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INTEGRAL OPERATORS ON THE UCD(α)-CLASS

by
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Abstract. We consider the class of functions $f = z + a_2 z^2 + a_3 z^3 + \ldots$, that are analytically and univalent in the unit disk.
We consider the class of functions *UCD(*α*)*, $\alpha \geq 0$ with the properties $\operatorname{Re} f'(z) \geq \alpha |zf''(z)|$ $(\forall) z \in U$.

In this paper we present some results from integrals operators at this class.

1.Introduction

Theorem A. [4] A sufficient condition for a function f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \qquad (1)$$

so that it belongs to the $UCD(\alpha), \alpha \geq 0$ class is

$$\sum_{k=2}^{\infty} k[1 + \alpha(k-1)] \cdot |a_k| \leq 1. \qquad (2)$$

2.Main results

Theorem 1. Let be $f \in S$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, $z \in U$. We suppose that the coefficients of function f verify the condition

$$\sum_{k=2}^{\infty} k[1 + \alpha(k-1)] \cdot |a_k| \leq 1.$$

We consider the integral operator

$$F(z) = \int_0^z \frac{f(t)}{t} dt,$$

which is the Alexander operator.
In this conditions, we have $F \in UCD(\alpha)$, $\alpha \geq 0, z \in U$.

Proof

Let be f of the form (1), verifying the condition (2).

We have:

$$F(z)=\int_0^z \frac{t+a_2t^2+a_3t^3+\dots}{t}dt=\int_0^z\left(1+a_2t+a_3t^2+\dots\right)dt=\left(t+\frac{a_2t^2}{2}+\frac{a_3t^3}{3}+\dots\right)\Bigg|_0^z=$$

$$=z+\frac{a_2}{2}z^2+\frac{a_3}{3}z^3\dots.$$

Denoting

$$b_k=\frac{a_k}{k},\ k\geq 2,$$

we can write, $F(z)=z+b_2z^2+b_3z^3+\dots$.

In that following, we evaluate the relation (2), for the function F, with the coefficients b_k.

$$\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot|b_k|=\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot\left|\frac{a_k}{k}\right|=\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot|a_k|\cdot\frac{1}{k}\leq$$

$$\sum_{k=2}^{\infty}[1+\alpha(k-1)]\cdot|a_k|\leq\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot|a_k|<1.$$

According to the relation (2), from the Theorem A, we obtain:

$F\in UCD(\alpha),\ \alpha>0, z\in U$.

Theorem 2. Let be $f\in A,\ f(z)=z+a_2z^2+a_3z^3+\dots,\alpha\geq 0,\ z\in U$. We suppose that the condition (2) is satisfied and we consider the Libera operator,

$$L(f)(z)=\frac{2}{z}\int_0^z f(t)dt.$$

Then $L\in UCD(\alpha),\ \alpha\geq 0,\ z\in U$.

Proof

Let be f of the form (1) verifying the condition (2).

We have:

$$F(z)=\frac{2}{z}\int_0^z\left(t+a_2t^2+a_3t^3+\ldots\right)dt=\frac{2}{z}\left(\frac{t^2}{2}+a_2\frac{t^3}{3}+a_3\frac{t^4}{4}+\ldots\right)\Bigg|_0^z=$$
$$=\frac{2}{z}\left(\frac{z^2}{2}+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\ldots\right)=$$
$$=z+\frac{2a_2}{3}z^2+\frac{2a_3}{4}z^3+\ldots=z+\sum_{k=2}^{\infty}\frac{2a_k}{k+1}z^k .$$

Denoting

$$b_k=\frac{2a_k}{k+1},$$

we can write $L(f)(z)=z+b_2z^2+b_3z^3+\ldots$.
We evaluate the relation (2), for the function obtained after integration with the coefficients b_k..
Since $k\geq 2$, we have

$$\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot|b_k|=\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot\left|\frac{2a_k}{k+1}\right|=\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot|a_k|\cdot\frac{2}{k+1}\leq$$
$$\leq\sum_{k=2}^{\infty}k[1+\alpha(k-1)]\cdot|a_k|\cdot\frac{2}{3}\leq\frac{2}{3}<1 .$$

that implies

$$L(f)\in UCD(\alpha),\ \alpha\geq 0,$$

so, the Libera operator preserves this class.

Theorem 3. Let be $f\in A$, $f(z)=z+a_2z^2+a_3z^3+\ldots$, $z\in U$. We suppose that the condition (2) is satisfied and we consider the Bernardi operator,

$$F(z)=\frac{1+\gamma}{z^\gamma}\int_0^z f(t)\cdot t^{\gamma-1}dt,\ \gamma\geq -1 .$$

Then, we have $F\in UCD(\alpha)$, $\alpha\geq 0$, $z\in U$.

Proof
Let be f of the form (1), verifying the condition (2).
We have

$$F(z)=\frac{1+\gamma}{z^\gamma}\int_0^z\left(t+a_2t^2+a_3t^3+\ldots\right)\cdot t^{\gamma-1}dt=\frac{1+\gamma}{z^\gamma}\int_0^z\left(t^\gamma+a_2t^{\gamma+1}+a_3t^{\gamma+2}+\ldots\right)dt=$$

$$= \frac{1+\gamma}{z^{\gamma}}\left(\frac{t^{\gamma+1}}{\gamma+1} + a_2\frac{t^{\gamma+2}}{\gamma+2} + \ldots\right)\Bigg|_0^z = \frac{1+\gamma}{z^{\gamma}}\left(\frac{z^{\gamma+1}}{\gamma+1} + \frac{a_2 z^{\gamma+2}}{\gamma+2} + \frac{a_3 z^{\gamma+3}}{\gamma+3}\ldots\right) =$$

$$= z + a_2\frac{\gamma+1}{\gamma+2}z^2 + a_3\frac{\gamma+1}{\gamma+3}z^3 + \ldots = z + \sum_{k=2}^{\infty} a_k \cdot \frac{\gamma+1}{\gamma+k}\cdot z^k \;.$$

Let be $b_k = a_k \cdot \dfrac{\gamma+1}{\gamma+k}$.

We evaluate the relation (2) for the function $F(z) = z + b_2 z^2 + b_3 z^3 + \ldots$

We have:

$$\sum_{k=2}^{\infty} k[1+\alpha(k-1)]\cdot|b_k| = \sum_{k=2}^{\infty} k[1+\alpha(k-1)]\cdot\left|a_k\frac{\gamma+1}{\gamma+k}\right| = \sum_{k=2}^{\infty} k[1+\alpha(k-1)]\cdot|a_k|\cdot\left|\frac{\gamma+1}{\gamma+k}\right| \le$$

$$\le \sum_{k=2}^{\infty} k[1+\alpha(k-1)]\cdot|a_k|\cdot\left|\frac{\gamma+1}{\gamma+1}\right| \le \sum_{k=2}^{\infty} k[1+\alpha(k-1)]\cdot|a_k| \le 1,$$ so:

$F \in UCD(\alpha)$.

Theorem 4. Let

$$F(z) = \frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)\cdot t^{\gamma-1}dt\,.$$

If f is of the form (1), $f \in S$, $\alpha > 0$ and $\operatorname{Re} f'(z) \ge \alpha|zf''(z)|$ $(\forall) z \in U$, then

$$|(1+\gamma)F'(z) + zF''(z)| \ge \alpha|z[(2+\gamma)F'' + zF''']|.$$

Proof

$$F(z) = \frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)\cdot t^{\gamma-1}dt \Leftrightarrow \frac{z^{\gamma}}{1+\gamma}F(z) = \int_0^z f(t)\cdot t^{\gamma-1}dt\,.$$

After successive derivations, we obtain:

$$\frac{\gamma}{1+\gamma}\cdot z^{\gamma-1}\cdot F(z) + \frac{z^{\gamma}}{1+\gamma}\cdot F'(z) = f(z)\cdot z^{\gamma-1} \Leftrightarrow \frac{\gamma}{1+\gamma}\cdot F(z) + \frac{zF'(z)}{1+\gamma} = f(z) \quad\Rightarrow$$

$$\frac{\gamma}{1+\gamma}\cdot F'(z) + \frac{F'(z)}{1+\gamma} + \frac{zF''(z)}{1+\gamma} = f'(z) \Leftrightarrow F'(z) + \frac{1}{1+\gamma}\cdot z\cdot F''(z) = f'(z) \Leftrightarrow$$

$$\Leftrightarrow F''(z) + \frac{1}{1+\gamma}F''(z) + \frac{zF'''(z)}{1+\gamma} = f''(z).$$

So, we have $f''(z) = \dfrac{2+\gamma}{1+\gamma}F''(z) + \dfrac{1}{1+\gamma}zF'''(z)$.

If $\operatorname{Re} f'(z) \geq \alpha |zf''(z)|$ $(\forall) z \in U, \alpha > 0$, then
$f \in UCD(\alpha) \Rightarrow |f'(z)| \geq \alpha |zf''(z)|$.
In this inequality, we put the expressions of f' and f'', and we obtain:

$$\left|F'(z) + \frac{1}{1+\gamma} zF''(z)\right| \geq \alpha \left| z\left(\frac{2+\gamma}{1+\gamma}\right)F''(z) + \frac{1}{1+\gamma} zF'''(z)\right| \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{|1+\gamma|}\left|(1+\gamma)F'(z) + zF''(z)\right| \geq \frac{\alpha}{|1+\gamma|}\left|z(2+\gamma)F''(z) + z^2F'''(z)\right| \Leftrightarrow$$

$$\Leftrightarrow \left|(1+\gamma)F'(z) + zF''(z)\right| \geq \alpha \left| z\left[(2+\gamma)F''(z) + z^2F'''(z)\right]\right|.$$

Remark 5. If f is of the form (1), $f \in S$, $\alpha > 0$ and $\operatorname{Re} f'(z) \geq \alpha |zf''(z)|$ $(\forall) z \in U$, then

$$|\log f'(z)| \leq \frac{1}{\alpha |z|}.$$

Proof

According to the Theorem 4 we have:

$$\left|F'(z) + \frac{1}{1+\gamma} zF''(z)\right| \geq \alpha \left| z\left(\frac{2+\gamma}{1+\gamma}\right)F''(z) + \frac{1}{1+\gamma} z^2F'''(z)\right| \Leftrightarrow$$

$$\Leftrightarrow 1 \geq \alpha \cdot |z| \cdot \left| \frac{\frac{2+\gamma}{1+\gamma}F''(z) + \frac{1}{1+\gamma} zF'''(z)}{F'(z) + \frac{1}{1+\gamma} zF''(z)} \right| = \alpha \cdot |z| \cdot \left| \left[\log\left(F'(z) + \frac{1}{1+\gamma} zF''(z)\right)\right]' \right| \Leftrightarrow$$

$$\Leftrightarrow 1 \geq \alpha \cdot |z| \cdot \left|[\log f'(z)]'\right| \Leftrightarrow \left|[\log f'(z)]'\right| \leq \frac{1}{\alpha \cdot |z|}.$$

References

1.Miller S.S., Mocanu P.T.- Integral operators on certain classes of analytic functions, Univalent Functions, Fractional Calculus, and their Applications, Halstead Pres, John Wiley (1989), 153-166.
2. Miller S.S., Mocanu P.T., Reade M.O.- Starlike integral operators, Pacific Journal of Mathematics, vol. 79, No. 1/1978, 157-168.

3. Miller S.S., Mocanu P.T.- Differential subordinations and univalent functions, Michigan, Math. J. 28(1981), 157-171.
4. Rosy T., Stephen A.B., Subramanian K.G., Silverman H.- Classes of convex functions, Internat. J. Math.& Math. Sci., vol 23, No. 12(2000), 819-825.
5. Silverman H.- Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51(1975), 109-116.

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