# APPROXIMATION PROPERTIES OF A STANCU S-VARIATE OPERATOR OF BETA TYPE

**by**
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**Abstract**: In this paper the main result consist in introducing and investigating the approximation properties of a new beta operator of a second kind $S_{n_1,n_2,\dots,n_s} = T_{n_1x_1,\dots,n_sx_s}^{n_1+1,\dots,n_s+1}$ which is an integral linear positive operator reproducing the linear functions. Then we have estimations of the orders of approximation, by using the modulus of continuity of first and second orders.

## 1. INTRODUCTION [1]

In his recent paper [18], Professor D. D. Stancu has introduced a new beta operator of second kind $S_n$ defined by the following formula:

(1) $$(S_n f)(x) = \frac{1}{B(nx,n+1)}\int_0^\infty f(t)\frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt, (x \in (0,\infty)),$$

where $f \in M[0,\infty)$ and n is a natural number $\geq 2$.

Here $M[0,\infty)$ denotes the space of bounded and locally integral functions on $[0,\infty)$. Actually, $S_n$ is applicable to functions of polynomial growth if we restrict ourselves to $n \geq n_0$ with $n_0 \in N$ sufficiently great.

The operator (1) is constructed, roughly speaking, by applying a beta second kind transform

$$T_{p,q}g = \frac{1}{B(p,q)}\int_0^\infty g(t)\frac{t^{p-1}}{(1+t)^{p+q}}dt, (g \in M[0,\infty)),$$

to the Baskakov operator

(2) $$B_n(f,x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right) \qquad (x \in (0,\infty))$$

and then choosing p=nx and q=n+1.

$S_n$ is a positive linear operator reproducing the linear functions. It is mentioned in [18] that $S_n$ is distinct from the other beta-type operators considered earlier in the papers [10], [19], [8] and [2].

Furthermore, Professor D. D. Stancu has given estimations for the order of approximation by using the moduli of continuity of first and second orders. He has also established an asymptotic formula of Voronovskaja-type

(3) $$\lim_{n\to\infty} n((S_n f)(x) - f(x)) = \frac{x(1+x)}{2} f''(x)$$

for all f $\in$ M[0,$\infty$) admitting a bounded derivative of second order at x (x >0).

## 2. THE BETA SECOND-KIND S-VARIATE TRANSFORM $T_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s}$

Let us denote by M([0,∞)×[0,∞)×…×[0,∞)) the linear space of functions g($t_1$, $t_2$, …, $t_s$), defined for $t_i \geq 0$, i ≥ 0, bounded and Lebesque measurable in $I_1 \times I_2 \times \ldots \times I_s$, where $I_s$ = [0,∞).

We shall define a linear transform by using the beta s-variate distribution of second kind, with the positive parameters $p_1$, $p_2$,…, $p_s$ and $q_1$, $q_2$,…, $q_s$, which has the probability density

(4) $$b_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s}(t_1,t_2,\ldots,t_s) = \prod_{i=1}^{s} \frac{t_i^{p_i-1}}{B(p_i,q_i)(1+t_i)^{p_i+q_i}},$$

where $t_i > 0$, i = 1,2,…, s and $b_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s}$ ($t_1$,$t_2$,…,$t_s$) =0 otherwise, by B(p,q), i=1,2,…,s is denoted the beta function. By using distribution (4) we can define the beta s-variate second-kind transform $T_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s}$, of a function g ∈ M([0,∞)×[0,∞)×…×[0,∞)), by

(5) $T_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s}$ g =

$$= \int_0^\infty \int_0^\infty \ldots \int_0^\infty g(t_1,t_t,\ldots,t_s) b_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s}((t_1,t_t,\ldots,t_s) dt_1 dt_2 \ldots dt_s =$$

$$= \prod_{i=1}^{s} \frac{1}{B(p,q)} \int_0^\infty \int_0^\infty \ldots \int_0^\infty g(t_1,t_2,\ldots,t_s) \prod_{i=1}^{s} \frac{t_i^{p_i-1}}{(1+t_i)^{p_i+q_i}} dt_1 dt_2 \ldots dt_s$$

such that $T_{p_1,p_2,\ldots,p_s}^{q_1,q_2,\ldots,q_s} |g| < \infty$.

One observes that $T^{q_1,q_2,...,q_s}_{p_1,p_2,...,p_s}$ is a linear positive functional.

We need to state and prove:

THEOREM 1. The moment of order ($r_1$, $r_2$,…,$r_s$) ($r_1$, $r_2$,…,$r_s$ ≥ 1) of the functional $T^{q_1,q_2,...,q_s}_{p_1,p_2,...,p_s}$ has the following value

(6) $$v_{r_1,r_2,...,r_s}(p_1,q_1;p_2,q_2;...;p_s,q_s) = T^{q_1,q_2,...,q_s}_{p_1,p_2,...,p_s} e_{r_1,r_2,...,r_s} =$$
$$= \prod_{i=1}^{s} \frac{p_i(p_i+1)...(p_i+r_i-1)}{(q_i-1)(q_i-2)...(q_i-r_i)}$$

*Proof:*

We have

$$T^{q_1,q_2,...,q_s}_{p_1,p_2,...,p_s} e_{r_1,r_2,...,r_s} = \prod_{i=1}^{s} \frac{1}{B(p_i,q_i)} \int_0^\infty \int_0^\infty ... \int_0^\infty \prod_{i=1}^{s} \frac{t_i^{p_i+r_i-1}}{(1+t_i)^{p_i+q_i}} dt_1...dt_s =$$
$$= \prod_{i=1}^{s} \frac{1}{B(p_i,q_i)} \int_0^\infty \frac{t_i^{p_i+r_i-1}}{(1+t_i)^{p_i+q_i}} dt_i \ .$$

If we make the change of integration variable $y_i = t_i / (1+t_i)$, we have $t_i = y_i / (1-y_i)$, $dt_i = (1-y_i)^{-2} dy_i$ (i = 1, 2, …., s) and we obtain

$$T^{q_1,q_2,...,q_s}_{p_1,p_2,...,p_s} e_{r_1,r_2,...,r_s} = \prod_{i=1}^{s} \frac{B(p_i+r_i,q_i-r_i)}{B(p_i,q_i)} \ .$$

By using the relation

$$B(p+r,q-r) = \frac{p(p+1)...(p+r-1)}{(q-1)(q-2)...(q-r)} B(p,q)$$

we obtain the desired result (6).

## 3. THE FUNCTIONAL $F^{f}_{n_1,n_2,...,n_s}(p_1,q_1;p_2,q_2;...;p_s,q_s) = T^{q_1,q_2,...,q_s}_{p_1,p_2,...,p_s}(B_{n_1,n_2,...,n_s} f)$

Now let us apply the transform (5) to the s-variant Baskakov operator $B_{n_1,n_2,...,n_s}$, defined by

$$(B_{n_1,n_2,...,n_s} f)(t_1,t_2,...,t_s) =$$

(7) $$= \sum_{k_1,k_2,...,k_s=0}^{\infty} \prod_{i=1}^{s} \binom{n_i+k_i-1}{k_i} \frac{t_i^{k_i}}{(1+t_i)^{k_i}} f\left(\frac{k_1}{n_1},...,\frac{k_s}{n_s}\right)$$

We may state and prove

THEOREM 2. The $T_{p_1,p_2,...,p_s}^{q_1,q_2,...,q_s}$ transform of $B_{n_1,n_2,...,n_s} f$ can be expressed under the following form

(8) $$F_{n_1,n_2,...,n_s}^{f}(p_1,q_1;p_2,q_2;...;p_s,q_s) = T_{p_1,p_2,...,p_s}^{q_1,q_2,...,q_s}(B_{n_1,n_2,...,n_s} f) =$$
$$\sum_{k_1,k_2,...,k_s=0}^{\infty} \prod_{i=1}^{s} \binom{n_i+k_i-1}{k_i} \frac{p_i(p_i+1)...(p_i+q_i-1)q_i(q_i+1)...(q_i+n_i-1)}{(p_i+q_i)(p_i+q_i-1)...(p_i+q_i+n_i+k_i-1)} f\left(\frac{k_1}{n_1},...,\frac{k_s}{n_s}\right)$$

*Proof:*

We can write successively

$$T_{p_1,p_2,...,p_s}^{q_1,q_2,...,q_s}(B_{n_1,n_2,...,n_s} f) = \sum_{k_1,k_2,...,k_s=0}^{\infty} \prod_{i=1}^{s} \binom{n_i+k_i-1}{k_i} \frac{1}{B(p_i,q_i)} \cdot$$
$$\cdot \left[\int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} \prod_{i=1}^{s} \frac{t_i^{p_i+k_i-1}}{(1+t_i)^{n_i+p_i+q_i+k_i}} dt_1 dt_2 ... dt_s\right] f\left(\frac{k_1}{n_1},\frac{k_2}{n_2},...,\frac{k_s}{n_s}\right)$$

.

If we make the change of variable $y_i = t_i / (1+t_i)$, i = 1,2,…,s in the integral

$$I_{n_i,k_i}(p_i,q_i) = \int_0^{\infty} \frac{t_i^{p_i+k_i-1} dt_i}{(1+t_i)^{p_i+q_i+n_i+k_i}} ,$$

we get

$$I_{n_i,k_i}(p_i,q_i) = \int_0^1 y_i^{k_i+p_i-1}(1-y_i)^{n_i+q_i-1} dy_i = B(k_i+p_i, n_i+q_i) =$$
$$= \frac{p_i(p_i+1)...(p_i+q_i-1)q_i(q_i+1)...(q_i+n_i-1)}{(p_i+q_i)(p_i+q_i-1)...(p_i+q_i+n_i+k_i-1)} B(p_i,q_i)$$

and we obtain the formula (8).

Now we can make the remark that if we select $p_i = x_i / \alpha_i$, $q_i = 1/ \alpha_i$, i = 1,2,…,s where $\alpha_i$ is a positive parameter, then formula (8) leads us to the parameterdependent operator $L_{n_1,n_2,...,n_s}^{\langle \alpha_1,\alpha_2,...,\alpha_s \rangle}$, as a generalization of the Baskakov operator. By using the factorial powers, with the step $h_i = -\alpha_i$, it can be expressed under the following compact form

$$\left(L_{n_1,n_2,...,n_s}^{\langle \alpha_1,\alpha_2,...,\alpha_s \rangle} f\right)(x_1, x_2,..., x_s) = \sum_{k_1,k_2,...,k_s=0}^{\infty} \prod_{i=1}^{s} \binom{n_i + k_i - 1}{k_i} \cdot$$
$$\cdot \frac{x_i^{[k_i,-\alpha_i]} \cdot 1^{[n_i,-\alpha_i]}}{(1+x_i)^{[n_i+k_i,-\alpha_i]}} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2},..., \frac{k_s}{n_s}\right)$$

## 4. THE BETA SECOND-KIND LINEAR POSITIVE OPERATOR $T_{n_1x_1,...,n_sx_s}^{n_1+1,...,n_s+1}$

Now we introduce a new beta second-kind s-variate approximating operator. If in (5) we choose $p_i = n_ix_i$ and $q_i = n_i + 1$, then to any function $f \in M([0,\infty)\times[0,\infty)\times\ldots\times[0,\infty))$ we associate the linear positive operator $S_{n_1,n_2,...,n_s}$, defined by

(10)

$$\left(S_{n_1,n_2,...,n_s} f\right)(x_1, x_2,..., x_s) = T_{n_1x_1,...,n_sx_s}^{n_1+1,...,n_s+1} f =$$
$$= \prod_{i=1}^{s} \frac{1}{B(n_u x_i, n_i + 1)} \int_0^\infty \int_0^\infty ... \int_0^\infty \prod_{i=1}^{s} \frac{t_i^{n_ix_i-1}}{(1+t_i)^{n_ix_i+n_i+1}} f(t_1, t_2,..., t_s) dt_1 dt_2 ... dt_s$$

This represents an extension to s-variables of the operator (1) of D. D. Stancu. Because

$$\left(S_{n_1,n_2,...,n_s} e_{0,0,...,0}\right)(x, y) = \int_0^\infty \int_0^\infty ... \int_0^\infty b_{n_1x_1,n_2x_2,...,n_sx_s}^{n_1+1,n_2+1,...,n_s+1}(t_1, t_2,..., t_s) dt_1 dt_2 ... dt_s = 1$$

and according to (6) we have
$\left(S_{n_1,n_2,...,n_s} e_{1,0,...,0}\right)(x_1, x_2,..., x_s) = x_1$, $\left(S_{n_1,n_2,...,n_s} e_{0,1,...,0}\right)(x_1, x_2,..., x_s) = x_2$ ,…,
$\left(S_{n_1,n_2,...,n_s} e_{0,0,...,1}\right)(x_1, x_2,..., x_s) = x_s$ and $\left(S_{n_1,n_2,...,n_s} e_{1,1,...,1}\right)(x_1, x_2,..., x_s) = x_1 x_2 ... x_s$.

For evaluating the corresponding orders of approximation, it is convenient to make use of the modulus of continuity of f, defined by

$$\varpi(f;\delta_1,\delta_2,...,\delta_s) = \sup\left\{\left|f(x_1,x_2,...,x_s) - f(x_1^{'},x_2^{'},...,x_s^{'})\right|\right\},$$
$$\left|x_i - x_i^{'}\right| \le \delta_i$$

i=1,2,…,s where $(x_1,x_2,\dots,x_s)$ and $(x_1^{'},x_2^{'},...,x_s^{'})$ are the points of $(0,\infty)\times(0,\infty)\times\ldots\times(0,\infty)$ and $\delta_i \in R_+$, i = 1,2,…,s.

By using a basic property of the modulus of continuity we can write

$$\left|f(x_1,x_2,...,x_s) - (S_{n_1,n_2,...,n_s} f)(x_1,x_2,...,x_s)\right| \le$$
$$\le \left[1 + \sum_{i=1}^{s}\frac{1}{\delta_i^2} S_{n_i}(x_i - t_i)^2\right]\varpi(f;\delta_1,...,\delta_s)$$

Because

$$\delta_{n_i}^2(x) = S_{n_i}(t_i - x_i)^2 = \frac{x_i(x_i+1)}{n_i - 1}, \text{ i = 1,2,…,s}$$

if we take $\delta_i = \dfrac{1}{\sqrt{n_i - 1}}$ we can state

THEOREM 3. If f $\in$ C($[0,\infty)\times[0,\infty)\times\ldots\times[0,\infty)$), such that $S_{n_1,n_2,...,n_s}(|f|;x_1,x_2,...,x_s)\langle\infty$, then for any $n_i >$ , i = 1,2,…,s we have the following inequality

$$\left|f(x_1,x_2,...,x_s) - (S_{n_1,n_2,...,n_s} f)(x_1,x_2,...,x_s)\right| \le$$
$$\le \left(1 + \sum_{i=1}^{s}\sqrt{x_i(x_i+1)}\right)\varpi\left(f;\frac{1}{\sqrt{n_1-1}},\frac{1}{\sqrt{n_2-1}},...,\frac{1}{\sqrt{n_s-1}}\right)$$

A similar inequality can be obtained if we assume that the function f has a bounded continuous derivative for $x_i \ge 0$, i = 1,2,…,s.

According to the Bohman-Korovkin convergence criterion, we can deduce

COROLLARY 1. If f $\in$ M($[0,\infty)\times[0,\infty)\times\ldots\times[0,\infty)$) and is continuous at all points of $I_1\times I_2\times\ldots\times I_s$ ($I_i \subset [0,\infty)$, i = 1,2,…,s), then $S_{n_1,n_2,...,n_s} f$ converges uniformly on $I_1\times I_2\times\ldots\times I_s$ to the function f when $n_i\to\infty$, i = 1,2,…,s.

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