Acta Universitatis Apulensis No. 36/2013
ISSN: 1582-5329 pp. 15-30

CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS ASSOCIATED WITH CERTAIN INTEGRAL OPERATOR

R. M. EL-Ashwah, M. K. Aouf, H. M. Zayed

Abstract. *In this paper we introduce and study two subclasses* $\sum_p(\beta,\alpha,\lambda;A,B)$ *and* $\sum_p^+(\beta,\alpha,\lambda;A,B)$ *of meromorphic p-valent functions of order* λ $(0 \leq \lambda < p)$ *defined by certain integral operator. We investigate the various important properties and characteristics of the problem. Finally, we derive many interesting results for the Hadamard products of functions belonging to the class* $\sum_p^+(\beta,\alpha,\lambda;A,B)$.

2000 *Mathematics Subject Classification*: 30C45.

Keywords: meromorphic, starlike and convex functions, Hadamard product.

1. Introduction

Let Σ_p denote the class of meromorphic functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \ (p \in \mathbb{N} = \{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. Let $g(z) \in \Sigma_p$, be given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p}, \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.3}$$

Aqlan et al. [4] defined the operator $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p$ by:

$$
\begin{aligned}
Q^{\alpha}_{\beta,p}f(z) &= \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)}\frac{1}{z^{\beta+p}}\int\limits_0^z t^{\beta+p-1}\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt \\
&= z^{-p}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)}a_{k-p}z^{k-p} \\
&= \left(z^{-p}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)}z^{k-p}\right)*f(z),
\end{aligned}
\tag{1.4}
$$

where $\alpha>0,\ \beta>-1,\ p\in\mathbb{N}$ and $f\in\sum_p$, and

$$
Q^{0}_{\beta,p}f(z)=f(z)\ (\alpha=0,\beta>-1). \tag{1.5}
$$

One can easily verify from the definition that

$$
z(Q^{\alpha}_{\beta,p}f(z))'=(\beta+\alpha-1)(Q^{\alpha-1}_{\beta,p}f(z))-(p+\beta+\alpha-1)Q^{\alpha}_{\beta,p}f(z)\ (\alpha\geq 1). \tag{1.6}
$$

For fixed parameters $A,\ B$ and λ with $-1\leq B<A\leq 1,\ -1\leq B<0,\ 0\leq\lambda<p$ and $p\in\mathbb{N}$, we say that a function $f(z)\in\sum_p$ is in the class $\sum_p(\beta,\alpha,\lambda;A,B)$ of meromorphically $p-$valent functions in U if it also satisfies the following inequality:

$$
\left|\frac{z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+p}{Bz^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+[pB+(A-B)(p-\lambda)]}\right|<1\ (z\in U). \tag{1.7}
$$

Furthermore, we say that a functions $f(z)\in\sum_p^{+}(\beta,\alpha,\lambda;A,B)$ whenever $f(z)$ is of the form:

$$
f(z)=z^{-p}+\sum_{k=p}^{\infty}|a_k|\,z^k\ (p\in\mathbb{N}). \tag{1.8}
$$

We note that:

(i)

$\sum_p^{+}(\beta,\alpha,\lambda;\delta,-\delta)=\sum_p^{+}(\beta,\alpha,\delta,\lambda)=$

$$
\left\{f(z)\in\sum\nolimits_p:\left|\frac{z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+p}{z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'-p+2\lambda}\right|<\delta\ (\beta>-1;\ \alpha\geq 0;\ 0<\delta\leq 1;\right.
$$

$$0 \le \lambda < p;\ p \in \mathbb{N})\}\,; \tag{1.9}$$

(ii)
$\sum_p^+(\beta,\alpha,\lambda;-(2\gamma-1)\delta,\delta)=\sum_p^+(\beta,\alpha,\gamma,\delta,\lambda)=$

$$\left\{f(z)\in\sum\nolimits_p:\left|\frac{z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+p}{(2\gamma-1)z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'-p+2\gamma\lambda}\right|<\delta\ (\beta>-1;\ \alpha\ge 0;\ \frac{1}{2}\le\right.$$

$$\gamma\le 1;\ 0<\delta\le 1;\ 0\le\lambda<p;\ p\in\mathbb{N})\}\,. \tag{1.10}$$

Meromorphically multivalent functions have been extensively studied by (for example) Mogra [8,9], Uralegaddi and Ganigi [14], Uralegaddi and Somanatha [15], Aouf [1,2], Srivastava et al. [12], Owa et al. [10], Joshi and Aouf [6], Joshi and Srivastava [7], Aouf et al. [3], Raina and srivastava [13] and Yang [16].

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our inclusion theorem.

2. Properties of the class $\sum_p(\beta,\alpha,\lambda;A,B)$

Lemma 1 [5]. *Let the (non-constant) function $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|<r<1$ at a point $z_0\in U$, then*

$$z_0 w'(z_0)=\gamma w(z_0), \tag{2.1}$$

where γ is a real number and $\gamma\ge 1$.

Theorem 1. *The following inclusion property holds true for the class $\sum_p(\beta,\alpha,\lambda;A,B)$, $\alpha>1,\ \beta>-1$:*

$$\sum\nolimits_p(\beta,\alpha-1,\lambda;A,B)\subset\sum\nolimits_p(\beta,\alpha,\lambda;A,B). \tag{2.2}$$

Proof. Let $f(z)\in\sum_p(\beta,\alpha-1,\lambda;A,B)$ and suppose that

$$z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'=-\frac{p+[pB+(A-B)(p-\lambda)]w(z)}{1+Bw(z)}, \tag{2.3}$$

where the function $w(z)$ is either analytic or meromorphic in U with $w(0)=0$. Then, by using (1.6) and (2.3), we have

$$z^{p+1}(Q^{\alpha-1}_{\beta,p}f(z))'=-\frac{p+[pB+(A-B)(p-\lambda)]}{1+Bw(z)}-\frac{(A-B)(p-\lambda)}{(\beta+\alpha-1)}\frac{zw'(z)}{[1+Bw(z)]^2}. \tag{2.4}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have $z_0 w'(z_0) = \gamma w(z_0)$ ($\gamma \geq 1$). Writing $w(z_0) = e^{i\theta} (0 \leq \theta \leq 2\pi)$ and taking $z = z_0$ in (2.4), we get

$$\begin{aligned} &\left| \frac{z_0^{p+1}(Q^{\alpha}_{\beta,p} f(z_0))' + p}{B z_0^{p+1}(Q^{\alpha}_{\beta,p} f(z_0))' + [pB + (A-B)(p-\lambda)]} \right|^2 - 1 \\ = &\frac{\left|(\beta+\alpha-1)(1+Be^{i\theta}) + \gamma\right|^2 - \left|(\beta+\alpha-1) + B(\beta+\alpha-\gamma-1)e^{i\theta}\right|^2}{|(\beta+\alpha-1) + B(\beta+\alpha-\gamma-1)e^{i\theta}|^2} \\ = &\frac{\gamma^2(1-B^2) + 2\gamma(\beta+\alpha-1)(1+B^2+2B\cos\theta)}{|(\beta+\alpha-1)(1+Be^{i\theta}) - \gamma B e^{i\theta}|^2} \geq 0, \end{aligned}$$

which obivously contradicts our hypothesis that $f(z) \in \sum_p(\beta, \alpha-1, \lambda; A, B)$. Thus we must have $|w(z)| < 1$ for $z \in U$, and so from (2.3), we conclude that $f(z) \in \sum_p(\beta, \alpha, \lambda; A, B)$, which evidently completes the proof of Theorem 1.

3. Properties of the class $\sum_p^+(\beta,\alpha,\lambda;\mathrm{A},\mathrm{B})$

Unless otherwise mentioned, we assume throughout this paper that

$$\alpha \geq 1,\ \beta > -1,\ -1 \leq B < A \leq 1,\ -1 \leq B < 0,\ 0 \leq \lambda < p,\ k \geq p \text{ and } p \in \mathbb{N}.$$

Theorem 2. *Let $f(z) \in \sum_p$ be given by (1.8). Then $f(z) \in \sum_p^+(\beta, \alpha, \lambda; A, B)$ if and only if*

$$\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=p}^{\infty} k(1-B) \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)} |a_k| \leq (A-B)(p-\lambda). \tag{3.1}$$

Proof. Let $f(z) \in \sum_p^+(\beta, \alpha, \lambda; A, B)$ be given by (1.8). Then, from (1.7) and (1.8), we have

$$\begin{aligned}
&\left|\frac{z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+p}{Bz^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+[pB+(A-B)(p-\lambda)]}\right| \\
=&\left|\frac{\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|\,z^{k+p}}{(A-B)(p-\lambda)+B\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|\,z^{k+p}}\right| \\
\leq&\ \frac{\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|}{(A-B)(p-\lambda)+B\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|}<1.
\end{aligned}$$

On simplification we easily arrive at the inequality (3.1).
Conversely, we assume that the inequality (3.1) holds true. Then,

$$\mathrm{Re}\left\{\frac{z^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+p}{Bz^{p+1}(Q^{\alpha}_{\beta,p}f(z))'+[pB+(A-B)(p-\lambda)]}\right\}\leq 1.$$

Hence

$$\mathrm{Re}\left\{\frac{\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|\,z^{k+p}}{(A-B)(p-\lambda)+B\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|\,z^{k+p}}\right\}\leq 1,$$

If we now choose z to be real and $z\to 1^-$, we obtain

$$\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k(1-B)\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|\leq(A-B)(p-\lambda).$$

This completes the proof of Theorem 2.

Corollary 1. *Let the function $f(z)$ defined by (1.8) be in the class $\sum_p^+(\beta,\alpha,\lambda;A,B)$. Then*

$$|a_k|\leq\frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)}{k(1-B)}. \tag{3.2}$$

The result is sharp for the function

$$f(z)=z^{-p}+\frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)}{k(1-B)}z^k. \tag{3.3}$$

4. Distortion theorems

Theorem 3. *Let the function $f(z)$ defined by (1.8) be in the class* $\sum_p^+(\beta, \alpha, \lambda; A, B)$. If the sequence $\{c_k\} = \{k \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}\}$ is nondecreasing, then

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(A-B)(p-\lambda)}{(1-B)C_p}\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\frac{p!}{(p-m)!}r^{2p}\right\} r^{-(p+m)}$$

$$\leq |f^m(z)| \leq$$

$$\left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(A-B)(p-\lambda)}{(1-B)C_p}\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\frac{p!}{(p-m)!}r^{2p}\right\} r^{-(p+m)}$$

$$(0 < |z| = r < 1;\ 0 \leq \lambda < p;\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ p > m). \tag{4.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{(A-B)(p-\lambda)}{k(1-B)}z^p. \tag{4.2}$$

Proof. By differentiating both sides of (1.8) m times with respect to z, we have

$$f^m(z) = (-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)} + \sum_{k=p}^{\infty}\frac{k!}{(k-m)!}|a_k|\, z^{k-m}. \tag{4.3}$$

It is easy to see from Theorem 2 that

$$\frac{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}p(1-B)\sum_{k=p}^{\infty}|a_k| \leq \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k(1-B)\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}|a_k|$$

$$\leq (A-B)(p-\lambda).$$

Then

$$\sum_{k=p}^{\infty}k!\,|a_k| \leq \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{p!(A-B)(p-\lambda)}{p(1-B)}. \tag{4.4}$$

Making use of (4.3) and (4.4), we have

$$\begin{aligned}|f^m(z)| &\geq \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)} - \frac{1}{(p-m)!}r^{p-m}\sum_{k=p}^{\infty}k!\,|a_k| \\ &\geq \left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(A-B)(p-\lambda)}{(1-B)C_p}\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\frac{p!}{(p-m)!}r^{2p}\right\} r^{-(p+m)},\end{aligned} \tag{4.5}$$

and

$$
\begin{aligned}
|f^m(z)| &\le \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)} + \frac{1}{(p-m)!}r^{p-m}\sum_{k=p}^{\infty} k!\,|a_k| \\
&\le \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{(A-B)(p-\lambda)}{(1-B)C_p}\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\frac{p!}{(p-m)!}r^{2p}\right\} r^{-(p+m)},
\end{aligned}
\tag{4.6}
$$

which proves the assertion (4.1). Finally, it is easy to see that the bounds in (4.1) are attained for the function $f(z)$ given by (4.2). This completes the proof of Theorem 3.

5. Closure theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$
f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k \ (a_{k,j} \ge 0). \tag{5.1}
$$

Theorem 4. *Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (5.1) be in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Then the function $h(z)$ defined by*

$$
h(z) = z^{-p} + \sum_{k=p}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}|a_{k,j}|\right) z^k, \tag{5.2}
$$

also belongs to the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$.

Proof. Since $f_j(z)$ $(j = 1, 2, ..., m)$ are in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$, it follows from Theorem 2, that

$$
\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty} k(1-B)\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}\,|a_{k,j}| \le (A-B)(p-\lambda),
$$

for every $j = 1, 2, ..., m$. Hence

$$
\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty} k(1-B)\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}\left(\frac{1}{m}\sum_{j=1}^{m}|a_{k,j}|\right)
$$

$$\begin{aligned} &= \frac{1}{m}\sum_{j=1}^{m}\left(\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k(1-B)\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}\left|a_{k,j}\right|\right) \\ &\leq (A-B)(p-\lambda). \end{aligned}$$

From Theorem 2, it follows that $h(z) \in \sum_p^+(\beta, \alpha, \lambda; A, B)$. This completes the proof of Theorem 4.

Theorem 5. *The class $\sum_p^+(\beta, \alpha, \lambda; A, B)$ is closed under convex linear combinations.*

Proof. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Then it is sufficient to show that the function

$$h(z) = \gamma f_1(z) + (1-\gamma) f_2(z) \ (0 \leq \gamma \leq 1),$$

is in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Since for $0 \leq \gamma \leq 1$,

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}[\gamma\left|a_{k,1}\right| + (1-\gamma)\left|a_{k,2}\right|]z^k, \tag{5.3}$$

with the aid of Theorem 2, we have

$$\begin{aligned} &\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=p}^{\infty}k(1-B)\frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)}[\gamma\left|a_{k,1}\right| + (1-\gamma)\left|a_{k,2}\right|] \\ &\leq \gamma(A-B)(p-\lambda) + (1-\gamma)(A-B)(p-\lambda) \\ &= (A-B)(p-\lambda), \end{aligned}$$

which implies that $h(z) \in \sum_p^+(\beta, \alpha, \lambda; A, B)$. This completes the proof of Theorem 5.

Theorem 6. *Let $f_p(z) = z^{-p}$ and*

$$f_k(z) = z^{-p} + \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)}{k(1-B)}z^k. \tag{5.4}$$

Then $f(z)$ is in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$ if and only if can be expressed in the form

$$f(z) = \sum_{k=p-1}^{\infty}\mu_k f_k(z), \tag{5.5}$$

where $\mu_k \geq 0$ *and* $\sum_{k=p-1}^{\infty} \mu_k = 1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=p-1}^{\infty} \mu_k f_k(z) \\ &= z^{-p} + \sum_{k=p}^{\infty} \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)} \frac{(A-B)(p-\lambda)}{k(1-B)} \mu_k z^k. \end{aligned} \tag{5.6}$$

Then it follows that

$$\sum_{k=p}^{\infty} \frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)} \frac{k(1-B)}{(A-B)(p-\lambda)} \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)} \frac{(A-B)(p-\lambda)}{k(1-B)} \mu_k$$

$$= \sum_{k=p}^{\infty} \mu_k = 1 - \mu_{p-1} \leq 1.$$

which implies that $f(z) \in \sum_p^+(\beta, \alpha, \lambda; A, B)$.

Conversely, assume that the function $f(z)$ defined by (1.8) be in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Then

$$|a_k| \leq \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)} \frac{(A-B)(p-\lambda)}{k(1-B)}.$$

Setting

$$\mu_k = \frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)} \frac{k(1-B)}{(A-B)(p-\lambda)} |a_k|,$$

where

$$\mu_{p-1} = 1 - \sum_{k=p}^{\infty} \mu_k ,$$

we can see that $f(z)$ can be expressed in the form (5.5). This completes the proof of Theorem 6.

Corollary 2. *The extreme points of the class* $\sum_p^+(\beta, \alpha, \lambda; A, B)$ *are the functions* $f_p(z) = z^{-p}$ *and*

$$f_k(z) = z^{-p} + \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)} \frac{(A-B)(p-\lambda)}{k(1-B)} z^k. \tag{5.7}$$

6. Radii of starlikeness and convexity

Theorem 7. *Let the function* $f(z)$ *defined by (1.8) be in the class* $\sum_p^+(\beta, \alpha, \lambda; A, B)$*, then we have:*
(i) $f(z)$ *is meromorphically p-valent starlike of order* δ $(0 \le \delta < p)$ *in the disc* $|z| < r_1$*, that is,*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta \ (|z| < r_1;\ 0 \le \delta < p), \tag{6.1}$$

where

$$r_1 = \inf_{k \ge p}\left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)(p-\delta)}{(k+\delta)(A-B)(p-\lambda)}\right\}^{\frac{1}{k+p}}, \tag{6.2}$$

(ii) $f(z)$ *is meromorphically p-valent convex of order* δ $(0 \le \delta < p)$ *in the disc* $|z| < r_2$*, that is,*

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \delta \ (|z| < r_2;\ 0 \le \delta < p), \tag{6.3}$$

where

$$r_2 = \inf_{k \ge p}\left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{p(1-B)(p-\delta)}{(k+\delta)(A-B)(p-\lambda)}\right\}^{\frac{1}{k+p}}. \tag{6.4}$$

The result is sharp, the extremal function given by (3.3).
Proof. (i) From the definition (1.8), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \le \frac{\sum\limits_{k=p}^{\infty}(k+p)\,|a_k|\,|z|^{k+p}}{2(p-\delta)-\sum\limits_{k=p}^{\infty}(k-p+2\delta)\,|a_k|\,|z|^{k+p}}. \tag{6.5}$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \le 1,$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\delta}{p-\delta}\right)|a_k|\,|z|^{k+p} \le 1. \tag{6.6}$$

But by using Theorem 2, (6.6) will be true if

$$\left(\frac{k+\delta}{p-\delta}\right)|z|^{k+p} \leq \frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}.$$

Then

$$|z| \leq \left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)(p-\delta)}{(k+\delta)(A-B)(p-\lambda)}\right\}^{\frac{1}{k+p}}. \tag{6.7}$$

(ii) In order to prove the second assertion of Theorem 7, we find from the definition (1.8) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq \frac{\sum\limits_{k=p}^{\infty} k(k+p)\,|a_k|\,|z|^{k+p}}{2p(p-\delta)-\sum\limits_{k=p}^{\infty} k(k-p+2\delta)\,|a_k|\,|z|^{k+p}}. \tag{6.8}$$

Thus, we have the desired inequality

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq 1,$$

if

$$\sum_{k=p}^{\infty}\frac{k(k+\delta)}{p(p-\delta)}\,|a_k|\,|z|^{k+p} \leq 1. \tag{6.9}$$

But by using Theorem 2, (6.9) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p} \leq \frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}.$$

Then

$$|z| \leq \left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{p(1-B)(p-\delta)}{(k+\delta)(A-B)(p-\lambda)}\right\}^{\frac{1}{k+p}}. \tag{6.10}$$

The last inequality (6.10) readily yields the disc $|z| < r_2$, where r_2 is defined by (6.4) and the proof of Theorem 7 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (3.3).

7. Hadamard products

Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1). The Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}|\,|a_{k,2}|\, z^k = (f_2 * f_1)(z). \tag{7.1}$$

Theorem 8. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Then $(f_1 * f_2)(z) \in \sum_p^+(\beta, \alpha, \eta; A, B)$, where*

$$\eta = p - \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)} \frac{(A-B)(p-\lambda)^2}{p(1-B)}. \tag{7.2}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)} \frac{(A-B)(p-\lambda)}{p(1-B)} z^p \ (j = 1, 2). \tag{7.3}$$

Proof. Employing the techinque used earlier by Schild and Silverman [11], we need to find the largest η such that

$$\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=p}^{\infty} \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)} \frac{k(1-B)}{(A-B)(p-\eta)} |a_{k,1}|\,|a_{k,2}| \le 1. \tag{7.4}$$

Since $f_j(z) \in \sum_p^+(\beta, \alpha, \lambda; A, B)$ $(j = 1, 2)$, we readily see that

$$\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=p}^{\infty} \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)} \frac{k(1-B)}{(A-B)(p-\lambda)} |a_{k,1}| \le 1, \tag{7.5}$$

and

$$\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=p}^{\infty} \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)} \frac{k(1-B)}{(A-B)(p-\lambda)} |a_{k,2}| \le 1. \tag{7.6}$$

By the Cauchy Schwarz inequality we have

$$\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=p}^{\infty} \frac{\Gamma(k+p+\beta)}{\Gamma(k+p+\beta+\alpha)} \frac{k(1-B)}{(A-B)(p-\lambda)} \sqrt{|a_{k,1}|\,|a_{k,2}|} \le 1. \tag{7.7}$$

This implies that, we need only to prove that

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \le \left(\frac{p-\eta}{p-\lambda}\right). \tag{7.8}$$

Hence, in light of the inequality (7.7), it is sufficient to prove that

$$\frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)}{k(1-B)} \leq \left(\frac{p-\eta}{p-\lambda}\right). \tag{7.9}$$

It follows from (7.9) that

$$\eta \leq p - \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)^2}{k(1-B)}. \tag{7.10}$$

Now defining the function $D(k)$ by

$$D(k) = p - \frac{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)^2}{k(1-B)}. \tag{7.11}$$

We see that $D(k)$ is an increasing function of k $(k \geq p)$. Therefore, we conclude that

$$\eta \leq D(p) = p - \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{(A-B)(p-\lambda)^2}{p(1-B)}, \tag{7.12}$$

which evidently completes the proof of Theorem 8.

Using arguments similar to those in the proof of Theorem 8, we obtain the following theorem.

Theorem 9. *Let the function $f_1(z)$ defined by (5.1) be in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Suppose also that the function $f_2(z)$ defined by (5.1) be in the class $\sum_p^+(\beta, \alpha, \varphi; A, B)$. Then $(f_1 * f_2)(z) \in \sum_p^+(\beta, \alpha, \zeta; A, B)$ where*

$$\zeta = p - \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{(A-B)(p-\lambda)(p-\varphi)}{p(1-B)}. \tag{7.13}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{(A-B)(p-\lambda)}{p(1-B)}z^p\ , \tag{7.14}$$

and

$$f_2(z) = z^{-p} + \frac{\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{(A-B)(p-\varphi)}{p(1-B)}z^p\ . \tag{7.15}$$

Theorem 10. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\sum_p^+(\beta, \alpha, \lambda; A, B)$. Then the function*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}\left(|a_{k,1}|^2 + |a_{k,2}|^2\right)z^k, \tag{7.16}$$

belong to the class $\sum_p^+(\beta,\alpha,\psi;A,B)$, *where*

$$\psi = p - \frac{2\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)}\frac{(A-B)(p-\lambda)^2}{p(1-B)}. \tag{7.17}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ *defined by (7.3).*

Proof. By using Theorem 2, we obtain

$$\sum_{k=p}^{\infty}\left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}\right\}^2 |a_{k,1}|^2 \leq$$

$$\left\{\sum_{k=p}^{\infty}\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}|a_{k,1}|\right\}^2 \leq 1, \tag{7.18}$$

and

$$\sum_{k=p}^{\infty}\left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}\right\}^2 |a_{k,2}|^2 \leq$$

$$\left\{\sum_{k=p}^{\infty}\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}|a_{k,2}|\right\}^2 \leq 1. \tag{7.19}$$

It follow from (7.18) and (7.19) that

$$\sum_{k=p}^{\infty}\frac{1}{2}\left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}\right\}^2\left(|a_{k,1}|^2+|a_{k,2}|^2\right) \leq 1. \tag{7.20}$$

Therefore, we need to find the largest ψ such that

$$\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\psi)} \leq$$

$$\frac{1}{2}\left\{\frac{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}{\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}\frac{k(1-B)}{(A-B)(p-\lambda)}\right\}^2, \tag{7.21}$$

that is

$$\psi \leq p - \frac{2\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)^2}{k(1-B)}, \tag{7.22}$$

since

$$G(k) = p - \frac{2\Gamma(\beta)\Gamma(k+p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(k+p+\beta)}\frac{(A-B)(p-\lambda)^2}{k(1-B)}, \tag{7.23}$$

is an increasing function of k $(k \geq p)$, we obtain

$$\psi \leq G(p) = p - \frac{2\Gamma(\beta)\Gamma(2p+\beta+\alpha)}{\Gamma(\beta+\alpha)\Gamma(2p+\beta)} \frac{(A-B)(p-\lambda)^2}{p(1-B)}, \tag{7.24}$$

and hence the proof of Theorem 10 is completed.

Remarks.
(1) Putting $A = \delta$ and $B = -\delta$ $(0 < \delta \leq 1)$ in our results, we obtain a corresponding results for the class $\sum_p^+(\beta, \alpha, \delta, \lambda)$;
(2) Putting $A = -(2\gamma - 1)\delta$ and $B = -\delta$ $(\frac{1}{2} \leq \gamma \leq 1$ and $0 < \delta \leq 1)$ in our results, we obtain a corresponding results for the class $\sum_p^+(\beta, \alpha, \gamma, \delta, \lambda)$.

References

[1] M. K. Aouf, *On a class of meromorphic multivalent functions with positive coefficients*, Math. Japon., 35(1990), 603-608.

[2] M. K. Aouf, *A generalization of meromorphic multivalent functions with positive coefficients*, Math. Japon., 35(1990), 609-614.

[3] M. K. Aouf, H. M. Hossen and H. E. Elattar, *A certain class of meromorphic multivalent functions with positive and fixed second coefficients*, Punjab Univ. J. Math., 33(2000), 115-124.

[4] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni, *Certain integral operators applied to meromorphic p-valent functions*, J. Nat. Geom., 24(2003), 111-120.

[5] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Sci., 2(1971), no. 3, 469-474.

[6] S. B. Joshi and M. K. Aouf, *Meromorphic multivalent functions with positive and fixed second coefficients*, Kyungpook Math. J., 35(1995), 163-169.

[7] S. B. Joshi and H. M. Srivastava, *A certain family of meromorphically multivalent functions*, Comput. Math. Appl., 38(3-4)(1999), 201-211.

[8] M. L. Mogra, *Meromorphic multivalent functions with positive coefficients I*, Math. Japon., 35(1990), no. 1, 1-11.

[9] M. L. Mogra, *Meromorphic multivalent functions with positive coefficients II*, Math. Japon., 35(1990), no. 6, 1089-1098.

[10] S. Owa, H. E. Darwish and M. K. Aouf, *Meromorphic multivalent functions with positive and fixed second coefficients*, Math. Japon., 46(1997), no. 2, 231-236.

[11] A. Schild and H. Silverman, *Convolution of univalent functions with negative coefficients*, Ann. Univ. Mariae-Curie Sklodowska, Sect. A, 29(1975), 109-116.

[12] H. M. Srivastava, H. M. Hossen and M. K. Aouf, *A unified presentation of some classes of meromorphically multivalent functions*, Comput. Math. Appl., 38(1999), 63-70.

[13] R. K. Raina and H. M. Srivastava, *A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions*, Math. Comput. Modelling, 43(2006), 350-356.

[14] B. A. Uralegaddi and M. D. Ganigi, *Meromorphic multivalent functions with positive coefficients*, Nepali Math. Sci. Rep., 11(1986), no. 2, 95-102.

[15] B. A. Uralegaddi and C. Somanatha, *Certain classes of meromorphic multivalent functions*, Tamkang J. Math., 23(1992), 223-231.

[16] D. -G. Yang, *On new subclasses of meromorphic p-valent functions*, J. Math. Res. Exposition, 15(1995), 7-13.

R. M. El-Ashwah, S.M. El-Deeb
Department of Mathematics, Faculty of Science at Damietta,
Mansoura University, New Damietta 34517, Egypt
email: *r_elashwah@yahoo.com, shezaeldeeb@yahoo.com*

M. K. Aouf, A. Shamandy
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
email: *mkaouf127@yahoo.com, shamandy16@hotmail.com*