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APPLICATION OF GENERALIZED HADAMARD PRODUCT ON SPECIAL CLASSES OF ANALYTIC P-VALENT FUNCTIONS

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Abstract. In this paper the author established certain results concerning the quasi-Hadamard product for generalized subclasses of p-valent functions with positive coefficients.

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1. Introduction

Let $A(p)$ denote the class of analytic p-valent functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, ...\}). \tag{1}$$

A function $f(z) \in A(p)$ is called p-valent starlike of order α if $f(z)$ satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{2}$$

for $0 \leq \alpha < p$ and $z \in U$. We denote by $S_p^*(\alpha)$ the class of all starlike p-valent functions of order α. Also a function $f(z) \in A(p)$ is called p-valent convex of order α if $f(z)$ satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{3}$$

for $0 \leq \alpha < p$ and $z \in U$. We denote by $C_p(\alpha)$ the class of convex p-valent functions of order α.

For $p < \beta < p + \frac{1}{2}$ and $z \in U$, let $\mathfrak{M}_p(\beta)$ denote the subclass of $A(p)$ consisting of functions $f(z)$ of the form (1) and satisfying the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \tag{4}$$

and let $\mathfrak{N}_p(\beta)$ denote the subclass of $A(p)$ consisting of functions $f(z)$ of the form (1) and satisfying the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \tag{5}$$

it follows from (4) and (5) that

$$f(z) \in \mathfrak{N}_p(\beta) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathfrak{M}_p(\beta) \tag{6}$$

The subclasses $\mathfrak{M}_p(\beta)$ and $\mathfrak{N}_p(\beta)$ and some related classes have been studied by several authors (e.g. [5], [8], [10] and [11]).

Furthermore, let $V(p)$ denote the subclass of analytic p-valent functions of the form:

$$f(z) = a_p z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_p > 0; a_{n+p} \geq 0). \tag{7}$$

Also, let

$$f_i(z) = a_{p,i} z^p + \sum_{n=1}^{\infty} a_{n+p,i} z^{n+p} \quad (a_{p,i} > 0; a_{n+p,i} \geq 0), \tag{8}$$

and

$$g_j(z) = b_{p,j} z^p + \sum_{n=1}^{\infty} b_{n+p,j} z^{n+p} \quad (b_{p,i} > 0; b_{n+p,i} \geq 0), \tag{9}$$

the quasi-Hadamard product $(f_i * g_j)(z)$ of the functions $f_i(z)$
and $g_j(z)$ by

$$(f_i * g_j)(z) = a_{p,i} b_{p,j} z + \sum_{n=2}^{\infty} a_{n+p,i} b_{n+p,j} z^{n+p} \quad (i, j = 1, 2, 3, ...).$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

Also, let $V_p(\beta) = \mathfrak{M}_p(\beta) \cap V(p)$ and $U_p(\beta) = \mathfrak{N}_p(\beta) \cap V(p)$, following the technique of Uralegaddi et al. [12], we can obtain the following lemmas.

Lemma 1. *Let the function $f(z) \in V(p)$, then $f(z) \in V_p(\beta)$ $(p < \beta < p + \frac{1}{2})$ if and only if*

$$\sum_{n=1}^{\infty}(n+p-\beta)a_{n+p} \leq (\beta - p)a_p. \tag{10}$$

Lemma 2. *Let the function $f(z) \in V(p)$, then $f(z) \in U_p(\beta)$ $(p < \beta < p + \frac{1}{2})$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)(n+p-\beta)a_{n+p} \leq (\beta - p)a_p. \tag{11}$$

Let $\varphi(z)$ be a fixed function of the form:

$$\varphi(z) = c_p z + \sum_{n=2}^{\infty} c_{n+p} z^{n+p} \ \ (c_p, c_{n+p} \geq 0). \tag{12}$$

Using the function defined by (12), we now define the following new classes.

Definition 1. *A function $f(z) \in V_{p,\varphi}(c_{n+p}, \delta)$ $(c_n \geq c_2 > 0)$ if and only if*

$$\sum_{n=1}^{\infty} c_{n+p}a_{n+p} \leq \delta a_p \quad (\delta > 0). \tag{13}$$

Definition 2. *A function $f(z) \in U_{p,\varphi}(c_{n+p}, \delta)$ $(c_{n+p} \geq c_{p+1} > 0)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right) c_{n+p}a_{n+p} \leq \delta a_p \quad (\delta > 0). \tag{14}$$

Also, we introduce the following class of analytic p-valent functions which plays an important role in the discussion that follows.

Definition 3. *A function $f(z) \in V^k_{p,\varphi}(c_{n+p}, \delta)$ $(c_{n+p} \geq c_{p+1} > 0)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^k c_{n+p}a_{n+p} \leq \delta a_p \ \ (\delta > 0), \tag{15}$$

where k is any fixed nonnegative real number.

For suitable choices of c_n, δ, k and $a_0 = 1$ we obtain :

(i) $V^1_{p,\varphi}(\left(\frac{n+p}{p}\right)(n+p-\gamma)\theta(n,p), \gamma - p) = A_p^{h(z)}(q,s,\gamma)$ $(h(z) = \frac{z^p}{1-z}, \theta(n,p) = \frac{(\alpha_1)_n ...(\alpha_q)_n}{(\beta_1)_n ...(\beta s)_n}\frac{1}{(1)_n}, q \leq s+1 (\alpha_i > 0 \ for \ i = 1,2,....,q; \ \beta_j > 0 \ for \ j = 1,2,....,s), p < \gamma < p + \frac{1}{2})$ (Najafzadeh et al. [5]);

(ii) $V^0_{p,\varphi}((n+p-\lambda+|n+p-2\alpha+\lambda|), 2(\alpha-p)) = \mathfrak{M}_p(\alpha,\lambda)$ $(0<\lambda<p, \alpha>p)$ (Sun et al.[11]);

(iii) $V^1_{p,\varphi}(\left(\frac{n+p}{p}\right)(n+p-\lambda+|n+p-2\alpha+\lambda|), 2(\alpha-p)) = \mathfrak{N}_p(\alpha,\lambda)$ $(0<\lambda<p, \alpha>p)$ (Sun et al.[11]);

(iv) $V^0_{1,\varphi}((n-\beta),(\beta-1)) = V(\beta)$ $(1<\beta<\frac{4}{3})$ (Uralegaddi et al.[12]);

(v) $V^1_{1,\varphi}(n(n-\beta),(\beta-1)) = U(\beta)$ $(1<\beta<\frac{4}{3})$ (Uralegaddi et al. [12]);

(vi) $V^0_{1,\varphi}((n-1)+|n-2\beta+1|, 2(\beta-1)) = M(\beta)$ $(\beta>1, a_0=1)$ (Niswaki and Owa [3] and Owa and Niswaki [6]);

(vii) $V^1_{1,\varphi}(n\{(n-1)+|n-2\beta+1|\}, 2(\beta-1) = N(\beta)$ $(\beta>1, a_0=1)$ (Niswaki and Owa [3] and Owa and Niswaki [6]).

Evidently, $V^0_{p,\varphi}(c_n,\delta) = V_{p,\varphi}(c_n,\delta)$ and $V^1_{p,\varphi}(c_n,\delta) = U_{p,\varphi}(c_n,\delta)$. Further $V^{\gamma_1}_{p,\varphi}(c_n,\delta) \subset V^{\gamma_2}_{p,\varphi}(c_n,\delta)$ if $\gamma_1>\gamma_2\geq 0$, the containment being proper. moreover for any positive integer k, we have the following inclusion relation

$V^k_{p,\varphi}(c_n,\delta) \subset V^{k-1}_{p,\varphi}(c_n,\delta) \subset ... \subset V^2_{p,\varphi}(c_n,\delta) \subset U_p(c_n,\delta) \subset V_p(c_n,\delta)$.

We also note that for nonnegative real number k, the class $V^k_{p,\varphi}(c_n,\delta)$ is nonempty as the function

$$f(z) = a_p z^p + \sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^{-k}\frac{\delta a_p}{c_{n+p}}\lambda_{n+p}a_{n+p}z^{n+p}, \tag{16}$$

where $a_p>0, \lambda_{n+p}\geq 0$ and $\sum\limits_{n=1}^{\infty}\lambda_{n+p}\leq 1$, satisfy the inequality (15).

The quasi-Hadamard product of two or more p-valent functions has recently been defined and used by Aouf et al. [1], Hossen [3] and Sekine [9].

The object of this paper is to establish a results concerning the quasi-Hadamard product of functions in the classes $V^k_{p,\varphi}(c_n,\delta), U_{p,\varphi}(c_n,\delta)$ and $V_{p,\varphi}(c_n,\delta)$.

2. The Main Results

Theorem 3. *Let the functions $f_i(z)$ defined by (8) belong to the class $U_{p,\varphi}(c_n,\delta)$ for every $i=1,2,...,m$; and let the functions $g_j(z)$ defined by (9) belong to the class $V_{p,\varphi}(c_n,\delta)$ for every $i=1,2,...,q$. If $c_n\geq(\dfrac{n+p}{p})\delta$*

$(n\in\mathbb{N})$. Then the quasi-Hadamard product $f_1*f_2*....*f_m*g_1*g_2*....*g_q(z)$ belongs to the class $V^{2m+q-1}_{p,\varphi}(c_n,\delta)$.

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)^{2m+q-1} c_{n+p}\left(\prod_{i=1}^{m}|a_{n+p,i}|\,.\prod_{j=1}^{q}|b_{n+p,j}|\right)\right]$$

$$\leq \delta\left(\prod_{i=1}^{m}a_{p,i}.\prod_{j=1}^{q}b_{p,j}\right).$$

Since $f_i(z)\in U_{p,\varphi}(c_n,\delta)$, we have

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)c_{n+p}a_{n+p,i}\leq \delta a_{p,i} \tag{17}$$

for every $i=1,2,...,m.$ Therefore

$$a_{n+p,i}\leq\left(\frac{n+p}{p}\right)^{-1}\left(\frac{\delta}{c_{n+p}}\right)a_{p,i},$$

and hence

$$a_{n+p,i}\leq\left(\frac{n+p}{p}\right)^{-2}a_{p,i}, \tag{18}$$

the inequalities (17) and (18) hold for every $i=1,2,...,m.$ Further, since $g_j(z)$ $\in V_{\varphi}(c_n,\delta)$, we have

$$\sum_{n=1}^{\infty}c_{n+p}b_{n+p,j}\leq\delta b_{p,j}\ , \tag{19}$$

for every $j=1,2,....,q$. Hence we obtain

$$|b_{n+p,j}|\leq\left(\frac{n+p}{p}\right)^{-1}b_{0,j}, \tag{20}$$

for every $j=1,2,...,q.$

Using (18) for $i=1,2,...,m$, (20) for $j=1,2,....,q-1$ and (19) for $j=q$, we have

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)^{2m+q-1} c_{n+p}\left(\prod_{i=1}^{m}|a_{n+p,i}|\,.\prod_{j=1}^{q}|b_{n+p,j}|\right)\right]$$

$$\leq\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)^{2m+q-1}c_n\left(\left(\frac{n+p}{p}\right)^{-2m}\left(\frac{n+p}{p}\right)^{-(q-1)}\prod_{i=1}^{m}a_{p,i}.\prod_{j=1}^{q-1}b_{p,j}\right)|b_{n+p,q}|\right]$$

$$= \left(\prod_{i=1}^{m} a_{p,i}. \prod_{j=1}^{q-1} b_{p,j}\right) \sum_{n=1}^{\infty} c_{n+p} \left|b_{n+p,q}\right| \leq \delta \left(\prod_{i=1}^{m} a_{p,i}. \prod_{j=1}^{q} b_{p,j}\right).$$

Hence $f_1 * f_2 * .... * f_m * g_1 * g_2 * .... * g_q \in V_{\varphi}^{2m+q-1}(c_n, \delta)$.

We note that the required estimate can also be obtained by using (18) for $i = 1, 2, ..., m-1$, (20) for $j = 1, 2, ..., q$, and (17) for $i = m$.

Taking into account the quasi-Hadamard product functions $f_1(z), f_2(z), ..., f_m(z)$ only, in the proof of Theorem 1 and using (18) for $i = 1, 2, ..., m-1$, and (17) for $i = m$, we obtain

Corollary 4. *Let the functions $f_i(z)$ defined by (8) belong to the class $U\varphi(c_n, \delta)$ for every $i = 1, 2, ..., m$. If $c_n \geq n\delta$, $(n \in \mathbb{N})$, then the quasi-Hadamard product $f_1 * f_2 * ... * f_m(z)$ belongs to the class $V_{p,\varphi}^{2m-1}(c_n, \delta)$.*

Also taking into account the quasi-Hadamard product functions $g_1(z), g_2(z), ..., g_q(z)$ only, in the proof of Theorem 1 and using (20) for $j = 1, 2, ..., q-1$, and (19) for $j = q$, we obtain

Corollary 5. *Let the functions $g_i(z)$ defined by (9) belong to the class $V_{\varphi}(c_n, \delta)$ for every $i = 1, 2, ..., q$. If $c_n \geq n\delta$, $(n \in \mathbb{N})$. Then the quasi-Hadamard product $g_1 * g_2 * .... * g_q$ belongs to the class $V_{p,\varphi}^{q-1}(c_n, \delta)$.*

Remark 1. *(i) Putting $p = 1$ in the above results, we obtain the results obtained by El-Ashwah [2];*

(ii)Putting $c_{n+p} = (k + p - \gamma)\theta(k, p)$ and $\delta = \gamma - p$ $(p < \gamma < p + \frac{1}{2})$ in the above results we obtain results corresponding to the class $A_p^{h(z)}(m, n, \gamma)$ $(h(z) = \frac{z^p}{1-z}, , \theta(k, p) = \frac{(\alpha_1)_k...(\alpha_m)_k}{(\beta_1)_k....(\beta_n)_k} \frac{1}{(1)_k}, n \leq m + 1$

$(\alpha_i > 0$ for $i = 1, 2, ..., q;$ $\beta_j > 0$ for $j = 1, 2, ..., s), p < \gamma < p + \frac{1}{2})$;

(iii) Putting $c_{n+p} = (n + p - \lambda - |n + p - 2\alpha + \lambda|)$ and $\delta = 2(\gamma - p)$ in the above results we obtain results corresponding to the classes $\mathfrak{M}_p(\alpha, \lambda)$ and $\mathfrak{N}_p(\alpha, \lambda)(0 < \lambda < p, \alpha > p)$.

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