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A NEW TYPE OF COVERING DIMENSION

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Abstract. In this paper, we introduce and study a new type of covering dimension and find some relations to other concepts using the ω-open sets in topological spaces.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. Dimension theory play an important role in the applications of General Topology to Real Analysis and Functional Analysis. In this paper, we have study the some new properties of ω-closed sets in topological spaces.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be semiopen [2] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of a space (X,τ) is called an ω-closed set [5] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in (X,τ). The complement of an ω-closed set is called an ω-open set [5]. The intersection of all ω-closed sets containing $A \subset X$ is called the ω-closure [4] of A and is denoted by $\omega\,\mathrm{Cl}(E)$. The union of all ω-open sets contained in $A \subset X$ is called the ω-interior [4] of A and is denoted by $\omega\,\mathrm{Int}(E)$. Let (X,τ) be a topological space. The family of all ω-open (resp. ω-closed) sets of (X,τ) is denoted by $\omega O(X)$

(resp. $\omega C(X)$). The family of all ω-open (resp. ω-closed) sets of (X, τ) containing a point $x \in X$ is denoted by $\omega O(X, x)$ (resp. $\omega C(X, x)$). It is well known that $\omega O(X)$ forms a topology [4].

Definition 1. *[1] A family $\{A_\alpha : \alpha \in \Delta\}$ of subsets of a topological space (X, τ) is said to be locally finite family if for each point x of X, there exists a neighborhood G of x such that the set $\{\alpha \in \Delta : G \cap A_\alpha \neq \emptyset\}$ is finite.*

Lemma 1. *[1] If $\{A_\alpha : \alpha \in \Delta\}$ is a locally finite family of subsets of a topological space (X, τ), then the family $\{\mathrm{Cl}(A_\alpha) : \alpha \in \Delta\}$ is a locally finite family of X and $\mathrm{Cl}(\cup A_\alpha) = \cup \mathrm{Cl}(A_\alpha)$.*

Definition 2. *[1] A family $\{A_\alpha : \alpha \in \Delta\}$ of subsets of a topological space (X, τ) is said to be point-finite if for each point x of X, the set $\{\alpha \in \Delta : x \in A_\alpha\}$ is finite.*

Definition 3. *[1] An open cover $\{G_\alpha : \alpha \in \Delta\}$ of a space X is said to be shrinkable if there exists an open cover $\{H_\alpha : \alpha \in \Delta\}$ of X such that $\mathrm{Cl}(H_\alpha) \subset G_\alpha$ for each $\alpha \in \Delta$.*

Definition 4. *[3] The family $\{A_\alpha : \alpha \in \Delta\}$ and $\{B_\alpha : \alpha \in \Delta\}$ of subsets of a set X are said to be similar if for each finite subset Γ of Δ, the sets $\bigcap_{\alpha\in\Gamma} A_\alpha$ and $\bigcap_{\alpha\in\Gamma} B_\alpha$ are either both empty or both nonempty.*

Theorem 2. *[1] Let X be any topological space. The following statements are equivalent:*

1. *X is a normal space.*
2. *Each point-finite open cover of X is shrinkable.*
3. *Each finite open cover of X has a locally finite closed refinement.*

Theorem 3. *[3] Let $\{U_\alpha : \alpha \in \Delta\}$ be a locally finite family of open sets of a normal space X and $\{F_\alpha : \alpha \in \Delta\}$ a family of closed sets such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Delta$. Then there exists a family $\{G_\alpha : \alpha \in \Delta\}$ of open sets such that $F_\alpha \subset G_\alpha \subset \mathrm{Cl}(G_\alpha) \subset U_\alpha$ for each $\alpha \in \Delta$ and the families $\{F_\alpha : \alpha \in \Delta\}$ and $\{\mathrm{Cl}(G_\alpha) : \alpha \in \Delta\}$ are similar.*

Theorem 4. *[4] For subset A and $A_\alpha (\alpha \in \Delta)$ of a topological space (X, τ), the following hold:*

1. *$A \subset \omega\,\mathrm{Cl}(A)$.*
2. *If $A \subset B$, then $\omega\,\mathrm{Cl}(A) \subset \omega\,\mathrm{Cl}(B)$.*

3. $\omega \operatorname{Cl}(\cap\{A_\alpha : \alpha \in \Delta\}) \subset \cap\{\omega \operatorname{Cl} A_\alpha : \alpha \in \Delta\}$.

4. $\omega \operatorname{Cl}(\cup\{A_\alpha : \alpha \in \Delta\}) = \cup\{\omega \operatorname{Cl} A_\alpha : \alpha \in \Delta\}$.

Definition 5. *If A is a subset of a topological space (X, τ), then the ω-boundary of A is defined as $\omega \operatorname{Cl}(A) \backslash \omega \operatorname{Int}(A)$ and is denoted by $\omega Bd(A)$.*

3. On $\omega^\star$-Normal space

Definition 6. *A topological space (X, τ) is said to be $\omega^\star$-normal if whenever A and B are disjoint ω-closed sets in X, there exist disjoint ω-open sets U and V with $A \subset U$ and $B \subset V$.*

Remark 1. *It is easy to prove that each $\omega^\star$-normal space is normal but the converse is not true in general as shown in the following example.*

Example 1. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$. Then the topological space (X, τ) is normal but not $\omega^\star$-normal because $\{b\}$ and $\{c\}$ are ω-closed sets in X and we can not separate them by disjoint ω-open sets.*

Definition 7. *A family $\{A_\alpha : \alpha \in \Delta\}$ of subsets of a space X is said to be ω-locally finite if for each point x of X, there exists an ω-open set G of X such that the set $\{\alpha \in \Delta : G \cap A_\alpha \neq \emptyset\}$ is finite.*

Definition 8. *An open cover $\{G_\alpha : \alpha \in \Delta\}$ of a space X is said to be ω-shrinkable if there exists an ω-open cover $\{H_\alpha : \alpha \in \Delta\}$ of X such that $\omega \operatorname{Cl}(H_\alpha) \subset G_\alpha$ for each $\alpha \in \Delta$.*

Theorem 5. *For a topological space (X, τ), the following statements are equivalent*

1. *X is $\omega^\star$-normal.*
2. *For every pair of ω-open G and H whose union is X, then there exist ω-closed sets A and B such that $A \subset G, B \subset H$ and $A \cup B = X$.*
3. *For every ω-closed set A and every ω-open set G containing A, there exists an ω-open set H such that $A \subset H \subset \omega \operatorname{Cl}(H) \subset G$.*

Proof. The proof is clear.

Lemma 6. *If $\{A_\alpha : \alpha \in \Delta\}$ is a locally finite family of subsets of a space X, then the family $\{\omega \operatorname{Cl}(A_\alpha) : \alpha \in \Delta\}$ is ω-locally finite family of X. Moreover, $\omega \operatorname{Cl}(\cup A_\alpha) = \cup \omega \operatorname{Cl}(A_\alpha)$.*

Proof. From Lemma 1, we obtain that the family $\{\mathrm{Cl}(A_\alpha) : \alpha \in \Delta\}$ is a locally finite family of X whenever $\{A_\alpha : \alpha \in \Delta\}$ is locally finite. Since $\omega\,\mathrm{Cl}(A_\alpha) \subset \mathrm{Cl}(A_\alpha)$ for every $\alpha \in \Delta$, the family $\{\omega\,\mathrm{Cl}(A_\alpha) : \alpha \in \Delta\}$ is a locally finite family of X. To prove that $\omega\,\mathrm{Cl}(\cup A_\alpha) = \cup\omega\,\mathrm{Cl}(A_\alpha)$, we have $\cup\omega\,\mathrm{Cl}(A_\alpha) \subset \omega\,\mathrm{Cl}(\cup A_\alpha)$. Therefore, it is sufficient to prove that $\omega\,\mathrm{Cl}(\cup A_\alpha) \subset \cup\omega\,\mathrm{Cl}(A_\alpha)$. Suppose that $x \notin \cup\omega\,\mathrm{Cl}(A_\alpha)$, so $x \notin \omega\,\mathrm{Cl}(A_\alpha)$ for all $\alpha \in \Delta$. This means that there exists an ω-open set G such that $G \cap A_\alpha = \emptyset$ for all $\alpha \in \Delta$ and hence, $G \cap \omega\,\mathrm{Cl}(A_\alpha) = \emptyset$ for all $\alpha \in \Delta$. Since the family $\{A_\alpha : \alpha \in \Delta\}$ is locally finite, there exists an open set H which contains x and the set $\{\alpha \in \Delta : H \cap A_\alpha = \emptyset\}$ is finite. This means that there exists a finite subset M of Δ such that $H \cap A_\alpha = \emptyset$ for $\alpha \in M$. From above we obtain that x belongs to $X\backslash\omega\,\mathrm{Cl}(A_\alpha)$ for every $\alpha \in \Delta$. Since the family of ω-open sets forms a topology on X ([4]), the set $V = \cap[X\backslash\omega\,\mathrm{Cl}(A_\alpha) : \alpha \in \Delta]$ is an ω-open set in X containing x. But $H \cap V$ is an ω-open set containing x and $(H \cap V) \cap A_\alpha = \emptyset$ for all $\alpha \in \Delta$. Therefore, $(H \cap V) \cap (\cup A_\alpha) = \emptyset$. This implies that $x \in \omega\,\mathrm{Cl}(\cup A_\alpha)$ and this completes the proof.

Theorem 7. *For a topological space (X, τ), the following statements are equivalent*

1. *X is an $\omega^\star$-normal space.*
2. *Each point-finite open cover of X is ω-shrinkable.*
3. *Each finite ω-open cover of X has a locally finite ω-closed refinement.*

Proof. (1)⇒(2): Let $\{U_\alpha : \alpha \in \Delta\}$ be a point-finite open cover of an $\omega^\star$-normal space X, we may assume α is well-ordered. We shall construct an ω-shrinkable family of $\{U_\alpha : \alpha \in \Delta\}$ by transfinite induction. Let $\mu \in \alpha$ and suppose that for each $\alpha < \mu$ we have ω-open set V_α such that $\omega\,\mathrm{Cl}(V_\alpha) \subset U_\alpha$ and for each $\upsilon < \mu$, we have $(\underset{\alpha\leq\upsilon}{\cup} V_\alpha) \cup (\underset{\alpha>\upsilon}{\cup} U_\alpha) = X$. Let $x \in X$, then since $\{U_\alpha : \alpha \in \Delta\}$ is a point-finite, there exists the largest element $\gamma \in \alpha$ such that $x \in U_\gamma$. If $\gamma \geq \mu$, then $x \in \underset{\alpha\geq\mu}{\cup} U_\alpha$, and if $\gamma < \mu$, then $x \in \underset{\alpha\leq\gamma}{\cup} V_\alpha \subset \underset{\alpha<\gamma}{\cup} V_\alpha$. Hence, $(\underset{\alpha<\mu}{\cup} V_\alpha) \cup (\underset{\alpha\geq\gamma}{\cup} U_\alpha) = X$. Thus, U_α contains the complement of $(\underset{\alpha<\mu}{\cup} V_\alpha) \cup (\underset{\alpha>\gamma}{\cup} U_\alpha)$. Since X is $\omega^\star$-normal, there exists an ω-open set V_μ such that $X\backslash[(\underset{\alpha<\mu}{\cup} V\alpha) \cup (\underset{\alpha>\gamma}{\cup} U\alpha)] \subset V_\mu \subset \omega\,\mathrm{Cl}(V_\mu) \subset U_\mu$. Thus, $\omega\,\mathrm{Cl}(V_\mu) \subset U_\mu$ and $(\underset{\alpha\leq\mu}{\cup} V_\alpha) \cup (\underset{\alpha>\gamma}{\cup} U_\alpha) = X$. Hence, the construction of an ω-shrinkable family for $\{U_\alpha : \alpha \in \Delta\}$ is completed by transfinite induction.
(2)⇒(3): Obvious.
(3)⇒(1): Let X be a space such that each finite ω-open cover of X has a locally finite ω-closed refinement. Let A and B be two ω-closed sets in X. The ω-open cover $\{X\backslash A, X\backslash B\}$ of X has a locally finite ω-closed refinement υ. Let E be the

union of members of υ disjoint from A and let F be union of members of υ disjoint from B. Then, by Lemma 6, E and F are ω-closed sets and $E \cup F = X$. Thus, if $U = X \backslash E$ and $V = X \backslash F$, then U and V are disjoint ω-open sets such that $A \subset U$ and $B \subset V$. Therefore, X is an $\omega^{\star}$-normal space.

Theorem 8. *Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a locally finite family of ω-open sets of an $\omega^{\star}$-normal space X and $\{F_\alpha\}_{\alpha\in\Delta}$ a family of ω-closed sets such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Delta$. Then there exists a family $\{G_\alpha\}_{\alpha\in\Delta}$ of ω-open sets such that $F_\alpha \subset G_\alpha \subset \omega\,\mathrm{Cl}(U_\alpha) \subset U_\alpha$ and the families $\{F_\alpha\}_{\alpha\in\Delta}$ and $\{\omega\,\mathrm{Cl}(G_\alpha)\}_{\alpha\in\Delta}$ are similar.*

Proof. Let Δ be well-ordered with a least element. By transfinite induction, we shall construct a family $\{G_\alpha\}_{\alpha\in\Delta}$ of ω-open sets such that $F_\alpha \subset G_\alpha \subset \omega\,\mathrm{Cl}(G_\alpha) \subset U_\alpha$ and for each element υ in Δ the family

$$\{K_\alpha^\upsilon\}_{\alpha\in\Delta} = \begin{cases} \mathrm{Cl}(G_\alpha) & \text{if } \alpha \leq \upsilon, \\ F_\alpha & \text{if } \alpha < \upsilon. \end{cases}$$

is similar to $\{F_\alpha\}_{\alpha\in\Delta}$. Suppose that $\mu \in \alpha$ and that G_α are defined for $\alpha < \mu$ such that for each $\upsilon < \mu$ the family $\{K_\alpha^\upsilon\}_{\alpha\in\Delta}$ is similar $\{F\alpha\}_{\alpha\in\Delta}$. Let $\{L_\alpha\}_{\alpha\in\Delta}$ be the family given by:

$$\{L_\alpha\}_{\alpha\in\Delta} = \begin{cases} \mathrm{Cl}(G_\alpha) & \text{if } \alpha < \upsilon, \\ F_\alpha & \text{if } \alpha \geq \upsilon. \end{cases}$$

Then $\{F_\alpha\}_{\alpha\in\Delta}$ and $\{L_\alpha\}_{\alpha\in\Delta}$. For suppose that $\alpha_1, \alpha_2, ..., \alpha_r \in \Delta$ and $\alpha_1, \alpha_2, ..., \alpha_j < \mu < \alpha_{j+1} < ... < \alpha_r$, then $\bigcap\limits_{i=1}^{r}\{L_{\alpha i}\} = \bigcap\limits_{i=1}^{r}\{K_{\alpha i}^{\alpha j}\}$. Therefore $\bigcap\limits_{i=1}^{r}\{L_{\alpha i}\} = \emptyset$ if and only if $\bigcap\limits_{i=1}^{r}\{F_{\alpha i}\} = \emptyset$ because $\{K_\alpha^\upsilon\}_{\alpha\in\Delta}$ is similar to $\{F_\alpha\}_{\alpha\in\Delta}$. Since $L_\alpha \subset G_\alpha$ for each α, the family $\{L_\alpha\}_{\alpha\in\Delta}$ is locally finite. Thus if Γ is the set of finite subsets of Δ and for each $\gamma \in \Gamma$, $E_\gamma = \bigcap\limits_{\alpha\in\Delta} L_\alpha$, then $\{E_\gamma\}_{\gamma\in\Gamma}$ is locally finite family of ω-closed sets. Hence, by Lemma 6, $E = \cup\{E_\gamma : E_\gamma \cap F_\mu\}$ is an ω-closed set which is disjoint from F_μ. Therefore, there exists an ω-open set G_μ such that $F_\mu \subset G_\mu \subset \omega\,\mathrm{Cl}(G_\mu) \subset U_\mu$, and $\omega\,\mathrm{Cl}(G_\mu \cap E) = \emptyset$. Now the ω-open set G_α is defined for $\alpha \leq \mu$ and to complete the proof it remains to show that the family $\{K_\alpha^\upsilon\}_{\alpha\in\Delta}$ is similar to the family $\{F_\alpha\}_{\alpha\in\Delta}$. It is sufficient to show that the families $\{K_\alpha^\upsilon\}_{\alpha\in\Delta}$ and $\{L_\alpha\}_{\alpha\in\Delta}$ are similar. Suppose that $\alpha_1, \alpha_2, ..., \alpha_r \in \Delta$ and that $\bigcap\limits_{i=1}^{r}\{L_{\alpha i}\} = \emptyset$, we have to show that $\bigcap\limits_{i=1}^{r}\{K_{\alpha i}^{\alpha j}\} = \emptyset$. Suppose that $\alpha_1 < \alpha_2 < ... < \alpha_j < \mu < \alpha_{j+1} < ... < \alpha_r$ if $\alpha_j \neq \mu$ there is nothing to prove. If $\alpha_j = \mu$, then $L_{\alpha_1} \cap .... \cap L_{\alpha_{j-1}} \cap F\mu \cap L_{\alpha_{j+1}} \cap ... \cap L_{\alpha_r} = \emptyset$. Hence by the construction $L_{\alpha_1} \cap .... \cap L_{\alpha_{j-1}} \cap \omega\,\mathrm{Cl}(G_\mu \cap L_{\alpha_{j+1}}) \cap ... \cap L_{\alpha_r} = \emptyset$. Thus $\bigcap\limits_{i=1}^{r}\{K_{\alpha i}^{\alpha j}\} = \emptyset$.

4. On ω-covering dimension

In this section, we introduce a type of a covering dimension by using ω-open sets which we call the ω-covering dimension function.

Definition 9. *The ω-covering dimension of a topological space X is the least positive integer n such that every finite ω-open cover of X has an ω-open refinement of order not exceeding n or is ∞ if there is no such integer. We shall denote the ω-covering dimension of a space X by $\dim_\omega X$. If X is an empty set, then $\dim_\omega X = -1$ and $\dim_\omega X \leq n$ if each finite ω-open cover of X has an ω-open refinement of order not exceeding n. Also we have $\dim_\omega X = n$ if it is true that $\dim_\omega X \leq n$ but it is not true if $\dim_\omega X \leq n-1$. Finally, $\dim_\omega X = \infty$ if for every integer n there exists a finite ω-open cover which has no ω-open refinement of order not exceeding n.*

Theorem 9. *If Y is a clopen subset of a space X, then $\dim_\omega Y \leq \dim_\omega X$.*

Proof. It is sufficient to prove if $\dim_\omega X = n$, then $\dim_\omega Y \leq n$. Let $\{U_1, U_2, ..., U_k\}$ be an ω-open cover of the open set Y. Then U_i is ω-open in X for each i and since every open set is ω-open. Then the finite ω-open cover $\{U_1, U_2, ..., U_k, X\backslash Y\}$ of X has an ω-open refinement ω of order which not exceeding n. Let ε be all members of ω except those members associated with $X\backslash Y$, since every open set is ω-open, then each member of ε is ω-open in Y and also ε is a refinement of $\{U_1, U_2, ..., U_k\}$ of order not exceeding n. This implies that $\dim_\omega Y \leq n$.

Now we give some characterizations of the ω-covering dimension in topological spaces.

Theorem 10. *If X is a topological space, then the following statements about X are equivalent:*

1. $\dim_\omega X \leq n$,
2. *For any finite ω-open cover $\{U_1, U_2, ..., U_k\}$ of X, then there exists an ω-open cover $\{V_1, V_2, ..., V_k\}$ of order not exceeding n such that $V_i \subset U_i$ for $i = 1, 2, ..., k$.*
3. *If $\{U_1, U_2, ..., U_{n+2}\}$ is an ω-open cover of X, then there exists an ω-open cover $\{V_1, V_2, ..., V_{n+2}\}$ such that $V_i \subset U_i$ and $\bigcap_{i=1}^{n+2} = \emptyset$.*

Proof. (1)⇒(2): Suppose that $\dim_\omega X \leq n$ and the ω-open cover $\{U_1, U_2, ..., U_k\}$ of X has an β-open refinement β of order not exceeding n. If $W \in \beta$, then $W \subset U_i$ for some i. Let each W in β be associated with one of the sets U_i containing it and let V_i be the union of those members of β thus associated with U_i, since every open set

is ω-open, then V_i is ω-open and $V_i \subset U_i$ and each point of X is in some member of β and hence in some V_i. Each point x of X is in at most $n+1$ members of ω, each of which is associated with a unique U_j and hence is in at most $n+1$ members of $\{V_i\}$. Thus, $\{V_i\}$ is an ω-open cover of X of order not exceeding n.
(2)⇒(3): Obvious.
(3)⇒(2): Let X be a space satisfying (3) and $\{U_1, U_2, ..., U_k\}$ a finite ω-open cover of X, we can assume that $k > n+1$. Let $G_i = U_i$ if $i \leq n+1$ and $G_{n+2} = \overset{k}{\underset{i=n+2}{\cup}} U_i$, then $\{G_1, G_2, ..., G_{n+2}\}$ is an ω-open cover of X and so by hypothesis there is an ω-open cover $\{H_1, H_2, ..., H_k\}$ such that $H_i \subset G_i$ and $\overset{n+2}{\underset{i=1}{\cap}} H_i = \emptyset$. Let $W_i = U_i$ if $i \leq n+1$ and let $W_i = U_i \cap G_{n+2}$ if $i > n+1$. Then $\delta = \{W_1, W_2, ..., W_k\}$ is an ω-open cover of X each $W_i \subset U_i$ and $\overset{n+2}{\underset{i=1}{\cap}} W_i = \emptyset$. If there exists such set B of $\{1, 2, ..., k\}$ with $n+2$ elements such that $\overset{n+2}{\underset{i=1}{\cap}} W_i \neq \emptyset$, let the members of δ be renumbered to give a family $P = \{P_1, P_2, ..., P_k\}$ $\overset{n+2}{\underset{i=1}{\cap}} P_i \neq \emptyset$. By applying the above construction to P, we obtain the ω-open cover $W' = \{W_1', W_2', ..., W_k'\}$ such that $W' \subset P_i$ and $\overset{n+2}{\underset{i=1}{\cap}} W_i' = \emptyset$. Thus by a finite number of repetitions of this process we obtain an ω-open cover $\{V_1, V_2, ..., V_k\}$ of X, of order not exceeding n such that $V_i \subset U_i$.
(2)⇒(1): Obvious.

Theorem 11. *In a topological space X if $\dim_\omega X = 0$, then X is an $\omega^\star$-normal space.*

Proof. Suppose that $\dim_\omega X = 0$ and let F and E be any two disjoint ω-closed sets in X, then $\{X \backslash F, X \backslash E\}$ is an ω-open cover of X; hence, there exists an ω-open refinement β of order not exceeding 0. This means that all members of β are pairwise disjoint. Let G be the union of all member of β that are associated with $X \backslash E$ and H be the union of all members of β that are associated with $X \backslash F$, hence, G and H are ω-open sets such that $G \cup H = X, G \subset X \backslash E$ and $H \subset X \backslash F$ and $G \cap H = \emptyset$. Thus, G and H are disjoint ω-open sets such that $F \subset G$ and $E \subset H$. Hence, X is an $\omega^\star$-normal space.

In $\omega^\star$-normal spaces, ω-covering dimension can be defined in terms of the order of finite ω-closed refinements of finite ω-open cover.

Theorem 12. *If (X, τ) is a topological space, then the following statements about X are equivalent*

1. $\dim_\omega X \leq n$,

2. *For any finite ω-open cover $\{U_1, U_2, ..., U_k\}$ of X, there exists an ω-open cover $\{V_1, V_2, ..., V_k\}$ such that $\omega\,\mathrm{Cl}(V_i) \subset U_i$ and the order for the family $\{\omega\,\mathrm{Cl}(V_1), \omega\,\mathrm{Cl}(V_2), ..., \omega\,\mathrm{Cl}(V_k)\}$ does not exceed n.*

3. *For any finite ω-open cover $\{U_1, U_2, ..., U_k\}$ of X there is an ω-closed cover $\{F_1, F_2, ..., F_k\}$ such that $F_i \subset U_i$ and the order for $\{F_1, F_2, ..., F_k\}$ does not exceed n.*

4. *Every finite ω-open cover of X has a finite ω-closed refinement of order not exceeding n.*

5. *If $\{U_1, U_2, ..., U_k\}$ is an ω-open cover of X, then there exists an ω-closed cover $\{F_1, F_2, ..., F_k\}$ such that $F_i \subset U_i$ and $\bigcap_{i=1}^{n+2} F_i = \emptyset$.*

Proof. (1)⇒(2): Suppose that $\dim_\omega X \leq n$, and let $\{U_1, U_2, ..., U_k\}$ be ω-open cover of X, then by Theorem 10, there exists an ω-open cover $\{W_1, W_2, ..., W_k\}$ of order not exceeding n such that $W_i \subset U_i$. Since X is $\omega^\star$-normal, by Theorem 7, there exists an ω-open cover $\{V_1, V_2, ..., V_k\}$ such that $\omega\,\mathrm{Cl}(V_i) \subset W_i$ for each i. Then $\{V_1, V_2, ..., V_k\}$ is an ω-open cover with the required properties.
(2)⇒(3) and (3)⇒(4): Obvious.
(4) ⇒(5): Let X be a space satisfying (4) and let $\Pi = \{U_1, U_2, ..., U_{n+2}\}$ be an ω-open cover of X. Then the cover Π has a finite ω-closed refinement υ of order not exceeding n. If $E \in \upsilon$, then $E \subset U_i$ for some i. Let each E in υ be associated with one of the sets U_i containing it and let F_i be the union of those members of υ which associated with U_i, then F_i is ω-closed, $F_i \subset U_i$ and $\{F_1, F_2, ..., F_{n+2}\}$ is an ω-cover of X such that $\bigcap_{i=1}^{n+2} F_i = \emptyset$.
(5)⇒ (1): Let X be a space satisfying (5) and let $\{U_1, U_2, ..., U_{n+2}\}$ be an ω-open cover of X, by hypothesis there exists an ω-closed cover $\{F_1, F_2, ..., F_{n+2}\}$ such that each $F_i \subset U_i$ and $\bigcap_{i=1}^{n+2} F_i = \emptyset$. By Theorem 8, there exist ω-open sets $\{V_1, V_2, ..., V_{n+2}\}$ such that $F_i \subset V_i \subset U_i$ for each i and $\{V_i\}$ is similar to $\{F_i\}$. Thus $\{V_1, V_2, ..., V_{n+2}\}$ is an ω-open cover of X, each $V_i \subset U_i$ $\bigcap_{i=1}^{n+2} V_i = \emptyset$. Therefore, by Theorem 10, $\dim_\omega X \leq n$.

Theorem 13. *If (X, τ) is an $\omega^\star$-normal space, then the following statements about X are equivalent*

1. $\dim_\omega X \leq n$,

2. *For any finite ω-closed sets $\{F_1, F_2, ..., F_{n+1}\}$ and each family of ω-open sets $\{U_1, U_2, ..., U_{n+1}\}$ such that $F_i \subset U_i$, there exists a family $\{V_1, V_2, ..., V_{n+1}\}$ of ω-open sets such that $F_i \subset V_i \subset \omega\,\mathrm{Cl}(V_i) \subset U_i$ for each i, and $\overset{n+1}{\underset{i=1}{\cap}} \alpha BdV_i = \emptyset$,*

3. *For each family ω-closed sets $\{F_1, F_2, ..., F_k\}$ and each family of ω-open sets $\{U_1, U_2, ..., U_k\}$ such that $F_i \subset U_i$, there exists a family $\{V_1, V_2, ..., V_k\}$ and $\{W_1, W_2, ..., W_k\}$ of ω-open sets such that $F_i \subset V_i \subset \omega\,\mathrm{Cl}(V_i) \subset W_i \subset U_i$ for each i and the order of the family $\{\omega\,\mathrm{Cl}(W_1)\backslash V_1, \omega\,\mathrm{Cl}(W_2)\backslash V_2, ..., \omega\,\mathrm{Cl}(W_k)\backslash V_k\}$ dose not exceed $n-1$.*

4. *For each family ω-closed sets $\{F_1, F_2, ..., F_k\}$ and each family of ω-open sets $\{U_1, U_2, ..., U_k\}$ such that $F_i \subset U_i$, there exists a family $\{V_1, V_2, ..., V_k\}$ of ω-open sets $F_i \subset V_i \subset \omega\,\mathrm{Cl}(V_i) \subset U_i$ and the order of the family $\{\omega Bd(V_1), \omega Bd(V_2), ..., \omega Bd(V_k)\}$ does not exceed $n-1$.*

Proof. (1)$\Rightarrow$(2): Suppose that $\dim_\omega X \leq n$, and let $\{F_1, F_2, ..., F_{n+1}\}$ be ω-closed sets and $\{U_1, U_2, ..., U_{n+1}\}$ ω-open sets such that $F_i \subset U_i$. Since $\dim_\omega X \leq n$, the ω-open cover of X consisting of sets of the form $\overset{n+2}{\underset{i=1}{\cap}} H_i$, where $H_i = U_i$ or $H_i = X\backslash F_i$ for each i, has a finite ω-open refinement $\{W_1, W_2, ..., W_q\}$ of order not exceeding n. Since X is $\omega^\star$-normal, there is an ω-closed cover $\{K_1, K_2, ..., K_q\}$ such that $K_r \subset W_r$ for $r = 1, 2, ..., q$. Let N_r denoted the set i such that $F_i \cap W_r \neq \emptyset$. For $r = 1, 2, ..., q$, we can find ω-open sets V_{ir} for r in N_r such that $K_r \subset V_{ir} \subset \omega\,\mathrm{Cl}(V_{ir}) \subset W_r$ and $\omega\,\mathrm{Cl}(V_{i_r}) \subset W_{jr}$ if $i < j$. Now for each $i = 1, 2, ..., n+1$, let $V_r = \underset{r}{\cup}\{V_{ir} : i \in N_r\}$. Then V_i is ω-open and $F_i \subset V_i$ for if $x \in F_i$ and $x \in K_r$; then $i \in N_r$ so that $x \in V_{ir} \subset V_i$. Furthermore if $i \in N_r$ so that $F_i \cap W_r \neq \emptyset$, then W_r is not contained in $X\backslash F_i$ so that $W_r \subset U_i$. Thus, if $i \in N_r$, then $V_{ir} \subset U_i$ and since $\omega\,\mathrm{Cl}(V_i) = \underset{r}{\cup}\{\omega\,\mathrm{Cl}(V_{ir}) : i \in N_r\}$, it follows that $\omega\,\mathrm{Cl}(V_i) \subset U_i$. Finally suppose that $x \in \overset{n+2}{\underset{i=1}{\cap}} \omega Bd(V_i)$ and since $\omega Bd(V_i) \subset \underset{r}{\cup}\{\omega Bd(V_{i_r}) : i \in N_r\}$, it follows that for each i there exists i_r such that $x \in \omega Bd(V_{i_r})$ and if $i \neq j$, then $r_i \neq r_j$ for if $r_i = r_j = r$, then $x \in \omega\,\mathrm{Cl}(V_{i_r})$ and $x \in \omega\,\mathrm{Cl}(V_{j_r})$ but $x \in V_{i_r}$ and $x \in V_{j_r}$, which are absurd, since either $\omega\,\mathrm{Cl}(V_{i_r}) \subset V_{j_r}$ or $\omega\,\mathrm{Cl}(V_{j_r}) \subset V_{i_r}$. For each $i, x \notin V_{irj}$ so that $x \notin K_{r_j}$. But $\{K_r\}$ is an ω-cover of X and so there exists r_0 different form each of the r_i such that $x \in K_{r_o} \subset W_{r_o}$. Since $x \in V_{irj}$, it follows that $x \in W_{r_i}$ for $i = 1, 2..., n+1$ so that $x \overset{n+1}{\underset{i=0}{\cap}} V_n$. Since the order of $\{W_r\}$ does not exceed n, this is absurd. Hence $\overset{n+1}{\underset{i=1}{\cap}} \omega Bd(V_i) = \emptyset$.

(2)$\Rightarrow$(3): Let $F_1, F_2, ..., F_k$ be ω-closed sets and let $U_1, U_2, ..., U_k$ be ω-open sets such that $F_i \subset U_i$. We can assume that $k > n+1$; otherwise, there is nothing to prove. Let the subset $\{1, 2, ..., k\}$ containing $n+1$ elements be enumerated as $C_1, C_2, ..., C_q$, where $q = k_{C_{n+1}}$. By using (2), we can find ω-open sets $V_{i,r}$ for i in C_i such that $F_i \subset V_{i,1} \subset \omega\operatorname{Cl}(V_{i,1}) \subset U_i$ and $\underset{i=1}{\overset{n+1}{\cap}} \omega Bd V_{i,r} = \emptyset$. We have a finite family $\{\omega Bd(V_{i,r}) : i \in C_1\}$ of ω-closed sets of the $\omega^{\star}$-normal space X and $\omega Bd(V_{i,1}) \subset U_i$ for each i in C_1. Thus, by Theorem 8, for each i in C_1, there exists an ω-open set G_i such that $\omega Bd(V_{i,1}) \subset G_i \subset \omega\operatorname{Cl}(G_i) \subset U_i$, and $\{\omega\operatorname{Cl}(G_i)\}_{i\in C_1}$ is similar to $\{\omega Bd(V_{i,r})\}_{i\in C_1}$, so that in particular $\underset{i\in c_i}{\cap} \omega\operatorname{Cl}(G_i) = \emptyset$. Let $W_{i,1} = V_{i,1} \cup G_i$ if $i \in C_1$, then $\omega\operatorname{Cl}(V_{i,1}) \subset W_{i,1} \subset \omega\operatorname{Cl} W_{i,1} \subset U_i$ and since $(\omega\operatorname{Cl}(W_{i,1})\backslash V_{i,1}) \subset \omega\operatorname{Cl}(G_i)$, we have $\underset{i\in C_1}{\cap}(\omega\operatorname{Cl}(W_{i,1})\backslash V_{i,1}) = \emptyset$. If $i \notin C_1$, let $V_{i,1}$ be an ω-open set such that $F_i \subset V_{i,1} \subset \omega\operatorname{Cl}(V_{i,1}) \subset U_i$ and let $W_{i,1} = U_i$. Then for $i = 1, 2, ..., k$ we have ω-open sets $V_{i,1}$ and $W_{i,1}$ such that $F_i \subset V_{i,1} \subset \omega\operatorname{Cl}(V_{i,1}) \subset U_i$ and $\underset{i\in c}{\cap}(\omega\operatorname{Cl}(W_{i,1})\backslash V_{i,1}) = \emptyset$. Suppose that $1 < m \leq q$ and for $i = 1, 2, ..., k$, we find ω-open sets $V_{i,m-1}$ and $W_{i,m-1}$ such that $F_i \subset V_{i,m-1} \subset \omega\operatorname{Cl}(V_{i,m-1}) \subset W_{i,m-1} \subset U_i$ and $\underset{i\in c}{\cap}(\omega\operatorname{Cl}(W_{i,m-1})\backslash V_{i,m-1}) = \emptyset$ if $1 \leq j \leq m-1$. By the above argument we can find ω-open sets $V_{i,m}$ and $W_{i,m}$ such that $\omega\operatorname{Cl}(V_{i,m-1}) \subset V_{i,m} \subset \omega\operatorname{Cl}(V_{i,m}) \subset W_{i,m} \subset W_{i,m-1}$ and $\underset{i\in cm}{\cap}(\omega\operatorname{Cl}(W_{i,m})\backslash V_{i,m}) = \emptyset$. Since $\omega\operatorname{Cl}(W_{i,m})\backslash V_{i,m} \subset (\omega\operatorname{Cl}(W_{i,m-1})\backslash V_{i,m-1})$, we have $\underset{i\in C_j}{\cap}(\omega\operatorname{Cl}(W_{i,m})\backslash V_{i,m}) = \emptyset$ if $j \leq m$. Thus by induction for $i = 1, 2, ..., k$, we can find ω-open sets V_i and $W_i = (V_{i,q}$ and $W_{i,q}$ respectively) such that $F_i \subset V_i \subset \omega\operatorname{Cl}(V_i) \subset W_i \subset U_i$ and $\underset{i\in cj}{\cap}(\omega\operatorname{Cl}(W_i)\backslash V_i) = \emptyset$, for $j = 1, 2, ..., k$. Thus the order of the family $\{\omega Bd(W_1\backslash V_1), ..., \omega Bd(W_k\backslash V_k)\}$ does not exceed $n-1$.

(3)⇒(4): Obvious.

(4)⇒ (1): Let (4) hold and let $\{U_1, U_2, ..., U_{n+2}\}$ be an ω-open cover of X. Since X is $\omega^{\star}$-normal, there exists an ω-closed cover $\{F_1, F_2, ..., F_{n+1}\}$ of X such that $F_i \subset U_i$ for each i. By hypothesis there exists a family of ω-open sets $\{V_1, V_2, ..., V_{n+1}\}$ such that $F_i \subset V_i \subset \omega\operatorname{Cl}(V_i) \subset U_i$ for each i, and $\{\omega Bd(V_1), \omega Bd(V_2), ..., \omega Bd(V_{n+2})\}$ has order not exceeding $n-1$. Let $L_j = \omega\operatorname{Cl}(V_j)\backslash \underset{i<j}{\cup} V_i$ for $j = 1, 2, ..., n+2$. For each j, L_j is an ω-closed, and $\{L_1, L_2, ..., L_{n+2}\}$ is an ω-closed cover of X, for if $x \in X$, there exists j such that $x \in V_j$ and $x \notin V_i$ for $i < j$ so that $x \in L_j$. Now $L_j = \omega\operatorname{Cl}(V_j) \underset{i<j}{\cap}(X\backslash V_j)$ so that $\underset{j=1}{\overset{n+2}{\cap}} L_j = \underset{j=1}{\overset{n+2}{\cap}} \omega\operatorname{Cl} V_j \cap \underset{i=1}{\overset{n+1}{\cap}}(X\backslash V_j) \subset \underset{j=1}{\overset{n+1}{\cap}} \omega Bd(V_j) = \emptyset$. Thus $\{L_1, L_2, ..., L_{n+2}\}$ is an ω-closed cover of X, $L_j \subset \omega\operatorname{Cl}(V_j) \subset U_j$ and $\underset{j=1}{\overset{n+2}{\cap}} L_j = \emptyset$. Hence by Theorem 12, $\dim_{\omega} X \leq n$.

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