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INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION-II

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Abstract. In the present paper we have studied invariant submanifolds of Kenmotsu manifolds admitting a semi-symmetric metric connection and obtained some interesting results.

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1. Semi-symmetric metric connection

The study of the geometry of invariant submanifolds of Kenmotsu manifolds is carried out by C.S. Bagewadi and V.S. Prasad [9], S. Sular and C. Ozgur [12] and M. Kobayashi [8]. The author [8] has shown that the submanifold M of a Kenmotsu manifold $\widetilde{M}$ has parallel second fundamental form if and only if M is totally geodesic. The authors [8, 9, 12] have shown the equivalence of totally geodesicity of M with parallelism and semiparallelism of σ. Also they have shown that invariant submanifold of Kenmotsu manifold carried Kenmotsu structure and $K \leq \widetilde{K}$, then M is totally geodesic. Further the author [12] have shown the equivalence of totally geodesicity of M, if σ is recurrent, M has parallel third fundamental form and σ is generalized 2-recurrent. In this paper we extend the results to invariant submanifolds M of Kenmotsu Manifolds admitting Semi-symmetric metric connection.

We know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M if $\nabla g = 0$ otherwise it is non-metric. Further it is said to be semi-symmetric if its torsion tensor $T(X, Y) = 0$ i.e., $T(X, Y) = w(Y)X - w(X)Y$, where w is a 1-form. In 1924, A. Friedmann and J.A. Schouten [6] introduced the idea of semi-symmetric linear connection on differentiable manifold. In 1932, H.A. Hayden [7] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by K. Yano [13] in 1970. After that the properties of semi-symmetric metric connection have studied by many authors like K.S. Amur and S.S. Pujar [1], C.S. Bagewadi, D.G. Prakasha and Venkatesha [2, 3], A. Sharfuddin and S.I. Hussain [11], U.C. De and G. Pathak [5] etc. If $\overline{\nabla}$ denotes semi-symmetric metric connection on a contact metric manifold, then it is given by [2]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{1}$$

where $\eta(Y) = g(Y,\xi)$.

2. Isometric Immersion

Let $f : (M, g) \to (\widetilde{M}, \widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M, g) into $(n + d)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote by ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{3}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y), N) = g(A_N X, Y),$$

for tangent vector fields X, Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{4}$$

$$\begin{aligned}(\widetilde{\nabla}^2\sigma)(Z,W,X,Y) &= (\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z,W), \qquad (5)\\ &= \nabla^{\perp}_X((\widetilde{\nabla}_Y\sigma)(Z,W)) - (\widetilde{\nabla}_Y\sigma)(\nabla_X Z, W)\\ &\quad -(\widetilde{\nabla}_X\sigma)(Z,\nabla_Y W) - (\widetilde{\nabla}_{\nabla_X Y}\sigma)(Z,W)\end{aligned}$$

respectively, where $\widetilde{\nabla}$ is called the *vander Waerden-Bortolotti connection* of M [4]. If $\widetilde{\nabla}\sigma = 0$, then M is said to have *parallel second fundamental form* [4]. We next define endomorphisms $R(X,Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$\begin{aligned}R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\\ (X \wedge_B Y)Z &= B(Y,Z)X - B(X,Z)Y \qquad (6)\end{aligned}$$

respectively, where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0,2)$-tensor.

Now, for a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field B on (M,g), we define the tensor $Q(B,T)$ by

$$\begin{aligned} Q(B,T)(X_1,...,X_k;X,Y) &= -(T(X \wedge_B Y)X_1,...,X_k) \\ &\quad - \cdots - T(X_1,...,X_{k-1}(X \wedge_B Y)X_k). \end{aligned} \tag{7}$$

Putting into the above formula $T = \overline{\overline{\tilde{\nabla}}}\sigma$ and $B = g, S$, we obtain the tensors $Q(g, \overline{\overline{\tilde{\nabla}}}\sigma)$ and $Q(S, \overline{\overline{\tilde{\nabla}}}\sigma)$.

3. Kenmotsu Manifolds

Let $\widetilde{M}$ be a n-dimensional almost contact metric manifold with structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{9}$$

for all vector fields X, Y on M. If

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{10}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{11}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold.

Example of Kenmotsu manifold: Consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3 : z \neq 0\}$, where (x,y,z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = -e^z \frac{\partial}{\partial x}, \quad E_2 = -e^z \frac{\partial}{\partial y}, \quad E_3 = -\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1.$$

The (ϕ, ξ, η) is given by

$$\begin{aligned} \eta &= -dz, \quad \xi = E_3 = \frac{\partial}{\partial z}, \\ \phi E_1 &= E_2, \quad \phi E_2 = -E_1, \quad \phi E_3 = 0. \end{aligned}$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= 1, \qquad \phi^2 U = -U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U, W) - \eta(U)\eta(W), \end{aligned}$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

The Levi-Civita connection with respect to above metric g is given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Then we have,

$$\begin{aligned} \nabla_{E_1} E_1 &= -E_3, \qquad \nabla_{E_1} E_2 = 0, \qquad \nabla_{E_1} E_3 = E_1, \\ \nabla_{E_2} E_1 &= 0, \qquad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = E_2, \\ \nabla_{E_3} E_1 &= 0, \qquad \nabla_{E_3} E_2 = 0, \qquad \nabla_{E_3} E_3 = 0. \end{aligned}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations $(8), (9), (10)$ and (11). Thus M is a Kenmotsu manifold.

In Kenmotsu manifolds the following relations hold:

$$R(X,Y)Z = \{g(X,Z)Y - g(Y,Z)X\}, \tag{12}$$
$$R(X,Y)\xi = \{\eta(X)Y - \eta(Y)X\}, \tag{13}$$
$$R(\xi,X)Y = \{\eta(Y)X - g(X,Y)\xi\}, \tag{14}$$
$$R(\xi,X)\xi = \{X - \eta(X)\xi\}, \tag{15}$$
$$S(X,\xi) = -(n-1)\eta(X), \tag{16}$$
$$Q\xi = -(n-1)\xi. \tag{17}$$

4. Invariant submanifolds of Kenmotsu manifolds admitting Semi-symmetric metric connection

A submanifold M of a Kenmotsu manifold $\widetilde{M}$ with a semi-symmetric metric connection is called an invariant submanifold of $\widetilde{M}$ with a semi-symmetric metric connection, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. For an invariant submanifold of a Sasakian manifold with a semi-symmetric metric connection, we have

$$\sigma(X, \xi) = 0, \tag{18}$$

for any vector X tangent to M.

Let $\widetilde{M}$ be a Kenmotsu manifold admitting a semi-symmetric metric connection $\widetilde{\nabla}$.

Lemma 1. *Let M be an invariant submanifold of contact metric manifold $\widetilde{M}$ which admits semi-symmetric metric connection $\overline{\widetilde{\nabla}}$ and let σ and $\overline{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and semi-symmetric metric connection, then (1) M admits semi-symmetric metric connection, (2) the second fundamental forms with respect to $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ are equal.*

Proof. We know that the contact metric structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ on $\widetilde{M}$ induces (ϕ, ξ, η, g) on invariant submanifold. By virtue of (1), we get

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi. \tag{19}$$

By using (2) in (19), we get

$$\overline{\widetilde{\nabla}}_X Y = \nabla_X Y + \sigma(X, Y) + \eta(Y)X - g(X, Y)\xi. \tag{20}$$

Now Gauss formula (2) with respect to semi-symmetric metric connection is given by

$$\overline{\widetilde{\nabla}}_X Y = \overline{\nabla}_X Y + \overline{\sigma}(X, Y). \tag{21}$$

Equating (20) and (21), we get (1) and

$$\overline{\sigma}(X, Y) = \sigma(X, Y). \tag{22}$$

We know that an invariant submanifold of Kenmotsu manifold is also Kenmotsu. Also by the above lemma an invariant submanifold which admits semi-symmetric metric connection. Further second fundamental forms are equal. Hence by (18) and (22),

$$\overline{\sigma}(X, \xi) = \sigma(X, \xi). \tag{23}$$

Definition 1. *An immersion is said to be 2-semi, 2-pseudo and 2-Ricci-generalized pseudoparallel with respect to semi-symmetric metric connection, respectively, if the following conditions hold for all vector fields X, Y tangent to M*

$$\overline{\widetilde{R}} \cdot \sigma = 0, \tag{24}$$

$$\overline{\widetilde{R}} \cdot \sigma = L_1 Q(g, \overline{\widetilde{\nabla}}\sigma) \quad \text{and} \tag{25}$$

$$\overline{\widetilde{R}} \cdot \sigma = L_2 Q(S, \overline{\widetilde{\nabla}}\sigma), \tag{26}$$

where $\overline{\overline{R}}$ denotes the curvature tensor with respect to connection $\overline{\overline{\nabla}}$. Here L_1 and L_2 are functions depending on $\overline{\overline{\nabla}}\sigma$.

Lemma 2. *Let M be an invariant submanifold of contact manifold $\widetilde{M}$ which admits semi-symmetric metric connection. Then Gauss and Weingarten formulae with respect to semi-symmetric metric connection are given by*

$$\begin{aligned} & tan(\overline{\overline{R}}(X,Y)Z) = R(X,Y)Z + \eta(\nabla_Y Z)X - g(X,\nabla_Y Z)\xi + \eta(Z)\nabla_X Y \\ & +\eta(Z)\eta(Y)X - \eta(Z)g(X,Y)\xi - \eta(\nabla_X Z)Y + g(Y,\nabla_X Z)\xi - \eta(Z)\nabla_Y X \\ & -\eta(Z)\eta(X)Y + \eta(Z)g(Y,X)\xi + g([X,Y],Z)\xi - \eta(Z)[X,Y] \\ & +tan\left\{\overline{\overline{\nabla}}_X\{\sigma(Y,Z)\} - \overline{\overline{\nabla}}_Y\{\sigma(X,Z)\} - (\overline{\overline{\nabla}}_Y\eta(Z))X + (\overline{\overline{\nabla}}_X\eta(Z))Y \right. \\ & \left. -(\overline{\overline{\nabla}}_X g(Y,Z))\xi + (\overline{\overline{\nabla}}_Y g(X,Z))\xi\right\}, \end{aligned} \tag{27}$$

$$\begin{aligned} & nor(\overline{\overline{R}}(X,Y)Z) = \sigma(X,\nabla_Y Z) + \eta(Z)\sigma(X,Y) - \sigma(Y,\nabla_X Z) \\ & -\eta(Z)\sigma(Y,X) - \sigma([X,Y],Z) + nor\left\{\overline{\overline{\nabla}}_X\{\sigma(Y,Z)\} - \overline{\overline{\nabla}}_Y\{\sigma(X,Z)\}\right. \\ & \left. -(\overline{\overline{\nabla}}_Y\eta(Z))X + (\overline{\overline{\nabla}}_X\eta(Z))Y - (\overline{\overline{\nabla}}_X g(Y,Z))\xi + (\overline{\overline{\nabla}}_Y g(X,Z))\xi\right\}. \end{aligned} \tag{28}$$

Proof. The Riemannian curvature tensor $\widetilde{R}$ on $\widetilde{M}$ with respect to semi-symmetric metric connection is given by

$$\overline{\overline{R}}(X,Y)Z = \overline{\overline{\nabla}}_X\overline{\overline{\nabla}}_Y Z - \overline{\overline{\nabla}}_Y\overline{\overline{\nabla}}_X Z - \overline{\overline{\nabla}}_{[X,Y]}Z. \tag{29}$$

Using (1) and (2) in (29), we get

$$\begin{aligned} & \overline{\overline{R}}(X,Y)Z = R(X,Y)Z + \sigma(X,\nabla_Y Z) + \eta(\nabla_Y Z)X - g(X,\nabla_Y Z)\xi \\ & +\overline{\overline{\nabla}}_X\{\sigma(Y,Z)\} + (\overline{\overline{\nabla}}_X\eta(Z))Y + \eta(Z)\nabla_X Y + \eta(Z)\sigma(X,Y) + \eta(Z)\eta(Y)X \\ & -\eta(Z)g(X,Y)\xi - (\overline{\overline{\nabla}}_X g(Y,Z))\xi - \sigma(Y,\nabla_X Z) - \eta(\nabla_X Z)Y + g(Y,\nabla_X Z)\xi \\ & -\overline{\overline{\nabla}}_Y\{\sigma(X,Z)\} - (\overline{\overline{\nabla}}_Y\eta(Z))X - \eta(Z)\nabla_Y X - \eta(Z)\sigma(Y,X) - \eta(Z)\eta(X)Y \\ & +\eta(Z)g(Y,X)\xi + (\overline{\overline{\nabla}}_Y g(X,Z))\xi - \sigma([X,Y],Z) - \eta(Z)[X,Y] + g([X,Y],Z)\xi. \end{aligned} \tag{30}$$

Comparing tangential and normal part of (30), we obtain Gauss and Weingarten formulae (27) and (28).

We obtain the condition in the following lemma for 2-semi, 2-pseudo, 2-Ricci-generalized pseudoparallelism for invariant submanifold M of Kenmotsu manifold $\widetilde{M}$.

Lemma 3. *Let M be an invariant submanifold of contact manifold $\widetilde{M}$ which admits semi-symmetric metric connection. Then*

$$
\begin{aligned}
&(\overline{\widetilde{R}}(X,Y)\cdot\overline{\widetilde{\nabla}}\sigma)(U,V,W) = \overline{\widetilde{R}}(X,Y)\left\{\overline{\nabla}^{\perp}_{U}\sigma(V,W) - \sigma(\overline{\nabla}_U V,W)\right. \qquad (31)\\
&\left.-\sigma(V,\overline{\nabla}_U W)\right\} - \overline{\widetilde{\nabla}}\sigma(R(X,Y)U,V,W) - \overline{\widetilde{\nabla}}\sigma(U,R(X,Y)V,W)\\
&-\overline{\widetilde{\nabla}}\sigma(U,V,R(X,Y)W) - \eta(\nabla_Y U)\overline{\widetilde{\nabla}}\sigma(X,V,W) + g(X,\nabla_Y U)\overline{\widetilde{\nabla}}\sigma(\xi,V,W)\\
&-\overline{\widetilde{\nabla}}\sigma(tan(\overline{\widetilde{\nabla}}_X\{\sigma(Y,U)\}),V,W) - \overline{\widetilde{\nabla}}\sigma((\overline{\widetilde{\nabla}}_X\eta(U))Y,V,W) - \eta(U)\overline{\widetilde{\nabla}}\sigma(\nabla_X Y,V,W)\\
&-\eta(U)\eta(Y)\overline{\widetilde{\nabla}}\sigma(X,V,W) + \eta(U)g(X,Y)\overline{\widetilde{\nabla}}\sigma(\xi,V,W) + \overline{\widetilde{\nabla}}\sigma((\overline{\widetilde{\nabla}}_X g(Y,U))\xi,V,W)\\
&+\eta(\nabla_X U)\overline{\widetilde{\nabla}}\sigma(Y,V,W) - g(Y,\nabla_X U)\overline{\widetilde{\nabla}}\sigma(\xi,V,W) + \overline{\widetilde{\nabla}}\sigma(tan(\overline{\widetilde{\nabla}}_Y\{\sigma(X,U)\}),V,W)\\
&+\overline{\widetilde{\nabla}}\sigma((\overline{\widetilde{\nabla}}_Y\eta(U))X,V,W) + \eta(U)\overline{\widetilde{\nabla}}\sigma(\nabla_Y X,V,W) + \eta(U)\eta(X)\overline{\widetilde{\nabla}}\sigma(Y,V,W)\\
&-\eta(U)g(Y,X)\overline{\widetilde{\nabla}}\sigma(\xi,V,W) - \overline{\widetilde{\nabla}}\sigma((\overline{\widetilde{\nabla}}_Y g(X,U))\xi,V,W) + \eta(U)\overline{\widetilde{\nabla}}\sigma([X,Y],V,W)\\
&-g([X,Y],U)\overline{\widetilde{\nabla}}\sigma(\xi,V,W) - \eta(\nabla_Y V)\overline{\widetilde{\nabla}}\sigma(U,X,W) + g(X,\nabla_Y V)\overline{\widetilde{\nabla}}\sigma(U,\xi,W)\\
&-\overline{\widetilde{\nabla}}\sigma(U,tan(\overline{\widetilde{\nabla}}_X\{\sigma(Y,V)\}),W) - \overline{\widetilde{\nabla}}\sigma(U,(\overline{\widetilde{\nabla}}_X\eta(V))Y,W) - \eta(V)\overline{\widetilde{\nabla}}\sigma(U,\nabla_X Y,W)\\
&-\eta(V)\eta(Y)\overline{\widetilde{\nabla}}\sigma(U,X,W) + \eta(V)g(X,Y)\overline{\widetilde{\nabla}}\sigma(U,\xi,W) + \overline{\widetilde{\nabla}}\sigma(U,(\overline{\widetilde{\nabla}}_X g(Y,V))\xi,W)\\
&+\eta(\nabla_X V)\overline{\widetilde{\nabla}}\sigma(U,Y,W) - g(Y,\nabla_X V)\overline{\widetilde{\nabla}}\sigma(U,\xi,W) + \overline{\widetilde{\nabla}}\sigma(U,tan(\overline{\widetilde{\nabla}}_Y\{\sigma(X,V)\}),W)\\
&+\overline{\widetilde{\nabla}}\sigma(U,(\overline{\widetilde{\nabla}}_Y\eta(V))X,W) + \eta(V)\overline{\widetilde{\nabla}}\sigma(U,\nabla_Y X,W) + \eta(V)\eta(X)\overline{\widetilde{\nabla}}\sigma(U,Y,W)\\
&-\eta(V)g(Y,X)\overline{\widetilde{\nabla}}\sigma(U,\xi,W) - \overline{\widetilde{\nabla}}\sigma(U,(\overline{\widetilde{\nabla}}_Y g(X,V))\xi,W) + \eta(V)\overline{\widetilde{\nabla}}\sigma(U,[X,Y],W)\\
&-g([X,Y],V)\overline{\widetilde{\nabla}}\sigma(U,\xi,W) - \eta(\nabla_Y W)\overline{\widetilde{\nabla}}\sigma(U,V,X) + g(X,\nabla_Y W)\overline{\widetilde{\nabla}}\sigma(U,V,\xi)\\
&-\overline{\widetilde{\nabla}}\sigma(U,V,tan(\overline{\widetilde{\nabla}}_X\{\sigma(Y,W)\})) - \overline{\widetilde{\nabla}}\sigma(U,V,(\overline{\widetilde{\nabla}}_X\eta(W))Y) - \eta(W)\overline{\widetilde{\nabla}}\sigma(U,V,\nabla_X Y)\\
&-\eta(W)\eta(Y)\overline{\widetilde{\nabla}}\sigma(U,V,X) + \eta(W)g(X,Y)\overline{\widetilde{\nabla}}\sigma(U,V,\xi) + \overline{\widetilde{\nabla}}\sigma(U,V,(\overline{\widetilde{\nabla}}_X g(Y,W))\xi)\\
&+\eta(\nabla_X W)\overline{\widetilde{\nabla}}\sigma(U,V,Y) - g(Y,\nabla_X W)\overline{\widetilde{\nabla}}\sigma(U,V,\xi) + \overline{\widetilde{\nabla}}\sigma(U,V,tan(\overline{\widetilde{\nabla}}_Y\{\sigma(X,W)\}))\\
&+\overline{\widetilde{\nabla}}\sigma(U,V,(\overline{\widetilde{\nabla}}_Y\eta(W))X) + \eta(W)\overline{\widetilde{\nabla}}\sigma(U,V,\nabla_Y X) + \eta(W)\eta(X)\overline{\widetilde{\nabla}}\sigma(U,V,Y)\\
&-\eta(W)g(Y,X)\overline{\widetilde{\nabla}}\sigma(U,V,\xi) - \overline{\widetilde{\nabla}}\sigma(U,V,(\overline{\widetilde{\nabla}}_Y g(X,W)\xi)) + \eta(W)\overline{\widetilde{\nabla}}\sigma(U,V,[X,Y])\\
&-g([X,Y],W)\overline{\widetilde{\nabla}}\sigma(U,V,\xi),
\end{aligned}
$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X,\nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}.$$

Proof. We know, from tensor algebra, that

$$
\begin{aligned}
(\overline{\widetilde{R}}(X,Y)\cdot\overline{\widetilde{\nabla}}\sigma)(U,V,W) &= \overline{\widetilde{R}}(X,Y)\overline{\widetilde{\nabla}}\sigma(U,V,W) - \overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{R}}(X,Y)U,V,W) \qquad (32)\\
&-\overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{R}}(X,Y)V,W) - \overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{R}}(X,Y)W).
\end{aligned}
$$

We write the equation (4) with respect to semi-symmetric metric connection and in the form, we have the following equalities:

$$\overline{\overline{\nabla}}\sigma(U,V,W) = \overline{\nabla}^{\perp}_{U}\sigma(V,W) - \sigma(\overline{\nabla}_U V, W) - \sigma(V, \overline{\nabla}_U W). \tag{33}$$

By using (30) in $\overline{\overline{\nabla}}\sigma(\overline{\overline{R}}(X,Y)U,V,W)$, $\overline{\overline{\nabla}}\sigma(U,\overline{\overline{R}}(X,Y)V,W)$ and $\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{R}}(X,Y)W)$ to get

$$\begin{aligned}
&\overline{\overline{\nabla}}\sigma(\overline{\overline{R}}(X,Y)U,V,W) = \overline{\overline{\nabla}}\sigma(R(X,Y)U,V,W) + \eta(\nabla_Y U)\overline{\overline{\nabla}}\sigma(X,V,W) \\
&-g(X,\nabla_Y U)\overline{\overline{\nabla}}\sigma(\xi,V,W) + \overline{\overline{\nabla}}\sigma(tan(\overline{\overline{\nabla}}_X\{\sigma(Y,U)\}),V,W) \\
&+\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_X\eta(U))Y,V,W) + \eta(U)\overline{\overline{\nabla}}\sigma(\nabla_X Y,V,W) + \eta(U)\eta(Y)\overline{\overline{\nabla}}\sigma(X,V,W) \\
&-\eta(U)g(X,Y)\overline{\overline{\nabla}}\sigma(\xi,V,W) - \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_X g(Y,U))\xi,V,W) - \eta(\nabla_Y U)\overline{\overline{\nabla}}\sigma(Y,V,W) \\
&+g(Y,\nabla_X U)\overline{\overline{\nabla}}\sigma(\xi,V,W) - \overline{\overline{\nabla}}\sigma(tan(\overline{\overline{\nabla}}_Y\{\sigma(X,U)\}),V,W) \\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y\eta(U))X,V,W) - \eta(U)\overline{\overline{\nabla}}\sigma(\nabla_Y X,V,W) - \eta(U)\eta(X)\overline{\overline{\nabla}}\sigma(Y,V,W) \\
&+\eta(U)g(Y,X)\overline{\overline{\nabla}}\sigma(\xi,V,W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y g(X,U))\xi,V,W) - \eta(U)\overline{\overline{\nabla}}\sigma([X,Y],V,W) \\
&+g([X,Y],U)\overline{\overline{\nabla}}\sigma(\xi,V,W),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\overline{\overline{\nabla}}\sigma(U,\overline{\overline{R}}(X,Y)V,W) = \overline{\overline{\nabla}}\sigma(U,R(X,Y)V,W) + \eta(\nabla_Y V)\overline{\overline{\nabla}}\sigma(U,X,W) \\
&-g(X,\nabla_Y V)\overline{\overline{\nabla}}\sigma(U,\xi,W) + \overline{\overline{\nabla}}\sigma(U,tan(\overline{\overline{\nabla}}_X\{\sigma(Y,V)\}),W) \\
&+\overline{\overline{\nabla}}\sigma(U,(\overline{\overline{\nabla}}_X\eta(V))Y,W) + \eta(V)\overline{\overline{\nabla}}\sigma(U,\nabla_X Y,W) + \eta(V)\eta(Y)\overline{\overline{\nabla}}\sigma(U,X,W) \\
&-\eta(V)g(X,Y)\overline{\overline{\nabla}}\sigma(U,\xi,W) - \overline{\overline{\nabla}}\sigma(U,(\overline{\overline{\nabla}}_X g(Y,V))\xi,W) - \eta(\nabla_X V)\overline{\overline{\nabla}}\sigma(U,Y,W) \\
&+g(Y,\nabla_X V)\overline{\overline{\nabla}}\sigma(U,\xi,W) - \overline{\overline{\nabla}}\sigma(U,tan(\overline{\overline{\nabla}}_Y\{\sigma(X,V)\}),W) \\
&-\overline{\overline{\nabla}}\sigma(U,(\overline{\overline{\nabla}}_Y\eta(V))X,W) - \eta(V)\overline{\overline{\nabla}}\sigma(U,\nabla_Y X,W) - \eta(V)\eta(X)\overline{\overline{\nabla}}\sigma(U,Y,W) \\
&+\eta(V)g(Y,X)\overline{\overline{\nabla}}\sigma(U,\xi,W) + \overline{\overline{\nabla}}\sigma(U,(\overline{\overline{\nabla}}_Y g(X,V))\xi,W) - \eta(V)\overline{\overline{\nabla}}\sigma(U,[X,Y],W) \\
&+g([X,Y],V)\overline{\overline{\nabla}}\sigma(U,\xi,W)
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{R}}(X,Y)W) = \overline{\overline{\nabla}}\sigma(U,V,R(X,Y)W) + \eta(\nabla_Y W)\overline{\overline{\nabla}}\sigma(U,V,X) \\
&-g(X,\nabla_Y W)\overline{\overline{\nabla}}\sigma(U,V,\xi) + \overline{\overline{\nabla}}\sigma(U,V,tan(\overline{\overline{\nabla}}_X\{\sigma(Y,W)\})) \\
&+\overline{\overline{\nabla}}\sigma(U,V,(\overline{\overline{\nabla}}_X\eta(W))Y) + \eta(W)\overline{\overline{\nabla}}\sigma(U,V,\nabla_X Y) + \eta(W)\eta(Y)\overline{\overline{\nabla}}\sigma(U,V,X) \\
&-\eta(W)g(X,Y)\overline{\overline{\nabla}}\sigma(U,V,\xi) - \overline{\overline{\nabla}}\sigma(U,V,(\overline{\overline{\nabla}}_X g(Y,W))\xi) - \eta(\nabla_X W)\overline{\overline{\nabla}}\sigma(U,V,Y) \\
&+g(Y,\nabla_X W)\overline{\overline{\nabla}}\sigma(U,V,\xi) - \overline{\overline{\nabla}}\sigma(U,V,tan(\overline{\overline{\nabla}}_Y\{\sigma(X,W)\})) - \overline{\overline{\nabla}}\sigma(U,V,(\overline{\overline{\nabla}}_Y\eta(W))X)
\end{aligned} \tag{36}$$

$$\begin{aligned}
&-\eta(W)\overline{\overline{\nabla}}\sigma(U,V,\nabla_Y X) - \eta(W)\eta(X)\overline{\overline{\nabla}}\sigma(U,V,Y) + \eta(W)g(Y,X)\overline{\overline{\nabla}}\sigma(U,V,\xi) \\
&+\overline{\overline{\nabla}}\sigma(U,V,(\overline{\overline{\nabla}}_Y g(X,W))\xi) - \eta(W)\overline{\overline{\nabla}}\sigma(U,V,[X,Y]) + g([X,Y],W)\overline{\overline{\nabla}}\sigma(U,V,\xi).
\end{aligned}$$

Substituting (33) – (36) into (32), we get (31).

5. 2-Semiparallel, 2-pseudoparallel and 2-Ricci-generalized pseudoparallel Invariant submanifolds of Kenmotsu manifolds admitting Semi-symmetric metric connection

We consider invariant submanifolds of Kenmotsu manifolds admitting semi-symmetric metric connection satisfying the conditions $\overline{\overline{R}} \cdot \overline{\overline{\nabla}}\sigma = 0$, $\overline{\overline{R}} \cdot \overline{\overline{\nabla}}\sigma = L_1 Q(g, \overline{\overline{\nabla}}\sigma)$, $\overline{\overline{R}} \cdot \overline{\overline{\nabla}}\sigma = L_2 Q(S, \overline{\overline{\nabla}}\sigma)$. We write the equation (4) with respect to semi symmetric metric connection in the form

$$(\overline{\overline{\nabla}}_X \sigma)(Y, Z) = \overline{\nabla}^{\perp}_X(\sigma(Y, Z)) - \sigma(\overline{\nabla}_X Y, Z) - \sigma(Y, \overline{\nabla}_X Z), \tag{37}$$

and prove the following theorems

Theorem 4. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. Then M is 2-semiparallel with respect to semi-symmetric metric connection if and only if the derivative of the second fundamental form with respect to the characteristic vector ξ is zero $\Rightarrow \sigma(U, Y) = constant$.*

Proof. Let M be 2-semiparallel satisfying $\overline{\overline{R}} \cdot \overline{\overline{\nabla}}\sigma = 0$. Put $X = V = \xi$ and use (8), (11) and (23) in (31) to get

$$\begin{aligned}
0 = &-\overline{\overline{R}}(\xi, Y)\left\{\sigma(\overline{\nabla}_U \xi, W) + \sigma(\xi, \overline{\nabla}_U W)\right\} - \overline{\overline{\nabla}}\sigma(R(\xi, Y)U, \xi, W) \\
&-\overline{\overline{\nabla}}\sigma(U, R(\xi, Y)\xi, W) - \overline{\overline{\nabla}}\sigma(U, \xi, R(\xi, Y)W) - \overline{\overline{\nabla}}\sigma(tan(\overline{\overline{\nabla}}_\xi \{\sigma(Y, U)\}), \xi, W) \\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_\xi \eta(U))Y, \xi, W) - \eta(U)\overline{\overline{\nabla}}\sigma(\nabla_\xi Y, \xi, W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_\xi g(Y, U))\xi, \xi, W) \\
&+\eta(\nabla_\xi U)\overline{\overline{\nabla}}\sigma(Y, \xi, W) - g(Y, \nabla_\xi U)\overline{\overline{\nabla}}\sigma(\xi, \xi, W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y \eta(U))\xi, \xi, W) \\
&+\eta(U)\overline{\overline{\nabla}}\sigma(\nabla_Y \xi, \xi, W) + \eta(U)\overline{\overline{\nabla}}\sigma(Y, \xi, W) - \eta(U)\eta(Y)\overline{\overline{\nabla}}\sigma(\xi, \xi, W) \\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y \eta(U))\xi, \xi, W) + \eta(U)\overline{\overline{\nabla}}\sigma([\xi, Y], \xi, W) - g([\xi, Y], U)\overline{\overline{\nabla}}\sigma(\xi, \xi, W) \\
&-\overline{\overline{\nabla}}\sigma(U, \overline{\overline{\nabla}}_\xi Y, W) - \overline{\overline{\nabla}}\sigma(U, \nabla_\xi Y, W) + \overline{\overline{\nabla}}\sigma(U, (\overline{\overline{\nabla}}_\xi \eta(Y))\xi, W) + \overline{\overline{\nabla}}\sigma(U, \overline{\overline{\nabla}}_Y \xi, W) \\
&+\overline{\overline{\nabla}}\sigma(U, \nabla_Y \xi, W) + \overline{\overline{\nabla}}\sigma(U, Y, W) - \eta(Y)\overline{\overline{\nabla}}\sigma(U, \xi, W) - \overline{\overline{\nabla}}\sigma(U, \overline{\overline{\nabla}}_Y \xi, W) \\
&+\overline{\overline{\nabla}}\sigma(U, [\xi, Y], W) - \eta([\xi, Y])\overline{\overline{\nabla}}\sigma(U, \xi, W) - \overline{\overline{\nabla}}\sigma(U, \xi, tan(\overline{\overline{\nabla}}_\xi \{\sigma(Y, W)\})) \\
&-\overline{\overline{\nabla}}\sigma(U, \xi, (\overline{\overline{\nabla}}_\xi \eta(W))Y) - \eta(W)\overline{\overline{\nabla}}\sigma(U, \xi, \nabla_\xi Y) + \overline{\overline{\nabla}}\sigma(U, \xi, (\overline{\overline{\nabla}}_\xi g(Y, W))\xi)
\end{aligned} \tag{38}$$

$$+\eta(\nabla_\xi W)\overline{\overline{\nabla}}\sigma(U,\xi,Y) - g(Y,\nabla_\xi W)\overline{\overline{\nabla}}\sigma(U,\xi,\xi) + \overline{\overline{\nabla}}\sigma(U,\xi,(\overline{\overline{\nabla}}_Y\eta(W))\xi)$$
$$+\eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,\nabla_Y\xi) + \eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,Y) - \eta(W)\eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,\xi)$$
$$-\overline{\overline{\nabla}}\sigma(U,\xi,(\overline{\overline{\nabla}}_Y\eta(W))\xi) + \eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,[\xi,Y]) - g([\xi,Y],W)\overline{\overline{\nabla}}\sigma(U,\xi,\xi).$$

In view of (1), (8), (11), (14), (15), (23) and (37), we have the following equalities:

$$\begin{aligned}\overline{\overline{\nabla}}\sigma(R(\xi,Y)U,\xi,W) &= (\overline{\overline{\nabla}}_{R(\xi,Y)U}\sigma)(\xi,W),\\ &= \overline{\nabla}^{\perp}_{R(\xi,Y)U}\sigma(\xi,W) - \sigma(\overline{\nabla}_{R(\xi,Y)U}\xi,W) - \sigma(\xi,\overline{\nabla}_{R(\xi,Y)U}W),\\ &= -2\eta(U)\sigma(Y,W),\end{aligned} \tag{39}$$

$$\begin{aligned}\overline{\overline{\nabla}}\sigma(U,R(\xi,Y)\xi,W) &= (\overline{\overline{\nabla}}_U\sigma)(R(\xi,Y)\xi,W),\\ &= \overline{\nabla}^{\perp}_U\sigma(R(\xi,Y)\xi,W) - \sigma(\overline{\nabla}_U R(\xi,Y)\xi,W) - \sigma(R(\xi,Y)\xi,\overline{\nabla}_U W),\\ &= \overline{\nabla}^{\perp}_U\sigma(\{Y-\eta(Y)\xi\},W) - \sigma(\overline{\nabla}_U\{Y-\eta(Y)\xi\},W)\\ &\quad -\sigma(Y,\overline{\nabla}_U W)\end{aligned} \tag{40}$$

and

$$\begin{aligned}\overline{\overline{\nabla}}\sigma(U,\xi,R(\xi,Y)W) &= (\overline{\overline{\nabla}}_U\sigma)(\xi,R(\xi,Y)W),\\ &= \overline{\nabla}^{\perp}_U\sigma(\xi,R(\xi,Y)W) - \sigma(\overline{\nabla}_U\xi,R(\xi,Y)W) - \sigma(\xi,\overline{\nabla}_U R(\xi,Y)W),\\ &= -2\eta(W)\sigma(U,Y).\end{aligned} \tag{41}$$

Substituting (39 – 41) into (38) and $W=\xi$, using (1), (2), (8), (11), (23) and (37), we get

$$\sigma(\nabla_\xi U,Y) = 0. \tag{42}$$

Interchanging Y and U in (42), we get

$$\sigma(\nabla_\xi Y,U) = 0. \tag{43}$$

Adding (42) and (43), we get $\xi\sigma(U,Y)=0$ that is the derivative of the second fundamental form with respect to the characteristic vector ξ is zero $\Rightarrow \sigma(U,Y) =$ constant.

Theorem 5. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. Then M is 2-pseudoparallel with respect to semi-symmetric metric connection if and only if the derivative of the second fundamental form with respect to the characteristic vector ξ is zero $\Rightarrow \sigma(U,Y) =$ constant.*

Proof. Let M be 2-pseudoparallel satisfying $\overline{\overline{R}} \cdot \overline{\overline{\nabla}}\sigma = L_1 Q(g, \overline{\overline{\nabla}}\sigma)$. Put $X = V = \xi$ and use (8), (11) and (23) in (7), (31) to get

$$\begin{aligned}
&-\overline{\overline{R}}(\xi, Y)\left\{\sigma(\overline{\nabla}_U \xi, W) + \sigma(\xi, \overline{\nabla}_U W)\right\} - \overline{\overline{\nabla}}\sigma(R(\xi, Y)U, \xi, W) \\
&-\overline{\overline{\nabla}}\sigma(U, R(\xi, Y)\xi, W) - \overline{\overline{\nabla}}\sigma(U, \xi, R(\xi, Y)W) - \overline{\overline{\nabla}}\sigma(tan(\overline{\overline{\nabla}}_\xi \{\sigma(Y, U)\}), \xi, W) \\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_\xi \eta(U))Y, \xi, W) - \eta(U)\overline{\overline{\nabla}}\sigma(\nabla_\xi Y, \xi, W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_\xi g(Y, U))\xi, \xi, W) \\
&+\eta(\nabla_\xi U)\overline{\overline{\nabla}}\sigma(Y, \xi, W) - g(Y, \nabla_\xi U)\overline{\overline{\nabla}}\sigma(\xi, \xi, W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y \eta(U))\xi, \xi, W) \\
&+\eta(U)\overline{\overline{\nabla}}\sigma(\nabla_Y \xi, \xi, W) + \eta(U)\overline{\overline{\nabla}}\sigma(Y, \xi, W) - \eta(U)\eta(Y)\overline{\overline{\nabla}}\sigma(\xi, \xi, W) \\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y \eta(U))\xi, \xi, W) + \eta(U)\overline{\overline{\nabla}}\sigma([\xi, Y], \xi, W) - g([\xi, Y], U)\overline{\overline{\nabla}}\sigma(\xi, \xi, W) \\
&-\overline{\overline{\nabla}}\sigma(U, \overline{\overline{\nabla}}_\xi Y, W) - \overline{\overline{\nabla}}\sigma(U, \nabla_\xi Y, W) + \overline{\overline{\nabla}}\sigma(U, (\overline{\overline{\nabla}}_\xi \eta(Y))\xi, W) + \overline{\overline{\nabla}}\sigma(U, \overline{\overline{\nabla}}_Y \xi, W) \\
&+\overline{\overline{\nabla}}\sigma(U, \nabla_Y \xi, W) + \overline{\overline{\nabla}}\sigma(U, Y, W) - \eta(Y)\overline{\overline{\nabla}}\sigma(U, \xi, W) - \overline{\overline{\nabla}}\sigma(U, \overline{\overline{\nabla}}_Y \xi, W) \\
&+\overline{\overline{\nabla}}\sigma(U, [\xi, Y], W) - \eta([\xi, Y])\overline{\overline{\nabla}}\sigma(U, \xi, W) - \overline{\overline{\nabla}}\sigma(U, \xi, tan(\overline{\overline{\nabla}}_\xi \{\sigma(Y, W)\})) \\
&-\overline{\overline{\nabla}}\sigma(U, \xi, (\overline{\overline{\nabla}}_\xi \eta(W))Y) - \eta(W)\overline{\overline{\nabla}}\sigma(U, \xi, \nabla_\xi Y) + \overline{\overline{\nabla}}\sigma(U, \xi, (\overline{\overline{\nabla}}_\xi g(Y, W))\xi) \\
&+\eta(\nabla_\xi W)\overline{\overline{\nabla}}\sigma(U, \xi, Y) - g(Y, \nabla_\xi W)\overline{\overline{\nabla}}\sigma(U, \xi, \xi) + \overline{\overline{\nabla}}\sigma(U, \xi, (\overline{\overline{\nabla}}_Y \eta(W))\xi) \\
&+\eta(W)\overline{\overline{\nabla}}\sigma(U, \xi, \nabla_Y \xi) + \eta(W)\overline{\overline{\nabla}}\sigma(U, \xi, Y) - \eta(W)\eta(Y)\overline{\overline{\nabla}}\sigma(U, \xi, \xi) \\
&-\overline{\overline{\nabla}}\sigma(U, \xi, (\overline{\overline{\nabla}}_Y \eta(W))\xi) + \eta(W)\overline{\overline{\nabla}}\sigma(U, \xi, [\xi, Y]) - g([\xi, Y], W)\overline{\overline{\nabla}}\sigma(U, \xi, \xi) \\
&= -L_1\left[\eta(W)\left\{\overline{\nabla}^\perp_\xi \sigma(Y, U) - \sigma(\overline{\nabla}_\xi Y, U) - \sigma(Y, \overline{\nabla}_\xi U)\right\} - \overline{\nabla}^\perp_W \sigma(Y, U)\right. \\
&+\sigma(\overline{\nabla}_W Y, U) + \sigma(Y, \overline{\nabla}_W U) - \eta(Y)\left\{\overline{\nabla}^\perp_\xi \sigma(W, U) - \sigma(\overline{\nabla}_\xi W, U) - \sigma(W, \overline{\nabla}_\xi U)\right\} \\
&\left.-\eta(U)\left\{\overline{\nabla}^\perp_\xi \sigma(Y, W) - \sigma(\overline{\nabla}_\xi Y, W) - \sigma(Y, \overline{\nabla}_\xi W)\right\}\right].
\end{aligned} \tag{44}$$

Substituting $(39 - 41)$ into (44) and $W = \xi$, using (1), (2), (8), (11), (18) and (37), we get

$$\sigma(\nabla_\xi U, Y) = 0. \tag{45}$$

Interchanging Y and U in (45), we get

$$\sigma(\nabla_\xi Y, U) = 0. \tag{46}$$

Adding (45) and (46), we get $\xi\sigma(U, Y) = 0$ that is the derivative of the second fundamental form with respect to the characteristic vector ξ is zero $\Rightarrow \sigma(U, Y) =$ constant.

Theorem 6. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting a semi-symmetric metric connection. Then M is 2-Ricci-generalized pseudoparallel with respect to semi-symmetric metric connection if and only if the derivative* *of the second fundamental form with respect to the characteristic vector ξ is zero $\Rightarrow \sigma(U,Y) = constant$.*

Proof. Let M be 2-Ricci-generalized pseudoparallel satisfying $\overline{\overline{R}}\cdot\overline{\overline{\nabla}}\sigma = L_2 Q(S,\overline{\overline{\nabla}}\sigma)$. Put $X = V = \xi$ and use (8), (11), (16) and (23) in (7), (31) to get

$$\begin{aligned}
&-\overline{\overline{R}}(\xi,Y)\left\{\sigma(\overline{\nabla}_U\xi,W)+\sigma(\xi,\overline{\nabla}_U W)\right\} - \overline{\overline{\nabla}}\sigma(R(\xi,Y)U,\xi,W) \qquad (47)\\
&-\overline{\overline{\nabla}}\sigma(U,R(\xi,Y)\xi,W) - \overline{\overline{\nabla}}\sigma(U,\xi,R(\xi,Y)W) - \overline{\overline{\nabla}}\sigma(tan(\overline{\overline{\nabla}}_\xi\{\sigma(Y,U)\}),\xi,W)\\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_\xi\eta(U))Y,\xi,W) - \eta(U)\overline{\overline{\nabla}}\sigma(\nabla_\xi Y,\xi,W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_\xi g(Y,U))\xi,\xi,W)\\
&+\eta(\nabla_\xi U)\overline{\overline{\nabla}}\sigma(Y,\xi,W) - g(Y,\nabla_\xi U)\overline{\overline{\nabla}}\sigma(\xi,\xi,W) + \overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y\eta(U))\xi,\xi,W)\\
&+\eta(U)\overline{\overline{\nabla}}\sigma(\nabla_Y\xi,\xi,W) + \eta(U)\overline{\overline{\nabla}}\sigma(Y,\xi,W) - \eta(U)\eta(Y)\overline{\overline{\nabla}}\sigma(\xi,\xi,W)\\
&-\overline{\overline{\nabla}}\sigma((\overline{\overline{\nabla}}_Y\eta(U))\xi,\xi,W) + \eta(U)\overline{\overline{\nabla}}\sigma([\xi,Y],\xi,W) - g([\xi,Y],U)\overline{\overline{\nabla}}\sigma(\xi,\xi,W)\\
&-\overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_\xi Y,W) - \overline{\overline{\nabla}}\sigma(U,\nabla_\xi Y,W) + \overline{\overline{\nabla}}\sigma(U,(\overline{\overline{\nabla}}_\xi\eta(Y))\xi,W) + \overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_Y\xi,W)\\
&+\overline{\overline{\nabla}}\sigma(U,\nabla_Y\xi,W) + \overline{\overline{\nabla}}\sigma(U,Y,W) - \eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,W) - \overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_Y\xi,W)\\
&+\overline{\overline{\nabla}}\sigma(U,[\xi,Y],W) - \eta([\xi,Y])\overline{\overline{\nabla}}\sigma(U,\xi,W) - \overline{\overline{\nabla}}\sigma(U,\xi,tan(\overline{\overline{\nabla}}_\xi\{\sigma(Y,W)\}))\\
&-\overline{\overline{\nabla}}\sigma(U,\xi,(\overline{\overline{\nabla}}_\xi\eta(W))Y) - \eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,\nabla_\xi Y) + \overline{\overline{\nabla}}\sigma(U,\xi,(\overline{\overline{\nabla}}_\xi g(Y,W))\xi)\\
&+\eta(\nabla_\xi W)\overline{\overline{\nabla}}\sigma(U,\xi,Y) - g(Y,\nabla_\xi W)\overline{\overline{\nabla}}\sigma(U,\xi,\xi) + \overline{\overline{\nabla}}\sigma(U,\xi,(\overline{\overline{\nabla}}_Y\eta(W))\xi)\\
&+\eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,\nabla_Y\xi) + \eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,Y) - \eta(W)\eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,\xi)\\
&-\overline{\overline{\nabla}}\sigma(U,\xi,(\overline{\overline{\nabla}}_Y\eta(W))\xi) + \eta(W)\overline{\overline{\nabla}}\sigma(U,\xi,[\xi,Y]) - g([\xi,Y],W)\overline{\overline{\nabla}}\sigma(U,\xi,\xi)\\
&= -L_2\Big[-(n-1)\eta(W)\Big\{\overline{\nabla}^\perp_\xi\sigma(Y,U) - \sigma(\overline{\nabla}_\xi Y,U) - \sigma(Y,\overline{\nabla}_\xi U)\Big\}\\
&+(n-1)\Big\{\overline{\nabla}^\perp_W\sigma(Y,U) - \sigma(\overline{\nabla}_W Y,U) - \sigma(Y,\overline{\nabla}_W U)\Big\}\\
&+(n-1)\eta(Y)\Big\{\overline{\nabla}^\perp_\xi\sigma(W,U) - \sigma(\overline{\nabla}_\xi W,U) - \sigma(W,\overline{\nabla}_\xi U)\Big\}\\
&+(n-1)\eta(U)\Big\{\overline{\nabla}^\perp_\xi\sigma(Y,W) - \sigma(\overline{\nabla}_\xi Y,W) - \sigma(Y,\overline{\nabla}_\xi W)\Big\}\Big].
\end{aligned}$$

Substituting $(39-41)$ into (47) and $W=\xi$, using (1), (2), (8), (11), (18) and (37), we get

$$\sigma(\nabla_\xi U,Y) = 0. \qquad (48)$$

Interchanging Y and U in (48), we get

$$\sigma(\nabla_\xi Y,U) = 0. \qquad (49)$$

Adding (48) and (49), we get $\xi\sigma(U, Y) = 0$ that is the derivative of the second fundamental form with respect to the characteristic vector ξ is zero $\Rightarrow \sigma(U, Y) =$ constant.

Remark. Let M be an invariant submanifold of a Kenmotsu manifold which admits semi-symmetric metric connection. If M is 2-semi, 2-pseudo and 2-Ricci-generalized pseudoparallel, then we have obtained conditions connecting ξ. These conditions need further investigation and are to be interpreted geometrically.

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