NONUNIFORM BEHAVIORS FOR LINEAR DISCRETE-TIME SYSTEMS IN BANACH SPACES

Ioan-Lucian Popa, Mihail Megan and Traian Ceauşu

Abstract. The aim of this paper is to emphasize two concepts of exponential behaviors for linear discrete-time systems in Banach spaces. Characterizations and relations between these concepts are given.

2000 *Mathematics Subject Classification*:39A11, 34D05.

Keywords: linear discrete-time systems, uniform exponential stability, exponential stability.

1. Introduction and preliminaries.

In the theory of difference equations both in finite and infinite dimensional spaces, the concepts of exponential stability play a central role in the study of the asymptotical behaviors of solutions of discrete-time systems.

The purpose of this paper is to characterize the nonuniform exponential stability of linear discrete-time systems in Banach spaces. The concept of exponential stability is a direct generalization of the concept of uniform exponential stability.

Our main objectives are to establish relations between these concepts and to give necessary and sufficient conditions for the nonuniform exponential stability and respectively uniform exponential stability. The obtained results are generalizations of some well known results due to K.M. Przyluski & S. Rolewicz ([11]), E.A. Barbashin ([2]) and A. Lyapunov ([8]).

Let X be a real or complex Banach space and X^* its dual space. By $\mathcal{B}(X)$ will be denoted the Banach algebra of all linear and bounded operators from X into itself. The norms on X, X^* and $\mathcal{B}(X)$ shall be denoted by $\| \, . \, \|$. Let Δ be the set of all pairs (m,n) of positive integers satisfying the inequality $m \geq n$. We also denote by T the set of all triplets (m,n,p) of positive integers with (m,n) and $(n,p) \in \Delta$.

In this paper we consider linear discrete-time systems of the form

$$x_{n+1} = A(n)x_n. \tag{$\mathfrak{A}$}$$

For system $(\mathfrak{A})$ we consider the mapping $A : \Delta \to \mathcal{B}(X)$ defined by

$$\mathcal{A}_m^n = \begin{cases} A(m) \cdot \ldots \cdot A(n+1), & m \geq n+1 \\ \quad I \qquad\qquad\qquad , & m = n. \end{cases} \tag{1}$$

where I is the identity operator on X.

Definition 1. The linear discrete-time system $(\mathfrak{A})$ is said to be *uniformly exponentially stable* (and denote u.e.s) if there exist two constants $N \geq 1$ and $\alpha > 0$ such that:

$$\| A(m) \cdot \ldots \cdot A(n+1)x \| \leq Ne^{-\alpha(m-n)} \| x \| \tag{2}$$

for all $(m, n, x) \in \Delta \times X$.

Remark 1. For every linear discrete-time system $(\mathfrak{A})$ the following statements are equivalent:

i) $(\mathfrak{A})$ is uniformly exponentially stable;

ii) there exist two constants $N \geq 1$ and $\alpha > 0$ such that

$$\| \mathcal{A}_m^p x \| \leq Ne^{-\alpha(m-n)} \| \mathcal{A}_n^p x \|$$

for all $(m, n, p, x) \in T \times X$.

iii) there exist two constants $N \geq 1$ and $\alpha > 0$ such that

$$\| \mathcal{A}_m^n x \| \leq Ne^{-\alpha(m-n)} \| x \|$$

for all $(m, n, x) \in \Delta \times X$.

Example 1. Let $X = \mathbb{R}$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R})$ defined by

$$A(n)(x) = \frac{x}{a_n}$$

where $a_n = e^{n+\frac{1}{2}}$, for all $(n, x) \in \mathbb{N} \times \mathbb{R}$. According to (1) we have that

$$\mathcal{A}_m^n x = \begin{cases} e^{\frac{(n+1)^2-(m+1)^2}{2}} x & m > n \\ x & m = n \end{cases}.$$

Hence, for $N = 1$ and $\alpha = \frac{1}{2}$ we have that system $(\mathfrak{A})$ is u.e.s.

Definition 2. The linear discrete-time system $(\mathfrak{A})$ is said to be *nonuniformly exponentially stable* (and denote n.e.s) if there exists a nondecreasing sequence of real numbers $\phi : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ such that:

$$\| A(m) \cdot \ldots \cdot A(n+1)x \| \leq \phi(n) e^{-\alpha(m-n)} \| x \| \tag{3}$$

for all $(m, n, x) \in \Delta \times X$.

Remark 2. For every linear discrete-time system $(\mathfrak{A})$ the following statements are equivalent:

i) $(\mathfrak{A})$ is nonuniformly exponentially stable;

ii) there exists a nondecreasing sequence of real numbers $\phi : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ such that

$$\| \mathcal{A}_m^p x \| \leq \phi(n) e^{-\alpha(m-n)} \| \mathcal{A}_n^p x \|$$

for all $(m, n, p, x) \in T \times X$.

iii) there exists a nondecreasing sequence of real numbers $\phi : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ such that

$$\| \mathcal{A}_m^n x \| \leq \phi(n) e^{-\alpha(m-n)} \| x \|$$

for all $(m, n, x) \in \Delta \times X$.

Definition 3. A mapping $L : \mathbb{N} \times X \longrightarrow \mathbb{R}_+$ is called a *Lyapunov function* for the system $(\mathfrak{A})$ if

i) there is a sequence $\varphi : \mathbb{N} \longrightarrow [1, \infty)$ such that

$$\| x \| \leq L(n, x) \leq \varphi(n) \| x \|,$$

and

ii) there is a constant $a \in (1, \infty)$ such that

$$L\left(m, \mathcal{A}_m^n x\right) - aL\left(m+1, \mathcal{A}_{m+1}^n x\right) \geq \| \mathcal{A}_m^n x \| \tag{$\mathcal{L}$}$$

for all $(m, n, x) \in \Delta \times X$.

2. Main results

It is obvious that if system $(\mathfrak{A})$ it is uniformly exponentially stable then it is nonuniformly exponentially stable. The following example shows that the converse implication is not valid.

Example 2. Let $X = \mathbb{R}$ and $A : \mathbb{N} \longrightarrow \mathcal{B}(\mathbb{R})$ given by:

$$A(n) = ca_n I, \quad \text{where} \quad a_n = \begin{cases} \frac{1}{(n+2)^b} & \text{if } n = 2k \\ (n+1)^b & \text{if } n = 2k+1 \end{cases} \tag{4}$$

with $b \in (0, 1)$ and $c > 1$, for all $(n, x) \in \mathbb{N} \times X$. According to (1) we have that:

$$\mathcal{A}_m^n x = \begin{cases} c^{m-n} a_{mn} x & m > n \\ x & m = n \end{cases},$$

where

$$a_{mn} = \begin{cases} 1 & \text{if } m = 2q+1 \text{ and } n = 2p+1 \\ (n+2)^b & \text{if } m = 2q+1 \text{ and } n = 2p \\ \dfrac{1}{(m+2)^b} & \text{if } m = 2q \text{ and } n = 2p+1 \\ \left(\dfrac{n+2}{m+2}\right)^b & \text{if } m = 2q \text{ and } n = 2p \end{cases} \tag{5}$$

We observe that previous inequality it is satisfied for $\phi(n) = (n+1)^c$ and $\alpha = -\ln b$. Now, using Remark 2 we can conclude that system $(\mathfrak{A})$ is n.e.s. If we suppose that system $(\mathfrak{A})$ is u.e.s. then there are two constants $N \geq 1$ and $\alpha > 0$ such that

$$b^{m-n} a_{mn} \leq N e^{-\alpha(m-n)},$$

for all $(m, n) \in \Delta$. In particular, for $m = 2q + 1$ and $n = 2q$ we have that

$$b e^{\alpha} (2q+2)^c \leq N,$$

for all $q \in \mathbb{N}$, which is a contradiction. Hence, system $(\mathfrak{A})$ is not u.e.s.

Remark 3. If system $(\mathfrak{A})$ is nonuniformly exponentially stable then using Remark 2 there are two sequences of real numbers $\theta, \gamma : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ with $\lim\limits_{n\to\infty} \theta(n) = \infty$ such that

$$\theta(m-n) \parallel \mathcal{A}_m^n x \parallel \leq \gamma(n) \parallel x \parallel, \tag{6}$$

for all $(m, n, x) \in \Delta \times X$. For the case of uniform exponential stability the converse implication is true, but for the nonuniform exponential stability it is not, fact illustrated by the following example.

Example 3. Let $X = \mathbb{R}$ and $A : \mathbb{N} \longrightarrow \mathcal{B}(\mathbb{R})$ defined by:

$$A(n) = \frac{n}{n+1} I,$$

for all $n \in \mathbb{N}$. According to (5) we have that $\mathcal{A}_m^n = \frac{n+1}{m+1}$, for all $m > n$. If we suppose that system $(\mathfrak{A})$ is n.e.s. then there exists a constant $\alpha > 0$ and a nondecreasing sequence of real numbers $\phi : \mathbb{N} \longrightarrow \mathbb{R}_+^*$ such that

$$\| \mathcal{A}_m^n \| \leq \phi(n) e^{-\alpha(m-n)},$$

for all $(m, n) \in \Delta$. In particular for $n = 0$ we have that

$$\frac{1}{m+1} \leq \phi(0) e^{-\alpha m},$$

which is false. But, for $\theta(n) = \gamma(n) = n + 1$ inequality (6) is verified and that completes the proof.

Theorem 1. *The linear discrete-time system $(\mathfrak{A})$ is nonuniformly exponentially stable if and only if there exists a constant $d > 0$ and a nondecreasing sequences of real numbers $N : \mathbb{N} \to [1, \infty)$ such that:*

$$\sum_{k=n}^{\infty} e^{d(k-n)} \| \mathcal{A}_k^p x \| \leq N(n) \| \mathcal{A}_n^p x \|, \tag{7}$$

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. By a simple computation for $d \in (0, \alpha)$ we have that

$$\begin{aligned} \sum_{k=n}^{\infty} e^{d(k-n)} \| \mathcal{A}_k^p x \| &\leq \phi(n) \| \mathcal{A}_n^p x \| \sum_{k=n}^{\infty} e^{(d-\alpha)(k-n)} \\ &= \tfrac{e^{\alpha} \phi(n)}{e^{\alpha} - e^{d}} \| \mathcal{A}_n^p x \| . \end{aligned}$$

Hence, for $N(n) = \frac{e^{\alpha}\phi(n)}{e^{\alpha} - e^{d}}$ we obtain relation (7), with α and sequence $\phi(n)$ given by Definition 2.

Sufficiency. The inequality (7) implies that:

$$e^{d(m-n)} \| \mathcal{A}_m^n x \| \leq N(n) \| x \|,$$

for all $(m, n, x) \in \Delta \times X$. Using Remark 2 it results that $(\mathfrak{A})$ is n.e.s. $\square$

Theorem 2. *If there is a constant $b > 0$ and a nondecreasing sequence of real numbers $N : \mathbb{N} \to [1, \infty)$ such that:*

$$\sum_{k=n}^{m} e^{b(m-k)} \parallel (\mathcal{A}_m^k)^* x^* \parallel \leq N(n) \parallel x^* \parallel, \tag{8}$$

for all $(m, n, x^*) \in \Delta \times X^*$, then the linear discrete-time system $(\mathfrak{A})$ is nonuniformly exponentially stable.

Proof. By (8) we have that:

$$\parallel \mathcal{A}_m^n \parallel \leq N(n) e^{-b(m-n)},$$

for all $(m, n) \in \Delta$, which implies that $(\mathfrak{A})$ is n.e.s. $\square$

Remark 4. The characterizations given by previous Theorem can be considered as a variants for the discrete-time case of a result due to E.A. Barbasin ([2]) in the continuous case.

Theorem 3. *The linear discrete-time system $(\mathfrak{A})$ is nonuniformly exponentially stable if and only if there exists a Lyapunov function for $(\mathfrak{A})$.*

Proof. Necessity. We define $L : \Delta \times \mathbb{N} \longrightarrow \mathbb{R}_+$ by

$$L(n, x) = \sum_{k=n}^{\infty} e^{d(k-n)} \parallel \mathcal{A}_k^n x \parallel,$$

for all $d > 0$ and all $(m, n, x) \in \Delta \times X$.

For $d \in (0, \alpha)$ we have that

$$\begin{aligned} L(n, x) &= \sum_{k=n}^{\infty} e^{d(k-n)} \parallel \mathcal{A}_k^n x \parallel \\ &\leq \sum_{k=n}^{\infty} e^{d(k-n)} e^{-\alpha(k-n)} N(n) \parallel x \parallel \\ &= \frac{N(n) e^{\alpha}}{e^{\alpha} - e^{d}} = \varphi(n) \parallel x \parallel . \end{aligned}$$

Hence,

$$\parallel x \parallel \leq L(n, x) \leq \varphi(n) \parallel x \parallel .$$

for all $(n,x) \in \mathbb{N} \times X$.

On the other hand we have that

$$\begin{aligned} L(m, \mathcal{A}_m^n x) &= \sum_{k=m}^{\infty} e^{d(k-m)} \parallel \mathcal{A}_k^n x \parallel \\ &= a_m^n + q a_{m+1}^n + q^2 a_{m+2}^n + \dots \end{aligned} \tag{9}$$

where $a_m^n = \parallel \mathcal{A}_m^n x \parallel$ and $q = e^d > 1$.

Also,

$$\begin{aligned} L(m+1, \mathcal{A}_{m+1}^n x) &= \sum_{k=m+1}^{\infty} e^{d(k-m-1)} \parallel \mathcal{A}_k^n x \parallel \\ &= a_{m+1}^n + q a_{m+2}^n + q^2 a_{m+3}^n + \dots \end{aligned} \tag{10}$$

According to (9) and (10) we have that

$$L(m, \mathcal{A}_m^n x) = a_m^n + qL(m+1, \mathcal{A}_{m+1}^n x).$$

Hence

$$L(m, \mathcal{A}_m^n x) - aL(m+1, \mathcal{A}_{m+1}^n x) \geq \parallel \mathcal{A}_m^n x \parallel,$$

for every $a \in (1, e^d)$ and $(m,n,x) \in \Delta \times X$.

Sufficiency. According to $(\mathcal{L})$ we have that

$$\begin{aligned} L(m, \mathcal{A}_m^n x) &- aL(m+1, \mathcal{A}_{m+1}^n x) \geq \parallel \mathcal{A}_m^n x \parallel \\ L(m+1, \mathcal{A}_{m+1}^n x) &- aL(m+2, \mathcal{A}_{m+2}^n x) \geq \parallel \mathcal{A}_{m+1}^n x \parallel \\ L(m+2, \mathcal{A}_{m+2}^n x) &- aL(m+3, \mathcal{A}_{m+3}^n x) \geq \parallel \mathcal{A}_{m+2}^n x \parallel \\ \dots \quad \dots & \quad \dots \quad \dots \\ & \vdots \end{aligned}$$

which implies

$$\sum_{j=m}^{\infty} a^{j-m} \parallel A_j^m x \parallel = \sum_{k=0}^{\infty} a^k \parallel A_{m+k}^n x \parallel \leq L(m, \mathcal{A}_m^n x) \leq \varphi(n) \parallel x \parallel$$

for all $(m,n,x) \in \Delta \times X$.

Therefore for $\alpha = \ln a > 0$ we obtain

$$\sum_{k=n}^{\infty} e^{\alpha(k-n)} \parallel \mathcal{A}_k^n x \parallel \leq L(n,x) \leq \varphi(n) \parallel x \parallel .$$

for every $(n,x) \in \mathbb{N} \times X$. By Theorem 1 we conclude that the system $(\mathfrak{A})$ is n.e.s. $\square$

Acknoledgement: This work is financially supported from the Exploratory Research Grant PN II ID 1080 No. 508/2009 of the Romanian Ministry of Education, Research and Innovation.

References

[1] R.P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992

[2] E.A. Barbasin, Introduction in Stability Theory, Izd. Nauka, Moskva, 1967.

[3] L. Barreira, C. Valls, Stability of Nonautonomous Differential Equations, Lecture Notes in Mathematics 1926, Springer 2008

[4] L. Barreira, C. Valls, Stability in delay difference equations with nonuniform exponential behavior, J. Diff. Equations 238 (2007), 470-490.

[5] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal. 3 (1972), 428-445.

[6] M.I. Gil, Difference equation in Normed space, Stability and Oscilation, Nort-Holland Mathematics-Studies, 2007.

[7] W. Hahn, Stability of Motion, Grundlehren Math. Wiss., vol. 138, Springer, 1967.

[8] A. Lyapunov The General Problem of the Stability of Motion, Taylor & Francis, 1992.

[9] I.-L. Popa, T. Ceauşu, M. Megan, On exponential stability for linear discrete-time systems in Banach spaces, to appear

[10] A. Pogan, C. Preda, P. Preda, A discrete Lyapunov theorem for the exponential stability of evolution families, New York J. Math. 11 (2005) 457463.

[11] K.M. Przyluski, S. Rolewicz, On stability of linear time-varying infinit-dimensional discrete-time systems, Syst. Control Lett. 4(1984), 307-315

Ioan-Lucian Popa
Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara,
V. Pârvan Blv. No. 4, 300223-Timişoara, Romania
email:*popa@math.uvt.ro*

Mihail Megan
Academy of Romanian Scientists,
Independenţei 54, 050094 Bucharest, Romania
email:*megan@math.uvt.ro*

Traian Ceauşu
Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara,
V. Pârvan Blv. No. 4, 300223-Timişoara, Romania
email:*ceausu@math.uvt.ro*