa)$-contact metric manifolds. In [5], quasi-conformal curvature tensor on Kenmotsu manifolds was studied by Özgür

and De. In a recent paper [22], De and Sarkar studied quasi-conformally flat and extended quasi-conformally flat (κ, μ)-contact metric manifolds.
Motivated by the above studies, we characterize a 3-dimensional generalized (κ, μ)-contact metric manifolds satisfying certain curvature conditions on the quasi-conformal curvature tensor. The present paper is organized as follows:
After preliminaries in section 3, we characterize quasi-conformally flat generalized (κ, μ)-contact metric manifolds. In the next section, we prove that a generalized (κ, μ)-contact metric manifold is locally φ-quasicoformally symmetric if and only if the generalized (κ, μ)-contact metric manifold is a (κ, μ)-contact metric manifold provided $a+b \neq 0$. Besides these, we prove that a ξ-quasiconformally flat generalized (κ, μ)-contact metric manifold is an N(κ)-contact metric manifold provided $(a+b) \neq 0$. Finally, it is shown that generalized (κ, μ)-contact metric manifold satisfying $\widetilde{C} \cdot S = 0$ is η-Einstein provided $(a+b) \neq 0$.

2. Preliminaries

An odd dimensional differentiable manifold M^n is called almost contact manifold if there is an almost contact structure (φ, ξ, η) consisting of a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η satisfying

$$\varphi^2(X) = -X + \eta(X)\xi, \ \eta(\xi) = 1. \tag{4}$$

From (4) it follows that

$$\varphi\xi = 0, \ \eta \circ \varphi = 0.$$

Let g be a compatible Reimannian metric with (φ, ξ, η), that is,

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \ for\ all\ X, Y \in \text{TM}. \tag{5}$$

An almost contact metric structure becomes a contact metric structure if

$$g(X, \varphi Y) = d\eta(X,Y), \ for\ all\ X, Y \in \text{TM}. \tag{6}$$

Given a contact metric manifold $M^n(\varphi, \xi, \eta, g)$ we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\text{L}_\xi\varphi$ where L denotes the Lie differentiation.Then h is symmetric and satisfies

$$h\xi = 0, \ h\varphi + \varphi h = 0, \ \nabla\xi = -\varphi - \varphi h, \ trace(h) = trace(\varphi h) = 0, \tag{7}$$

where ∇ is the Levi-Civita connection.
A contact metric manifold is said to be an η-Einstein manifold if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{8}$$

where a, b are smooth functions and $X, Y \in$ TM, S is the Ricci tensor.

Blair, Koufogiorgos and Papantoniou [9] considered the (κ, μ)-nullity condition and gave several reasons for studying it. The (κ, μ)-nullity distribution N(κ, μ) ([9], [3]) of a contact metric manifold M is defined by

$$N(\kappa,\mu) : p \mapsto N_p(\kappa,\mu) = [U \in T_pM \mid R(X,Y)U = (\kappa I + \mu h)(g(Y,U)X - g(X,U)Y)]$$

for all $X, Y \in$ TM, where $(\kappa, \mu) \in \mathbb{R}^2$.
A contact metric manifold M^n with $\xi \in N(\kappa,\mu)$ is called a (κ, μ)- contact metric manifold.Then we have

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{9}$$

For all $X, Y \in$ TM. If $\mu = 0$, then the (κ, μ)-nullity distribution N(κ, μ) is reduced to κ-nullity distribution N(κ) [21]. If $\xi \in N(\kappa)$, then we call contact metric manifold M an N(κ)- contact metric manifold.
In a (κ, μ)-contact metric manifold the following relations hold:

$$h^2 = (\kappa - 1)\varphi^2, \tag{10}$$

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{11}$$

$$R(\xi, X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{12}$$

$$S(X,\xi) = (n-1)\kappa\eta(X), \tag{13}$$

$$S(X,Y) = [(n-3) - \frac{n-1}{2}\mu]g(X,Y) + \tag{14}$$
$$[(n-3)+\mu]g(hX,Y) + [(3-n) + \frac{n-1}{2}(2\kappa+\mu)]\eta(X)\eta(Y),$$

$$r = (n-1)(n-3+\kappa - \frac{n-1}{2}\mu), \tag{15}$$

A (κ, μ)-contact metric manifold is called a generalized (κ, μ)-contact metric manifold if κ, μ are smooth functions. In [18], Koufogiorgos and Tsichlias proved its existence for 3-dimensional case, whereas greater than 3-dimensional, such manifold does not exist. In generalized (κ, μ)-contact metric manifold $M^3(\varphi, \xi, \eta, g)$ the following relations hold ([18], [3]):

$$\xi\kappa = 0, \tag{16}$$

$$\xi r = 0, \tag{17}$$

$$h\ grad\ \mu = grad\ \mu, \tag{18}$$

$$S(X,Y) = -\mu g(X,Y) + \mu g(hX,Y) + (2\kappa+\mu)\eta(X)\eta(Y), \tag{19}$$

$$S(X,hY) = -\mu g(X,hY) - (\kappa-1)\mu g(X,Y) + (\kappa-1)\mu\eta(X)\eta(Y), \tag{20}$$

$$S(X,\xi) = 2\kappa\eta(X), \tag{21}$$

$$QX = \mu(hX - X) + (2\kappa+\mu)\eta(X)\xi, \tag{22}$$

$$r = 2(\kappa - \mu). \tag{23}$$

$$\begin{aligned}(\nabla_X h)Y &= \{(1-\kappa)g(X,\varphi Y) \\ &\quad -g(X,\varphi hY)\}\xi - \eta(Y)\{(1-\kappa)\varphi X \\ &\quad +\varphi hX\} - \mu\eta(X)\varphi hY,\end{aligned} \tag{24}$$

$$(\nabla_X\varphi)Y = \{g(X,Y) + g(X,hY)\}\xi - \eta(Y)(X + hX). \tag{25}$$

3. Quasi-conformally flat generalized (κ, μ)-contact metric manifolds

Definition 1. *A generalized (κ, μ)-contact metric manifold M^3 is called quasi-conformally flat if the quasi-conformal curvature tensor $\widetilde{C} = 0$.*

It is known that conformal curvature tensor vanishes identically in a 3-dimensional Riemannian manifold. Hence, from (2) we obtain

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \\ &\quad \frac{r}{2}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{26}$$

Substituting $Y = Z = \xi$ in (26) we have

$$QX = \frac{1}{2}(r-2\kappa)X + \frac{1}{2}(6\kappa - r)\eta(X)\xi + \mu hX. \tag{27}$$

Taking inner product with Y of (27) we get

$$\begin{aligned}S(X,Y) &= \frac{1}{2}(r-2\kappa)g(X,Y) \\ &\quad +\frac{1}{2}(6\kappa - r)\eta(X)\eta(Y) + \mu g(hX,Y).\end{aligned} \tag{28}$$

From (1) we have

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - \\ &\quad g(X,Z)QY] - \frac{r}{3}[\frac{a}{2} + 2b][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{29}$$

Putting (26), (27) and (28) in (29) we have

$$\begin{aligned}\widetilde{C}(X,Y)Z &= (a+b)\{\frac{4\kappa+2\mu}{3}[g(X,Z)Y - g(Y,Z)X] + (\kappa+\mu) \\ &\quad [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \\ &\quad \eta(X)\eta(Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY + \\ &\quad g(hY,Z)X - g(hX,Z)Y]\}. \end{aligned} \tag{30}$$

Thus we have

Lemma 3.1. Let M be a 3-dimensional generalized $(\kappa,\ \mu)$ contact metric manifold. Then the quasi-conformal curvature tensor vanishes identically provided $a + b = 0$.

Next we assume that $a + b \neq 0$ and M is Quasi-conformally flat. Then from (30) we have

$$\begin{aligned}\frac{4\kappa+2\mu}{3} &[\ g(X,Z)Y - g(Y,Z)X] + (2\kappa+\mu) \\ &[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y\xi + \eta(Y)\eta(Z)X - \\ &\eta(Z)\eta(X)Y] + \mu[g(Y,Z)hX - g(X,Z)hY + \\ &g(hY,Z)X - g(hX,Z)Y] = 0. \end{aligned} \tag{31}$$

Taking inner product with W of (31) we get

$$\begin{aligned}\frac{4\kappa+2\mu}{3} &[\ g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] + (2\kappa+\mu) \\ &[g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) + \eta(Y)\eta(Z)g(X,W) - \\ &\eta(Z)\eta(X)g(Y,W)] + \mu[g(Y,Z)g(hX,W) - g(X,Z)g(hY,W) + \\ &g(hY,Z)g(X,W) - g(hX,Z)g(Y,W)] = 0. \end{aligned} \tag{32}$$

Putting $Y = Z = \xi$ we have

$$\mu\, g(hX,W) = -\frac{2\kappa+\mu}{3}g(X,W) + \frac{2\kappa+\mu}{3}\eta(X)\eta(W). \tag{33}$$

From (19) and (33) we obtain

$$S(X,W) = ag(X,W) + b\eta(X)\eta(W), \tag{34}$$

where

$$a = -\mu - \frac{2\kappa + \mu}{3}$$

and

$$b = (2\kappa + \mu) + \frac{2\kappa + \mu}{3}.$$

Hence from (34) we conclude the following:

Theorem 3.1. *A 3-dimensional quasi-conformally flat generalized* (κ, μ) *contact metric manifold is an* η*-Einstein manifold if* $a + b \neq 0$.

4. Locally φ-Quasiconformally symmetric generalized (κ, μ)-contact metric manifolds

Definition 2. *A contact metric manifold is said to be locally* φ*-symmetric if the manifold satisfy the following:*

$$\varphi^2((\nabla_X R)(Y, Z)W) = 0, \tag{35}$$

for all vector fields X, Y, Z, W *orthogonal to* ξ. *This notion was introduced for Sasakian manifolds by Takahashi [20].*
In this paper, we study locally φ*-quasiconformally symmetric 3-dimensional generalized* (κ, μ)*-contact metric manifolds. A generalized* (κ, μ)*-contact manifold is called* φ*-quasiconformally symmetric if the condition*

$$\varphi^2((\nabla_X \widetilde{C})(Y, Z)W) = 0, \tag{36}$$

holds on the manifold, where X, Y, Z, W *are orthogonal to* ξ.

Let us consider M be a 3-dimensional generalized (κ, μ)-contact metric manifold. Taking covariant differentiation of (30) we have

$$\begin{aligned} ((\nabla_W \widetilde{C})(X, Y)Z) \quad = \quad & (a + b)\big\{ - \big(\frac{4W\kappa + 2W\mu}{3}\big)[g(Y, Z)X - g(X, Z)Y] + (2\kappa + \mu) \\ & [g(Y, Z)g(W + hW, \varphi X) - g(X, Z)g(W + hW, \varphi Y)]\xi + \\ & (W\mu)[g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y] + \\ & \mu[(1 - \kappa)g(W, \varphi X) + g(W, h\varphi X)]g(Y, Z)\xi - \mu[(1 - \kappa) \\ & g(W, \varphi Y) + g(W, h\varphi Y)]g(X, Z)\xi\big\}, \end{aligned} \tag{37}$$

for all vector fields X, Y, Z, W orthogonal to ξ.
Operating φ^2 to the above equation, we obtain

$$\begin{aligned}\varphi^2((\nabla_W \widetilde{C})(X,Y)Z) &= (a+b)\big\{-(\frac{4W\kappa+2W\mu}{3})+ \\ &\quad (W\mu)[g(Y,Z)[g(Y,Z)hX - g(X,Z)hY + \\ &\quad g(hY,Z)X - g(hX,Z)Y]\big\}, \qquad (38)\end{aligned}$$

for all vector fields X, Y, Z, W orthogonal to ξ.
Thus from (38) we conclude that if κ and μ are constants, then M is locally φ-quasiconformally symmetric. Conversely, let us consider that M is locally φ-quasiconformally symmetric.

From (36) and (38) we have if $(a+b) \neq 0$

$$\begin{aligned}-(\frac{4W\kappa+2W\mu}{3}) \quad [\quad & g(Y,Z)X - g(X,Z)Y] + (W\mu)[g(Y,Z)hX - \\ & g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] = 0. \qquad (39)\end{aligned}$$

Taking inner product with U of (39) we get

$$\begin{aligned}(\frac{4W\kappa+2W\mu}{3}) \quad [\quad & g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] - (W\mu)[g(Y,Z)hX - \\ & g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y] = 0. \qquad (40)\end{aligned}$$

Contracting X and Z we obtain

$$2(\frac{4W\kappa+2W\mu}{3})Y - (W\mu)hY = 0. \qquad (41)$$

Applying h on both sides of (41) we have

$$2(\frac{4W\kappa+2W\mu}{3})hY - (W\mu)h^2Y = 0. \qquad (42)$$

Taking trace on both sides of (42) and using $trace(h) = 0$ we obtain μ is constant. Thus κ is also constant. Therefore, we can state the following:

Theorem 4.1. *Let M be a 3-dimensional generalized (κ, μ)-contact metric manifold.M is locally φ-quasiconformally symmetric if and only if M is a (κ, μ)-contact metric manifold provided $a+b \neq 0$.*

5. ξ-Quasiconformally flat generalized (κ, μ)-contact metric manifolds

Assume that M^3 is a ξ-quasi-conformally flat (κ, μ)-contact metric manifold. So we have

$$\widetilde{C}(X,Y)\xi = 0. \tag{43}$$

From (1) we have

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + \\ &\quad g(Y,Z)QX - g(X,Z)QY] - \frac{r}{3}\big(\frac{a}{2} + 2b\big) \\ &\quad \big[g(Y,Z)X - g(X,Z)Y\big].\end{aligned} \tag{44}$$

Using (26) in (44) we obtain

$$\begin{aligned}\widetilde{C}(X,Y)Z &= (a+b)\big\{[S(Y,Z)X - S(X,Z)Y + \\ &\quad g(Y,Z)QX - g(X,Z)QY] - 2r3 \\ &\quad [g(Y,Z)X - g(X,Z)Y]\big\}.\end{aligned} \tag{45}$$

Putting $Z = \xi$ and using (21) , (22) and (43) we have

$$(a+b)\big[\big(2\kappa - \mu - \frac{2r}{3}\big)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)\big] = 0. \tag{46}$$

Putting $Y = \xi$ in (46) we obtain

$$(a+b)\big[\big(2\kappa - \mu - \frac{2r}{3}\big)(X - \eta(X)\xi) + \mu hX\big] = 0. \tag{47}$$

Applying h on both sides of (47) we get

$$(a+b)\big[\big(2\kappa - \mu - \frac{2r}{3}\big)hX + \mu h^2\big] = 0. \tag{48}$$

Taking trace on both sides of (48) and using $trace(h) = 0$ we have

$$(a+b)\,\mu\,trace(h^2) = 0. \tag{49}$$

As $trace(h^2) \neq 0$ we can conclude that
if $(a+b) \neq 0$, then $\mu = 0$.
If $\mu = 0$, then M^3 is an $N(\kappa)$-contact metric manifold.
From the above discussion we can state the following:

Theorem 5.1. *Let M be a 3-dimensional ξ-quasi-conformally flat generalized $(\kappa$, $\mu)$-contact metric manifold. Then M is an $N(\kappa)$-contact metric manifold provided $(a+b) \neq 0$.*

6. Generalized $(\kappa,\ \mu)$-contact metric manifold satisfying $\widetilde{C}\cdot S=0$

Let M^3 be a generalized $(\kappa,\ \mu)$-contact metric manifold satisfying $\widetilde{C}\cdot S=0$, which implies that

$$S(\widetilde{C}(X,Y)U,V)+S(U,\widetilde{C}(X,Y)V)=0. \tag{50}$$

Putting $X=U=\xi$ in (50) and using (21) we have

$$S(\widetilde{C}(\xi,Y)\xi,V)=2\kappa\eta(\widetilde{C}(\xi,Y)V). \tag{51}$$

Putting $X=\xi$ in (37) and using (21) we obtain

$$\begin{aligned}\widetilde{C}(\xi,Y)V &= (a+b)\big\{\big[S(Y,V)\xi+2\kappa\eta(V)Y+g(Y,V)2\kappa\xi-\eta(V)QY\big]-\\ &\quad \frac{2r}{3}[g(Y,V)\xi-\eta(V)Y]\big\}.\end{aligned} \tag{52}$$

Taking inner product with ξ of (52) we get

$$\eta(\widetilde{C}(\xi,Y)V)=(a+b)\big\{[S(Y,V)+2\kappa g(Y,V)]-\frac{2r}{3}[g(Y,V)-\eta(V)\eta(Y)]\big\}. \tag{53}$$

Putting $V=\xi$ in (52) and using (21) and (22) we have

$$\widetilde{C}(\xi,Y)\xi=(a+b)\big[\big(2\kappa-\mu-\frac{2r}{3}\big)(\eta(Y)\xi-Y)-\mu hY\big], \tag{54}$$

which implies

$$\begin{aligned}S(\widetilde{C}(\xi,Y)\xi,V) &= (a+b)\big[-\big(2\kappa-\mu-\frac{2r}{3}\big)2\kappa\eta(Y)\eta(V)-\\ &\quad \big(2\kappa-\mu-\frac{2r}{3}\big)S(Y,V)-\mu S(hY,V)\big].\end{aligned} \tag{55}$$

Putting (53) and (55) in (51) we obtain

$$\begin{aligned}&(a+b)\big[\big(4\kappa-\mu-\frac{2r}{3}\big)S(Y,V)+\mu S(hY,V)+\big(4\kappa^2-\\ &\frac{4\kappa r}{3}\big)g(Y,V)+\big(2\kappa-\mu-\frac{2r}{3}+\frac{4\kappa r}{3}\big)\eta(V)\eta(Y)\big]=0,\end{aligned}$$

Thus if $(a+b)\neq 0$

$$\begin{aligned}&\big[\big(4\kappa-\mu-\frac{2r}{3}\big)S(Y,V)+\mu S(hY,V)+\big(4\kappa^2-\\ &\frac{4r\kappa}{3}\big)g(Y,V)+\big(2\kappa-\mu-\frac{2r}{3}+\frac{4\kappa r}{3}\big)\eta(V)\eta(Y)\big]=0.\end{aligned} \tag{56}$$

Using (19) and (20) in (56) we have

$$\mu g(hY,V) = a_1 g(Y,V) + b_1 \eta(Y)\eta(V), \tag{57}$$

where

$$a_1 = \frac{[3\mu^2\kappa - 4\mu^2 - 8\kappa^2]}{[8\kappa + 4\mu]},$$

and

$$b_1 = -\frac{[(8\kappa - 2\mu)(3\kappa + \mu) + 3\mu^2\kappa + 2\kappa + \mu]}{8\kappa + \mu}.$$

From (57) and (19) we obtain

$$S(Y,V) = ag(Y,V) + b\eta(Y)\eta(V), \tag{58}$$

where

$$a = -\mu + \frac{[3\mu^2\kappa - 4\mu^2 - 8\kappa^2]}{[8\kappa + 4\mu]},$$

and

$$b = (2\kappa + \mu) - \frac{[(8\kappa - 2\mu)(3\kappa + \mu) + 3\mu^2\kappa + 2\kappa + \mu]}{8\kappa + \mu}.$$

From (58) we can state the following:

Theorem 6.1. *Let M be a 3-dimensional generalized (κ, μ)-contact metric manifold satisfying $\widetilde{C} \cdot S = 0$. Then M is an η-Einstein manifold provided $(a + b) \neq 0$.*

References

[1] A. Sarkar, U.C. De and M. Sen, *Some results on generalized (κ, μ)-contact metric manifolds.* Acta Universitatis Apulensis. 32(2012), $49-59$.

[2] A. Yildiz, U.C. De and A. Cetinkaya, *On some classes of 3-dimensional generalized (κ, μ)-contact metric manifolds*, submitted.

[3] B.J. Papantoniou, *Contact Remannian manifolds satisfying $R(\xi, X) \cdot R = 0$ and $\xi \in (\kappa, \mu)$-nullity distribution*, Yokohama Math.J.40(1993), $149-161$.

[4] B. O'Neill, *Semi-Reimannian Geometry. Academic Press, New York*, 1983.

[5] C. Özgür and U.C. De, *On the quasi-conformal curvature tensor of a Kenmotsu Manifold, Mathematica Pannonica*, 17/2(2006), $221-228$.

[6] C.S. Bagewadi, D.G. Prakasha, and Venkatasha, *On pseudo projective curvature tensor of a contact metric manifold.* SUT J. Math. 43(2007), $115-126$.

[7] D.E. Blair, *Contact manifolds in Reimannian geometry*, Lecture notes in math., 509, Springer-verlag., 1976.

[8] D.E. Blair, *Two remarks on contact metric structures*, Tohoku Math. J. 29(1977), 319 – 324.

[9] D.E. Blair, T. Koufogiorgos and B.J. Papantoniou, *Contact metric manifold satisfying a nullity condition*, Israel J.Math.91(1995), 189 – 214.

[10] E. Boeckx, *A full classificstion of contact metric (κ, μ)-spaces*, Illinois J.Math. 44(2000), 212 – 219 .

[11] F. Gouli-Andreon and P.J. Xenos, *A class of contact metric 3-manifolds with $\xi \in N(\kappa, \mu)$ and κ, μ functions.* Algebras. Group and Geom. 17(200), 401 – 407.

[12] G. Zhen, *On conformal symmetric K-contact manifolds*, Chinese Quart. J. Math. 7(1992), 5 – 10.

[13] G. Zhen, J.L. Cabrerizo, L.M. Fernández and M. Fernández, *On ξ-conformally flat contact metric manifols*, Indian J. Pure Appl. Math. 28(1997), 725 – 734.

[14] J.L. Cabrerizo, L.M. Fernández., M. Fernández and G. Zhen, *The structure of a clss of K-contact manifolds*, Acta Math. Hungar. 82(4)(1999), 331 – 340.

[15] K. Yano, and M. Kon, *Structure on manifolds, Series in Pure Mathematics*, 3. World Scientific Publishing Co., Singapore, 1984.

[16] K. Yano and S. Sawaki, *Reimannian manifold a conformal transformation group*, J.Differential Geometry 2(1968), 161 – 184.

[17] S. Ghosh and U.C. De, *On a class of generalized (κ, μ)-contact metric manifolds.* Proceeding of the Jangjeon Math. Society. 13(2010), 337 – 347.

[18] T. Koufogiorgos and C. Tsichlilias, *On the existance of new class of contact metric manifolds*, Canad. Math.Bull., $XX(Y)$(2000), 1 – 8.

[19] T. Koufogiorgos and C. Tsichlilias, *Generalized (κ, μ)-contact metric manifolds with $\parallel grad\ \kappa \parallel = constant$*, J. Geom. 78(2003), 54 – 65.

[20] T. Takahashi, *Sasakian φ-symmetric spaces*, Tôhoku Math. J. (2)29(1977), 91 – 113.

[21] S. Tanno, *Ricci curvatures of contact Reimannian manifolds*, Tôhoku Math. J. 40(1988), 441 – 448.

[22] U.C. De and A. Sarkar, *On quasi-conformal curvature tensor of (κ, μ)-contact metric manifold*, Math. Reports 14(64), 2(2012), 115 – 129 .

[23] U.C. De, J.B. Jun and A.K. Gazi, *Sasakian manifolds with quasi-conformal curvature tensor*, Bull. Korean Math. Soc. 45(2008), 313 – 319.

[24] U.C. De and S. Biswas, *A note on ξ-conformally flat contact manifolds*, Bull. Malays. Math. Soc. 29(2006), 51 – 57.

[25] U.C. De and Y. Matsuyama, *Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor*. SUT. J. Math. 42(2006), 295 – 303.

Uday Chand De
Department of Pure Mathematics
Calcutta University
35, Ballygunge Circular Road
Kol 700019,West Bengal, India
email: *uc_de@yahoo.com*

Srimayee Samui
Umes Chandra College
13, surya sen street
kol 700012, West Bengal, India
email: *srimayee.samui@gmail.com*