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GENERALIZED COMMON FIXED POINT FOR COMPATIBLE MAPPINGS OF TYPE (γ) IN COMPLETE FUZZY METRIC SPACES

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Abstract. In this paper we first explain the concept fuzzy metric spaces and in this sequel explain the nation of Cauchy sequence and convergent in fuzzy metric spaces and in addition we explain the concept of compatible maps of type (γ) in fuzzy metric spaces and some of these maps and finally, we prove a common fixed point theorem for properties Compatible maps of type (γ) on complete fuzzy metric spaces.

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [20] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application.George and Veeramani [6] and Kramosil and Michalek [9] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\varepsilon^{(\infty)}$ theory which were given and studied by El Naschie [1, 2, 3, 4, 5]. Many authors [6, 10, 13] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

In this paper, we prove common fixed point theorems satisfying some conditions in fuzzy metric spaces in the sense of Sedghi, Turkoglu and Shobe [17]. Our main theorems extend, generalize and improvement some known results in fuzzy metric spaces, in particular produce a general style for prove common fixed point theorems.

Definition 1. *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-norm if it satisfies the following conditions*

1. $*$ *is associative and commutative,*

2. $*$ *is continuous,*

3. $a * 1 = a$ *for all* $a \in [0, 1]$,

4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = min\{a, b\}$.

Definition 2. *A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X $(non-empty)$ *set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X$ *and* $t, s > 0$,

1. $M(x, y, t) > 0$,

2. $M(x, y, t) = 1$ *if and only if* $x = y$,

3. $M(x, y, t) = M(y, x, t)$,

4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

5. $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous.*

6. $lim_{t\to\infty} M(x, y, t) = 1$.

Let $M(x, y, t)$ be a fuzzy metric space. For any $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$all with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable.A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, y, t) \to 1$, as $n \to \infty$, for all $t > 0$. it is called a Cauchy sequence if, for any $0 < \varepsilon < 1$ such that $M(x_n, y_m, t) > 1 - \varepsilon$ for any $n, m \geq n_0$. The F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 1. *Let* $X = \mathbb{R}$ *and denote* $a*b = ab$ *for all* $a, b \in [0, 1]$. *For any* $t \in (0, \infty)$, *define*

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$. *Then* M *is a fuzzy metric space in* X.

Lemma 1. *Let* $(X, M, *)$ *be a fuzzy metric space. Then* M *is non-decreasing with respect to t, for all* x, y *in* X.

Definition 3. *Let $(X, M, *)$ be a fuzzy metric space. M is said continuous if*

$$lim_{n\to\infty}M(x_n, y_n, t_n) = M(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e,

$$lim_{n\to\infty}M(x_n, y, t) = lim_{n\to\infty}M(x, y_n, t) = 1,$$

$$lim_{n\to\infty}M(x, y, t_n) = M(x, y, t).$$

Lemma 2. *Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.*

Proof. See proposition 1 of [28]. ■

Definition 4. *Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.*

Definition 5. *Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if for all $t > 0$,*

$$lim_{n\to\infty}M(ASx_n, SAx_n, t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that

$$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = x \in X.$$

Proposition 1. *If the self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible, then they are weak compatible.*

The converse is not true as seen in following example.

Example 2. *Let $(X, M, *)$ be fuzzy metric space, where $X = [0, 2]$, with t-norm defined $a * b = min\{a, b\}$, for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $t > 0$ and $x, y \in X$. Define the self-mappings A and S on as follows:*

$$Ax = \begin{cases} 2 & if \quad 0 \le x \le 1, \\ \frac{x}{2} & if \quad 1 \le x \le 2, \end{cases} \qquad Sx = \begin{cases} 2 \quad if & x = 1, \\ \frac{x+3}{5} & otherwise \end{cases}$$

and $x_n = 2 - \frac{1}{2n}$. *Then we have* $S1 = A1 = 2$ *and* $S2 = A2 = 1$. *Also* $SA1 = AS1 = 1$ *and* $SA2 = AS2 = 2$. *Thus* (A, S) *is weak compatible. Again,*

$$Ax_n = 1 - \frac{1}{4n}, \quad Sx_n = 1 - \frac{1}{10n}.$$

Thus we have

$$Ax_n \to 1, \quad Sx_n \to 1.$$

Further, it follows that

$$SAx_n = \frac{4}{5} - \frac{1}{20n}, \quad ASx_n = 2.$$

Therefore, we have

$$lim_{n\to\infty} M(ASx_n, SAx_n, t) = lim_{n\to\infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{t}{t + \frac{6}{5}} < 1$$

for all $t > 0$. *Hence* (A, S) *is not compatible.*

2. Compatible Maps of Type (γ)

In this section, we give the concept of compatible maps of type (γ) in fuzzy metric spaces and some properties of these maps.

Definition 6. *Let* A *and* S *be mappings from a fuzzy metric space* $(X, M, *)$ *into itself. Then the mappings are said to be compatible maps of type* (γ) *if satisfying:*

1. A *and* S *are compatible, that is*

$$lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \quad \forall t > 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x \in X,$$

2. *They are continuous at* x.

On the other hand we have,

$$A(x) = A(lim_{n\to\infty} Ax_n) = A(lim_{n\to\infty} Sx_n) = lim_{n\to\infty} ASx_n$$
$$= lim_{n\to\infty} SAx_n = S(lim_{n\to\infty} Ax_n) = S(x)$$

Definition 7. *Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. The maps A and S are said to be weak compatible maps of type (γ) if*

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$$

for some $x \in X$ implies that $Ax = Sx$.

Clearly if self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible maps of type (γ), then they are weak compatible of type (γ). But the converse is not true as seen in following example.

Example 3. *Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, with t-norm defined $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $t > 0$ and $x, y \in X$. Define self-maps A and S on X as follows:*

$$Ax = \begin{cases} 1 & if \quad x \in \mathbb{Q}, \\ \frac{1}{2} & otherwise, \end{cases} \qquad Sx = \begin{cases} 1, & if \quad x \in \mathbb{Q} \\ 0 & otherwise, \end{cases}$$

and $x_n = 2 - \frac{1}{2n}$. Then we have

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = 1,$$

and $S1 = A1 = 1$. That is (A, S) is weak compatible of type (γ) also (A, S) is compatible,for $Ax_n = A(2 - \frac{1}{2n}) = 1$ and $Sx_n = S(2 - \frac{1}{2n}) = 1$ hence

$$lim_{n\to\infty} ASx_n = lim_{n\to\infty} SAx_n = 1,$$

but A, S are not continuous at 1 and hence (A, S) is not compatible of type (γ).

Lemma 3. *Let $(X, M, *)$ be a fuzzy metric space.*

(i) If we define $E_{\lambda,M} : X^2 \to \mathbb{R}^+ \bigcup\{0\}$ by

$$E_{\lambda,M}(x, y) = inf\{t > 0 : M(x, y, t) > 1 - \lambda\}$$

for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ and $x, y \in X$ such that

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + ... + E_{\lambda,M}(x_{n-1}, x_n)$$

for any $x_1, x_2, ..., x_n \in X$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if it is a Cauchy sequence with $E_{\lambda,M}$.

Proof. (i) For any $\mu \in (0,1)$, we can find a $\lambda \in (0,1)$ such that

$$\underbrace{(1-\lambda) * (1-\lambda) * ... * (1-\lambda)}_{n} \geq 1-\mu$$

and so, by the triangular inequality, we have

$$\begin{aligned} & M(x_1, x_n, E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + ... + E_{\lambda,M}(x_{n-1},x_n) + n\delta) \\ & \geq M(x_1,x_2,E_{\lambda,M}(x_1,x_2)+\delta) * ... * M(x_{n-1},x_n,E_{\lambda,M}(x_{n-1},x_n)+\delta) \\ & \geq \underbrace{(1-\lambda) * (1-\lambda) * ... * (1-\lambda)}_{n} \geq 1-\mu \end{aligned}$$

for all $\delta > 0$, which implies that

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + ... + E_{\lambda,M}(x_{n-1},x_n) + n\delta.$$

Since $\delta > 0$ is arbitrary implies that

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + ... + E_{\lambda,M}(x_{n-1},x_n).$$

(ii) Note that since M is continuous in its third place and

$$E_{\lambda,M}(x,y) = inf\{t > 0 : M(x,y,t) > 1-\lambda\}.$$

Hence, we have

$$M(x_n,x,\eta) > 1-\lambda \quad \Longleftrightarrow \quad E_{\lambda,M}(x_n,x) < \eta$$

for every $\eta > 0$. ■

Lemma 4. *Let $(X,M,*)$ be fuzzy metric space. if a sequence $\{x_n\}$ in X is such that, for any $n \in \mathbb{N}$,*

$$M(x_n,x_{n+1},t) \geq M(x_0,x_1,k^n t)$$

for all $k > 1$, then sequence $\{x_n\}$ is a cauchy sequence.

Proof. For all $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} E_{\lambda,M}(x_{n+1},x_n) &= inf\{t > 0 : M(x_{n+1},x_n,t) > 1-\lambda\} \\ &\leq inf\{t > 0 : M(x_0,x_1,k^n t) > 1-\lambda\} \\ &= inf\{\frac{t}{k^n} : M(x_0,x_1,t) > 1-\lambda\} \\ &= \frac{1}{k^n} inf\{t > 0 : M(x_0,x_1,t) > 1-\lambda\} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0,x_1). \end{aligned}$$

By lemma 3, for all $\mu \in (0,1)$ such that

$$\begin{aligned} E_{\mu,M}(x_n,x_m) &\leq E_{\lambda,M}(x_n,x_{n+1}) + E_{\lambda,M}(x_{n+1},x_{n+2}) + ... + E_{\lambda,M}(x_{m-1},x_m) \\ &\leq \frac{1}{k^n}E_{\mu,M}(x_0,x_1) + \frac{1}{k^{n+1}}E_{\lambda,M}(x_0,x_1) + ... + \frac{1}{k^{m-1}}E_{\lambda,M}(x_0,x_1) \\ &= E_{\mu,M}(x_0,x_1)\sum_{j=n}^{m-1}\frac{1}{k^j} \to 0. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence. ■

Lemma 5. ([12]). *If for all $x, y \in X$, $t > 0$ and for a number $k \in (0,1)$*

$$M(x,y,kt) \geq M(x,y,t)$$

then $x = y$.

Proof. It is immediate from (6) definition (2). ■

3. Main Results

In this section, we prove some common fixed point theorems for compatible mappings of type (γ) under satisfying some conditions in fuzzy metric spaces.

Theorem 6. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let P, S, T and Q be self-mappings of a complete fuzzy metric space, such that:*

(i) $P(X) \subseteq T(X)$, $Q(X) \subseteq S(X)$,

(ii) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} M(Px,Qy,kt) \geq & M(Tx,Px,t) * M(Sy,Qy,t) * M(Sy,Px,\alpha t) \\ & * M(Tx,Qy,(2-\alpha)t) * M(Tx,Sy,t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0,2)$ and $t > 0$,

(iii) the pairs (P,T) and (Q,S) are weak compatible of type (γ).

Then P, S, T and Q have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point, there exist $x_1, x_2 \in X$ such that $Px_0 = Tx_1 = y_0$, $Qx_1 = Sx_2 = y_1$. Inductively, construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Tx_{2n+1} = Px_{2n} = y_{2n}$, $Sx_{2n+2} = Qx_{2n+1} = y_{2n+1}$ for $n = 0, 1, 2, ...$

Now, we prove $\{y_n\}$ is a Cauchy sequence. For $n = 0, 1, 2, ...$ by (ii) then, by $\alpha = 1 - q$ and $q \in (0, 1)$, if we set $x = x_{2n+1}$,$y = x_{2n+2}$ for all $t > 0$, we have

$$\begin{aligned} M(Px_{2n+1}, Qx_{2n+2}, kt) \geq & M(Tx_{2n+1}, Px_{2n+1}, t) \\ & * M(Sx_{2n+2}, Qx_{2n+2}, t) \\ & * M(Sx_{2n+2}, Px_{2n+1}, (1-q)t) \\ & * M(Tx_{2n+1}, Qx_{2n+2}, (1+q)t) \\ & * M(Tx_{2n+1}, Sx_{2n+2}, t) \end{aligned}$$

and

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * M(y_{2n+1}, y_{2n+1}, (1-q)t) \\ & * M(y_{2n}, y_{2n+2}, (1+q)t) \\ & * M(y_{2n}, y_{2n+1}, t) \end{aligned}$$

then

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * 1 * M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, qt) \\ & * M(y_{2n}, y_{2n+1}, t) \end{aligned}$$

Hence we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * M(y_{2n+1}, y_{2n+2}, qt). \end{aligned}$$

Since the $*$ and $M(x, y, .)$ are continuous, letting $q \to 1$, we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

Similarly we have also

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t).$$

Hence for $n = 1, 2, ...$ and so, for positive integers n, p

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, \frac{t}{k^p}).$$

Thus, since $M(y_{2n+1}, y_{2n+2}, \frac{t}{k^p}) \to 1$ as $p \to \infty$, we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we also have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

For $k \in (0,1)$ if set $k_1 = \frac{1}{k} > 1$ and set $t = k_1 t_1$, then we have

$$M(y_n, y_{n+1}, t_1) \geq M(y_{n-1}, y_n, k_1 t_1) \geq ... \geq M(y_0, y_1, k_1^n t_1).$$

Hence by lemma 4 $\{y_n\}$ is cauchy and the complete of X, $\{y_n\}$ converges to z in X. That is, $lim_{n\to\infty} y_n = z$. Hence

$$\begin{aligned} lim_{n\to\infty} Px_{2n} &= lim_{n\to\infty} Tx_{2n+1} = lim_{n\to\infty} y_{2n} \\ &= lim_{n\to\infty} Qx_{2n+1} = lim_{n\to\infty} Sx_{2n+2} \\ &= lim_{n\to\infty} y_{2n+1} = z. \end{aligned}$$

Since P, T are weak compatible of type (γ) we have get $Pz = Tz$.

Now, taking $x = z, y = x_{2n+1}$ and $\alpha = 1$ in (ii), we have

$$\begin{aligned} M(Pz, Qx_{2n+1}, kt) \geq & M(Tz, Pz, t) * M(Sx_{2n+1}, Qx_{2n+1}, t) \\ & * M(Sx_{2n+1}, Pz, t) * M(Tz, Qx_{2n+1}, t) \\ & * M(Tz, Sx_{2n+1}, t) \end{aligned}$$

as $n \to \infty$

$$\begin{aligned} M(Pz, z, kt) \geq & M(Tz, Pz, t) * M(z, z, t) * M(z, Pz, t) \\ & * M(Tz, z, t) * M(Tz, z, t) \end{aligned}$$

thus

$$M(Pz, z, kt) \geq M(Pz, z, t).$$

Hence by lemma 5, for all $t > 0$, $Pz = z$. Therefore $Pz = Tz = z$.

Similarly, since pair (Q, S) are weak compatible of type (γ) hence we get $Qz = Sz$. Now, we show that $Qz = z$. For this taking $x = x_{2n}$, $y = z$ and $\alpha = 1$ in (ii), we have

$$\begin{aligned} M(Px_{2n}, Qz, kt) \geq & M(Tx_{2n}, Px_{2n}, t) * M(Sz, Qz, t) * M(Sz, Px_{2n}, t) \\ & * M(Tx_{2n}, Qz, t) * M(Tx_{2n}, Sz, t) \end{aligned}$$

as $n \to \infty$

$$\begin{aligned} M(z, Qz, kt) \geq & M(z, z, t) * M(Sz, Qz, t) * M(Sz, z, t) \\ & * M(z, Qz, t) * M(z, Sz, t) \end{aligned}$$

thus

$$M(z, Qz, kt) \geq M(z, Qz, t).$$

Hence for all $t > 0$, we have $Qz = Sz = z$. Therefore, z is a common fixed point of P, S, T and Q.

For uniqueness, let $v(v \neq z)$ be another common fixed point of P, S, T and Q and $\alpha = 1$, then by (ii), we have

$$\begin{aligned} M(Pz, Qv, kt) \geq & M(Tz, Pz, t) * M(Sv, Qv, t) \\ & * M(Sv, Pz, t) * M(Tz, Qz, t) \\ & * M(Tz, Sv, t) \end{aligned}$$

so

$$\begin{aligned} M(z, v, kt) \geq & M(z, z, t) * M(v, v, t) * M(v, z, t) \\ & * M(z, z, t) * M(z, v, t). \end{aligned}$$

Hence $M(z, v, kt) \geq M(z, v, t)$. Therefore by using lemma 5 we have $z = v$. ■

Theorem 7. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t * t = t$ *for all* $t \in [0, 1]$. *Let* $P_1, P_2, ..., P_{2m}$ *and* Q_0, Q_1 *be self-mappings continuous of a complete fuzzy metric space, such that:*

(i) $Q_0(X) \subseteq P_1P_3...P_{2m-1}(X)$, $Q_1(X) \subseteq P_2P_4...P_{2m}(X)$,

(ii) there exists a constant $k \in (0, 1)$ *such that*

$$\begin{aligned} M(Q_0x, Q_1y, kt) \geq & M(P_1P_3...P_{2m-1}x, Q_0x, t) \\ & * M(P_2P_4...P_{2m}y, Q_1y, t) \\ & * M(P_2P_4...P_{2m}y, Q_0x, \alpha t) \\ & * M(P_1P_3...P_{2m-1}x, Q_1y, (2 - \alpha)t) \\ & * M(P_1P_3...P_{2m-1}x, P_2P_4...P_{2m}y, t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $t > 0$,

(iii) the pairs $(Q_0, P_1P_3...P_{2m-1})$ *and* $(Q_1, P_2P_4...P_{2m})$ *are weak compatible of type* (γ),

(iv) for all $1 \leq i = 2n - 1 \leq 2m$ *and* $2 \leq j = 2n \leq 2m$ *such that*

$P_iQ_0 = Q_0P_i$,
$P_iP_1P_3...P_{2m-1} = P_1P_3...P_{2m-1}P_i$,
$P_jQ_1 = Q_1P_j$,
$P_jP_2P_4...P_{2m} = P_2P_4...P_{2m}P_j$.

Then $P_1, P_2, ..., P_{2m}$ *and* Q_0, Q_1 *have a unique common fixed point in* X.

Proof. Let $x_0 \in X$. From the condition (i) there exists $x_1, x_2 \in X$ such that $Q_0x_0 = P_1P_3...P_{2m-1}x_1 = y_0$ and $Q_1x_1 = P_2P_4...P_{2m}x_2 = y_1$. Inductively, construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Q_0x_{2n} = P_1P_3...P_{2m-1}x_{2n+1} = y_{2n},$$
$$Q_1x_{2n+1} = P_2P_4...P_{2m}x_{2n+2} = y_{2n+1},$$

for $n = 0, 1, 2, ...$

Now, we prove $\{y_n\}$ is a cauchy sequence. For $n = 0, 1, 2, ...$ by (ii) then by $\alpha = 1 - q$ and $q \in (0.1)$, if we set $x = x_{2n+1}, y = x_{2n+2}, P_1^{'} = P_1P_3...P_{2m-1}$ and $P_2^{'} = P_2P_4...P_{2m}$ for $t > 0$, we have

$$\begin{aligned} M(Q_0x_{2n+1}, Q_1x_{2n+2}, kt) \geq & M(P_1^{'}x_{2n+1}, Q_0x_{2n+1}, t) \\ & * M(P_2^{'}x_{2n+2}, Q_1x_{2n+2}, t) \\ & * M(P_2^{'}x_{2n+2}, Q_0x_{2n+1}, (1-q)t) \\ & * M(P_1^{'}x_{2n+1}, Q_1x_{2n+2}, (1+q)t) \\ & * M(P_1^{'}x_{2n+1}, P_2^{'}x_{2n+2}, t) \end{aligned}$$

and

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * M(y_{2n+1}, y_{2n+1}, (1-q)t) \\ & * M(y_{2n}, y_{2n+2}, (1+q)t) \\ & * M(y_{2n}, y_{2n+1}, t) \end{aligned}$$

then

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * 1 * M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, qt) \\ & * M(y_{2n}, y_{2n+1}, t) \end{aligned}$$

Hence we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) \geq & M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \\ & * M(y_{2n+1}, y_{2n+2}, qt). \end{aligned}$$

Since the $*$ and $M(x, y, .)$ are continuous, letting $q \to 1$, we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

Similarly we have also

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t).$$

Hence for $n = 1, 2, ...$ and so, for positive integers n, p

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, \frac{t}{k^p}).$$

thus, since $M(y_{2n+1}, y_{2n+2}, \frac{t}{k^p}) \to 1$ as $p \to \infty$, we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we also have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

For $k \in (0, 1)$ if set $k_1 = \frac{1}{k} > 1$ and set $t = k_1 t_1$, then we have

$$M(y_n, y_{n+1}, t_1) \geq M(y_{n-1}, y_n, k_1 t_1) \geq ... \geq M(y_0, y_1, k_1^n t_1).$$

Hence by lemma 4 $\{y_n\}$ is cauchy and the complete of X, $\{y_n\}$ converges to z in X. That is, $lim_{n\to\infty} y_n = z$. Hence

$$\begin{aligned} lim_{n\to\infty} Q_0 x_{2n} &= lim_{n\to\infty} P_1^{'} x_{2n+1} = lim_{n\to\infty} y_{2n} \\ &= lim_{n\to\infty} Q_1 x_{2n+1} = lim_{n\to\infty} P_2^{'} x_{2n+2} \\ &= lim_{n\to\infty} y_{2n+1} = z. \end{aligned}$$

Since $Q_0, P_1^{'}$ are weak compatible of type (γ) we have get $Q_0 z = P_1^{'} z$.

Now, taking $x = P_1^{'} x_{2n+1}$ and $y = x_{2n+2}$ with $\alpha = 1$ in (ii), we have

$$\begin{aligned} M(Q_0 P_1^{'} x_{2n+1}, Q_1 x_{2n+2}, kt) \geq & M(P_1^{'} P_1^{'} x_{2n+1}, Q_0 P_1^{'} x_{2n+1}, t) \\ & * M(P_2^{'} x_{2n+2}, Q_1 x_{2n+2}, t) \\ & * M(P_2^{'} x_{2n+2}, Q_0 P_1^{'} x_{2n+1}, t) \\ & * M(P_1^{'} P_1^{'} x_{2n+1}, Q_1 x_{2n+2}, t) \\ & * M(P_1^{'} P_1^{'} x_{2n+1}, P_2^{'} x_{2n+2}, t) \end{aligned}$$

as $n \to \infty$

$$\begin{aligned} M(Q_0 z, z, kt) \geq & M(P_1^{'} z, Q_0 z, t) * M(z, z, t) \\ & * M(z, Q_0 z, t) * M(P_1^{'} z, z, t) \\ & * M(P_1^{'} z, z, t) \end{aligned}$$

thus

$$M(Q_0z, z, kt) \geq M(z, Q_0z, t).$$

Hence by lemma 5, for all $t > 0$, $Q_0z = z$. Therefore $Q_0z = P_1'z = z$. Now, by putting $x = P_iz$ and $y = x_{2n+1}$ with $\alpha = 1$ in (ii), for all $1 \leq i = 2n - 1 \leq 2m$, we have

$$\begin{aligned} M(Q_0P_ix_{2n+1}, Q_1x_{2n+2}, kt) \geq & M(P_1'P_ix_{2n+1}, Q_0P_ix_{2n+1}, t) \\ & * M(P_2'x_{2n+2}, Q_1x_{2n+2}, t) \\ & * M(P_2'x_{2n+2}, Q_0P_ix_{2n+1}, t) \\ & * M(P_1'P_ix_{2n+1}, Q_1x_{2n+2}, t) \\ & * M(P_1'P_ix_{2n+1}, P_2'x_{2n+2}, t) \end{aligned}$$

thus by (iv), as $n \to \infty$

$$\begin{aligned} M(P_iz, z, kt) \geq & M(P_iz, P_iz, t) * M(z, z, t) \\ & * M(z, P_iz, t) * M(P_iz, z, t) \\ & * M(P_iz, z, t) \end{aligned}$$

and

$$M(P_iz, z, kt) \geq M(P_iz, z, t).$$

Hence by lemma 5, $P_iz = z$ for all $1 \leq i = 2n - 1 \leq 2m$.

Similarly since pair (Q_1, P_2') weak compatible of type (γ) hence we have $Q_1z = P_2'z$. Now, we show $Q_1z = z$. For this taking $x = z$ and $y = P_2'x_{2n+2}$ with $\alpha = 1$ in (ii), we have

$$\begin{aligned} M(Q_0z, Q_1P_2'x_{2n+2}, kt) \geq & M(P_1'z, Q_0z, t) \\ & * M(P_2'P_2'x_{2n+2}, Q_1P_2'x_{2n+2}, t) \\ & * M(P_2'P_2'x_{2n+2}, Q_0z, t) \\ & * M(P_1'z, Q_1P_2'x_{2n+2}, t) \\ & * M(P_1'z, P_2'P_2'x_{2n+2}, t) \end{aligned}$$

as $n \to \infty$

$$\begin{aligned} M(z, Q_1z, kt) \geq & M(z, z, t) * M(P_2'z, z, t) \\ & * M(P_2'z, z, t) * M(z, Q_1z, t) \\ & * M(z, P_2'z, t). \end{aligned}$$

Thus
$$M(z, Q_1z, kt) \geq M(z, Q_1z, t).$$
Hence $Q_1z = z$. Therefore $Q_1z = P_2'z = z$.

Now, by putting $x = z$ and $y = P_jz$ with $\alpha = 1$ in (ii), for all $2 \leq j = 2n \leq 2m$, we have

$$\begin{aligned} M(Q_0z, Q_1P_jz, kt) \geq & M(P_1'z, Q_0z, t) \\ & * M(P_2'P_jz, Q_1P_jz, t) \\ & * M(P_2'P_jz, Q_0z, t) \\ & * M(P_1'z, Q_1P_jz, t) \\ & * M(P_1'z, P_2'P_jz, t) \end{aligned}$$

thus by (iv), as $n \to \infty$

$$\begin{aligned} M(z, P_jz, kt) \geq & M(z, z, t) * M(P_jz, P_jz, t) \\ & * M(P_jz, P_jz, t) * M(z, P_jz, t) \\ & * M(z, P_jz, t) \end{aligned}$$

and
$$M(P_jz, z, kt) \geq M(P_jz, z, t).$$
Hence by lemma 2.6 $P_jz = z$ for all $2 \leq j = 2n \leq 2m$. Therefore z is a common fixed point of $P_1, P_2, ..., P_{2m}$ and Q_0, Q_1.

For uniqueness, let $v(v \neq z)$ be another common fixed point of $P_1, P_2, ..., P_{2m}$ and Q_0, Q_1 and $\alpha = 1$, then by (ii), we have

$$\begin{aligned} M(Q_0z, Q_1v, kt) \geq & M(P_1'z, Q_0z, t) * M(P_2'v, Q_1v, t) \\ & * M(P_2'v, Q_0z, t) * M(P_1'z, Q_1z, t) \\ & * M(P_1'z, P_2'v, t) \end{aligned}$$

Hence $M(z, v, kt) \geq M(z, v, t)$. Therefore by using lemma 5 we have $z = v$. ■

Theorem 8. *Let $\{Q_\mu\}_{\mu \in A}$, $\{Q_\nu\}_{\nu \in B}$ and $\{P_k\}_{k=1}^{2m}$ be the set of all self-mappings a complete fuzzy metric spaces $(X, M, *)$ with $t * t = t$ for all $t \in [0, 1]$, such that:*

(i) $Q_\mu(X) \subseteq P_1, P_2, ..., P_{2m}(X)$ and $Q_\nu(X) \subseteq P_1, P_3, ..., P_{2m-1}(X)$ for all $\mu \in A, \nu \in B$,

(ii) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} M(Q_\mu x, Q_\nu y, kt) \geq & M(P_1P_3...P_{2m-1}x, Q_\mu x, t) \\ & * M(P_2P_4...P_{2m}y, Q_\nu y, t) \\ & * M(P_2P_4...P_{2m}y, Q_\mu x, \alpha t) \\ & * M(P_1P_3...P_{2m-1}x, Q_\nu y, (2-\alpha)t) \\ & * M(P_1P_3...P_{2m-1}x, P_2P_4...P_{2m}y, t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0,2)$, $\mu \in A, \nu \in B$ and $t > 0$,

(iii) there exists $\mu_0 \in A$, such that pairs $(Q_{\mu_0}, P_1P_3...P_{2m-1})$ and $(Q_\nu, P_2P_4...P_{2m})$ are weak compatible of type (γ),

(iv) for all $\mu \in A, \nu \in B$, $1 \leq i = 2n-1 \leq 2m$ and $2 \leq j = 2n \leq 2m$ such that

$P_iQ_\mu = Q_\mu P_i$,

$P_iP_1P_3...P_{2m-1} = P_1P_3...P_{2m-1}P_i$,

$P_jQ_\nu = Q_\nu P_j$,

$P_jP_2P_4...P_{2m} = P_2P_4...P_{2m}P_j$.

Then all P_k and $\{Q_\mu\}_{\mu\in A}$, $\{Q_\nu\}_{\nu\in B}$ have a unique common fixed point in X.

Proof. Let Q_{μ_0} be a fixed element in $\{Q_\mu\}_{\mu\in A}$. By theorem 3.2 with $Q_0 = Q_{\mu_0}$ and $Q_1 = Q_\nu$ it follows that there exists some $z \in X$ such that

$$Q_\nu z = Q_{\mu_0} z = P_2P_4...P_{2m}z = P_1P_3...P_{2m-1}z = z.$$

Let $\mu_0 \neq \mu \in A$ be arbitrary. Then from (ii) with $\alpha = 1$, we have,

$$\begin{aligned} M(Q_\mu z, Q_\nu z, kt) \geq & M(P_1P_3...P_{2m-1}z, Q_\mu z, t) \\ & * M(P_2P_4...P_{2m}z, Q_\nu z, t) \\ & * M(P_2P_4...P_{2m}z, Q_\mu z, t) \\ & * M(P_1P_3...P_{2m-1}z, Q_\nu z, t) \\ & * M(P_1P_3...P_{2m-1}z, P_2P_4...P_{2m}z, t) \end{aligned}$$

and hence

$$\begin{aligned} M(Q_\mu z, z, kt) \geq & M(z, Q_\mu z, t) * M(z, z, t) \\ & * M(z, Q_\mu x, t) * M(z, z, t) \\ & * M(z, z, t). \end{aligned}$$

Therefore $Q_\mu z = z$. Hence, z is a unique common fixed point for all P_k and $\{Q_\mu\}_{\mu\in A}$, $\{Q_\nu\}_{\nu\in B}$ in X. ■

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