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NEW PROOFS FOR THEOREMS PROVEN BY SHIRAISHI AND OWA

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Abstract. The object of the present paper is to give new proofs for theorems proven by Shiraishi and Owa [1]. They proved two theorems that are sufficient conditions for analytic functions $f(z)$ to be starlike in the unit disc. They proved their theorems by lemma of Jack but we prove them by Miller-Mocanu lemma.

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1. Introduction

Let A denote the class of functions $f(z)$ that are analytic in the open unit disk $U = \{z : |z| < 1\}$, so that $f(0) = f'(0) - 1 = 0$.

We denote by S the subclass of A consisting of univalent functions. Let $S^*(\alpha)$ be the subcalss of A consisting of all functions $f(z)$ which satisfy

$$Re\left(z\frac{f'(z)}{f(z)}\right) > \alpha$$

We denote $S^*(0)$ by S^*.

Also, let $K(\alpha)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > \alpha$$

We denote $K(0)$ by K.

From the definitions for $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$.

Let $f(z)$ and $g(z)$ be analytic in U. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in U satisfying $w(z) = 0$, $|w(z)| < 1$ $(z \in U)$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z)$$

The basic tool in proving our results is the following lemma due to Miller and Mocanu [2].

Lemma 1. [2] *Let $\varphi(u,v)$ be complex valued function such that*

$$\varphi : D \to C \quad (D \subset C \times C)$$

C being the complex plane and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ suppose that the function $\varphi(u,v)$ satisfies each of following condition: $Re\{\varphi(iu,v)\} \leq 0$ for all $(iu,v) \in D$ such that $v \leq -\frac{1}{2}(1+u^2)$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be regular in U, such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{\varphi(p(z), zp'(z)\} > 0$$

Then $Re\{p(z)\} > 0$.

2. Main results

Applying Lemma 1, we reprove the following result for the class C.

Theorem 2. [1] *If $f(z) \in A$ satisfies*

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) < \frac{\alpha+1}{2(\alpha-1)} \quad (z \in U)$$

for some $\alpha(2 \leq \alpha < 3)$, or

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) \prec \frac{5\alpha-1}{2(\alpha+1)} \quad (z \in U)$$

for some $\alpha(1 < \alpha \leq 2)$, then

$$\frac{zf'(z)}{f(z)} < \frac{\alpha(1-z)}{\alpha - z} \quad (z \in U)$$

and

$$\left|\frac{zf'}{f(z)} - \frac{\alpha}{\alpha+1}\right| < \frac{\alpha}{\alpha+1} \quad (z \in U)$$

This implies that $f(z) \in S^*$ and $\int_0^z \frac{f(t)}{t}dt \in K$.

Proof. By definition, we must prove:

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1 - w(z))}{\alpha - w(z)} \quad (w(z) \neq \alpha)$$

where $w(z)$ is analytic in U, $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ but since

$$|w(z)| < 1 \Leftrightarrow Re(p(z)) > 0$$

where $p(z) = \frac{1-w(z)}{1+w(z)}$ thus we prove that

$$\frac{zf'(z)}{f(z)} = \frac{2\alpha p(z)}{(\alpha + 1)p(z) + \alpha - 1}$$

where $p(z)$ is analytic, $p(z) = 0$ and $Re(p(z)) > 0$. Since

$$1 + z\frac{f''(z)}{f'(z)} = \frac{2\alpha p(z)}{(\alpha + 1)p(z) + \alpha - 1} + \frac{(\alpha - 1)zp'(z)}{(\alpha + 1)p^2(z) + (\alpha - 1)p(z)}$$

We see that

$$Re\left(\frac{\alpha + 1}{2(\alpha - 1)} - \frac{2\alpha p(z)}{(\alpha + 1)p(z) + \alpha - 1} - \frac{(\alpha - 1)zp'(z)}{(\alpha + 1)p^2(z) + (\alpha - 1)p(z)}\right) > 0$$

for $\alpha (2 \leq \alpha < 3)$, and

$$Re\left(\frac{5\alpha - 1}{2(\alpha + 1)} - \frac{2\alpha p(z)}{(\alpha + 1)p(z) + \alpha - 1} - \frac{(\alpha - 1)zp'(z)}{(\alpha + 1)p^2(z) + (\alpha - 1)p(z)}\right) > 0$$

for $\alpha (1 < \alpha \leq 2)$, we define $\varphi(u, v)$ and $\psi(u, v)$ as follow:

$$\varphi(u, v) = \frac{\alpha + 1}{2(\alpha - 1)} - \frac{2\alpha u}{(\alpha + 1)u + \alpha - 1} - \frac{(\alpha - 1)v}{(\alpha + 1)u + (\alpha - 1)u}$$

for $\alpha (2 \leq \alpha < 3)$ and

$$\psi(u, v) = \frac{5\alpha - 1}{2(\alpha + 1)} - \frac{2\alpha u}{(\alpha + 1)u + \alpha - 1} - \frac{(\alpha - 1)v}{(\alpha + 1)u^2 + (\alpha - 1)u}$$

for $\alpha (1 < \alpha \leq 2)$, and compute $Re\varphi(iu, v)$ and $Re\psi(iu, v)$ and estimate them for

$v \leq -\frac{1}{2}(1+u^2)$. We have:

$$\begin{aligned}
Re\varphi(iu,v) &= Re\left(\frac{\alpha+1}{2(\alpha-1)} - \frac{2\alpha iu}{(\alpha+1)iu+\alpha-1} + \frac{(\alpha-1)v}{(\alpha+1)u^2-(\alpha-1)iu}\right)\\
&= Re\left(\frac{\alpha+1}{2(\alpha-1)} - \frac{2\alpha(\alpha+1)u^2}{(\alpha+1)^2u^2+\alpha-1} + \frac{(\alpha-1)v}{(\alpha+1)u^2-(\alpha-1)iu}\right)\\
&= Re\left(\frac{\alpha+1}{2(\alpha-1)} - \frac{2\alpha(\alpha+1)u^2}{(\alpha+1)^2u^2+\alpha-1} + \frac{(\alpha-1)u^2v}{(\alpha+1)^2u^2-(\alpha-1)^2u^2}\right)\\
&\leq \left(\frac{\alpha+1}{2(\alpha-1)} - \frac{2\alpha(\alpha+1)u^2}{(\alpha+1)^2u^2+\alpha-1} - \frac{1}{2}\frac{(\alpha-1)u^2(1+u^2)}{2(\alpha+1)^2u^2-(\alpha-1)^2u^2}\right)\\
&= \frac{\alpha+1}{2(\alpha-1)} - \frac{(\alpha+1)(5\alpha-1)u^2+\alpha-1)}{2((\alpha+1)^2u^2+(\alpha-1)^2)}\\
&= \frac{(\alpha+1)^2-4\alpha-(\alpha-1)^2}{2(\alpha^2-1)}\\
&\leq \frac{\alpha+1}{2(\alpha-1)} - \frac{\alpha+1}{2(\alpha-1)} = 0
\end{aligned}$$

for $\alpha(2 \leq \alpha < 3)$ and $v \leq -\frac{1}{2}(1+u^2)$ and similarly we have:

$$\begin{aligned}
Re\psi(iu,v) &= Re\left(\frac{5\alpha-1}{2(\alpha+1)} - \frac{2\alpha iu}{(\alpha+1)iu+\alpha-1} + \frac{(\alpha-1)v}{(\alpha+1)u^2-(\alpha-1)iu}\right)\\
&= Re\left(\frac{5\alpha-1}{2(\alpha+1)} - \frac{2\alpha(\alpha+1)u^2}{(\alpha+1)^2u^2+\alpha-1} + \frac{(\alpha-1)v}{(\alpha+1)u^2-(\alpha-1)iu}\right)\\
&= Re\left(\frac{5\alpha-1}{2(\alpha+1)} - \frac{2\alpha(\alpha+1)u^2}{(\alpha+1)^2u^2+\alpha-1} + \frac{(\alpha-1)u^2v}{(\alpha+1)^2u^4-(\alpha-1)^2u^2}\right)\\
&\leq \left(\frac{5\alpha-1}{2(\alpha+1)} - \frac{2\alpha(\alpha+1)u^2}{(\alpha+1)^2u^2+\alpha-1} - \frac{1}{2}\frac{(\alpha-1)u^2(1+u^2)}{2(\alpha+1)^2u^4-(\alpha-1)^2u^2}\right)\\
&\leq \frac{5\alpha-1}{2(\alpha+1)} - \frac{\alpha+1}{2(\alpha-1)} - \frac{(\alpha+1)((5\alpha-1)u^2+\alpha-1)}{2((\alpha+1)^2u^2+(\alpha-1)^2)}\\
&\leq \frac{5\alpha-1}{2(\alpha+1)} - \frac{5\alpha-1}{2(\alpha+1)} = 0
\end{aligned}$$

for $\alpha(1 < \alpha \leq 2)$ and $v \leq -\frac{1}{2}(1+u^2)$. Thus $\varphi(u,v)$ and $\psi(u,v)$ satisfy in the condition of 1 hence $Re(p(z)) > 0$. This completes the proof of theorem.

Theorem 3. [1] *If $f(z) \in A$ satisfies*

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > -\frac{\alpha+1}{2\alpha(\alpha-1)} \quad (z \in U)$$

for some $\alpha(\alpha \leq -1)$*, or*

$$Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > \frac{3\alpha+1}{2\alpha(\alpha+1)} \quad (z \in U)$$

for some $\alpha(\alpha > 1)$*, then*

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \quad (z \in U)$$

and

$$f(z) \in S^*(\frac{\alpha+1}{2\alpha})$$

This implies that $\int_0^z \frac{f(t)}{t}dt \in K(\frac{\alpha+1}{2\alpha})$.

Proof. By definition, we must prove:

$$\frac{f(z)}{zf'(z)} = \frac{\alpha(1-w(z))}{\alpha - w(z)} \quad (w(z) \neq \alpha)$$

where $w(z)$ is analytic in U, $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$

$$\frac{f(z)}{zf'(z)} \prec \frac{\alpha(1-z)}{\alpha - z} \Leftrightarrow f(z) \in S^*(\frac{\alpha+1}{2\alpha})$$

thus we prove that

$$\frac{zf'(z)}{f(z)} = \frac{\alpha+1}{2\alpha} + \frac{\alpha-1}{2\alpha}p(z)$$

where $p(z)$ is analytic, $p(z) = 0$ and $Re(p(z)) > 0$. Since

$$1 + z\frac{f''(z)}{f'(z)} = \frac{\alpha+1}{2\alpha} + \frac{\alpha-1}{2\alpha}p(z) + \frac{(\alpha-1)zp'(z)}{(\alpha-1)p(z)+(\alpha+1)}$$

We see that

$$Re\left(\frac{\alpha+1}{2\alpha} + \frac{\alpha-1}{2\alpha}p(z) + \frac{(\alpha-1)zp'(z)}{(\alpha-1)p(z)+(\alpha+1)} + \frac{\alpha+1}{2\alpha(\alpha-1)}\right) > 0 \quad (z \in U)$$

for some $\alpha(\alpha \leq -1)$, and

$$Re\left(\frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} - \frac{\alpha-1}{2\alpha}p(z) - \frac{(\alpha-1)zp'(z)}{(\alpha-1)p(z)+(\alpha+1)}\right) > 0 \quad (z \in U)$$

for some $\alpha(\alpha > 1)$, we define $\varphi(u, v)$ and $\psi(u, v)$ as follow:

$$\varphi(u, v) = \frac{\alpha+1}{2\alpha} + \frac{\alpha-1}{2\alpha}u + \frac{(\alpha-1)v}{(\alpha-1)u+(\alpha+1)} + \frac{\alpha+1}{2\alpha(\alpha-1)}$$

for $\alpha(\alpha \leq -1)$, and

$$\psi(u,v) = \frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} - \frac{\alpha-1}{2\alpha}u - \frac{(\alpha-1)v}{(\alpha-1)u+(\alpha+1)}$$

for $\alpha(\alpha > 1)$, and compute $Re\varphi(iu,v)$ and $Re\psi(iu,v)$ and estimate them for $v \leq -\frac{1}{2}(1+u^2)$. We have:

$$\begin{aligned}
Re\varphi(iu,v) &= Re\left(\frac{\alpha+1}{2\alpha} + \frac{\alpha-1}{2\alpha}iu + \frac{(\alpha-1)v}{(\alpha-1)iu+(\alpha+1)} + \frac{\alpha+1}{2\alpha(\alpha-1)}\right)\\
&= \frac{\alpha+1}{2\alpha} + \frac{\alpha+1}{2\alpha(\alpha-1)} + Re\left(\frac{(\alpha-1)v}{(\alpha-1)iu+(\alpha+1)}\right)\\
&= \frac{\alpha+1}{2\alpha} + \frac{\alpha+1}{2\alpha(\alpha-1)} + Re\left(\frac{(\alpha^2-1)v}{(\alpha-1)^2u^2+(\alpha+1)^2}\right)\\
&\leq \frac{\alpha+1}{2\alpha} + \frac{\alpha+1}{2\alpha(\alpha-1)} - \frac{1}{2}\frac{(\alpha^2-1)(1+u^2)}{(\alpha-1)^2u^2+(\alpha+1)^2}\\
&\leq \frac{\alpha+1}{2\alpha} + \frac{\alpha+1}{2\alpha(\alpha-1)} - \frac{\alpha+1}{2(\alpha-1)}\\
&= \frac{\alpha^2+\alpha}{2\alpha(\alpha-1)} - \frac{\alpha+1}{2(\alpha-1)} = 0
\end{aligned}$$

for $\alpha(\alpha \leq -1)$ and $v \leq -\frac{1}{2}(1+u^2)$. Similary we have:

$$\begin{aligned}
Re\psi(iu,v) &= Re\left(\frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} - \frac{\alpha-1}{2\alpha}iu - \frac{(\alpha-1)v}{(\alpha-1)iu+(\alpha+1)}\right)\\
&= \frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} - Re\left(\frac{(\alpha-1)v}{(\alpha-1)iu+(\alpha+1)}\right)\\
&= \frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} - Re\left(\frac{(\alpha^2-1)v}{(\alpha-1)^2u^2+(\alpha+1)^2}\right)\\
&\leq \frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} + \frac{1}{2}\frac{(\alpha^2-1)(1+u^2)}{(\alpha-1)^2u^2+(\alpha+1)^2}\\
&\leq \frac{3\alpha+1}{2\alpha(\alpha+1)} - \frac{\alpha+1}{2\alpha} + \frac{\alpha-1}{2(\alpha+1)}\\
&= \frac{\alpha-\alpha^2}{2\alpha(\alpha+1)} + \frac{\alpha-1}{2(\alpha+1)} = 0
\end{aligned}$$

for $\alpha(\alpha > 1)$ and $v \leq -\frac{1}{2}(1+u^2)$. Thus $\varphi(u,v)$ and $\psi(u,v)$ satisfy in the condition of Lemma 1 hence $Re(p(z)) > 0$. Thus completes the proof of theorem.

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