REDUCING OF VARIANCE BY A COMBINED SCHEME BASED ON BERNSTEIN POLYNOMIALS

by
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Abstract. A combined scheme of the control variates and weighted uniform sampling methods for reducing of variance is investigated by using of the multivariate Bernstein operator on the unit hypercube. The new obtained estimators for the random numerical integration are numerically compared with the crude Monte Carlo, control variates, and weighted uniform sampling estimators.

1. INTRODUCTION

Definite integrals can be estimated by probabilistic considerations, and these are rather when multiple integrals are concerned. The integral is interpreted as the mean value of a certain random variable, which is an unknown parameter. To estimate this parameter, i.e. the definite integral, one regards the sample mean of the sampling from a suitable random variable. This sample mean is an unbiased estimator for the definite integral and is refered as *the crude Monte Carlo estimator*.

Generally, this method is not fast-converging ratio to the volume of sampling, and efficiency depends on the variance of the estimator, which is expressed by the variance of the integrand. Consequently, for improving the efficiency of Monte Carlo method, it must reduce as much as possible the variance of the integrated function. There is a lot of procedures for reducing of the variance in the Monte Carlo method. In the following we approach the reducing of variance by a *combined scheme of the control variates* and *weighted uniform sampling methods*, using the multivariate Bernstein operators on the unit hypercube.

Numerical experiments are considered comparatively with the crude Monte Carlo, control variates, and weighted uniform sampling estimates.

2. CRUDE MONTE CARLO METHOD

Let $\boldsymbol{X}$ be an n-dimensional random variable having the probability density function $\rho : IR^n \to IR$. In the random numerical integration the multidimensional integral

$$I[\rho; f] = \int_{IR^n} \rho(\mathbf{x}) f(\mathbf{x}) \mathbf{dx} \qquad (1)$$

is interpreted as the mean value of the random variable $f(\mathbf{X})$, where the function $f{:}IR^n \to IR$ usually belongs to $L^2_\rho(IR^n)$, in other words $\int_{IR^n} \rho(\mathbf{x}) f^2(\mathbf{x})\mathbf{dx}$ exists, and therefore the mean value $I[\rho; f]$ exists.

Using a basic statistical technique, the mean value given by (1) can be estimated by taking *N* independent samples (random numbers) $\mathbf{x}_i, i = \overline{1, N}$, with the probability density function ρ. These random numbers are regarded as values of the independent identically distributed random variables $\mathbf{X}_i, i = \overline{1, N}$, i.e. sample variables with the common probability function ρ.

We use the same notation $\overline{I}_N[\rho; f]$ for the sample mean of random variables $f(\mathbf{X}_i), i = \overline{1, N}$, and respectively for its value, i.e.

$$\overline{I}_N[f] = \overline{I}_N[\rho; f] = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{X}_i),$$

$$\overline{I}_N[f] = \overline{I}_N[\rho; f] = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i).$$

The estimator $\overline{I}_N[\rho; f]$ satisfies the following properties:

$$E\left(\overline{I}_N[\rho; f]\right) = I[\rho; f], \text{ (unbiased estimator of } I[\rho; f]\text{)},$$

$$Var\left(\overline{I}_N[\rho; f]\right) \to 0, \quad N \to \infty,$$

$$\overline{I}_N[\rho; f] \to I[\rho; f], \quad N \to \infty, \text{ (with probability 1).}$$

Taking into account these results, the crude Monte Carlo integration formula is defined by

$$I[\rho; f] = \int_{IR^n} \rho(\mathbf{x}) f(\mathbf{x})\mathbf{dx} \cong \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i). \qquad (2)$$

It must remark that in (1) the domain of integration is only apparently the whole *n*-dimensional Euclidean space. Thus, it is possible that the density $\rho(\mathbf{x}) = 0,\ \mathbf{x} \notin \mathbf{D}$, where $\mathbf{D}$ is a region of the *n*-dimensional Euclidean space IR^n, therefore the integral (1) becomes

$$I[\rho; f] = \int_{\mathbf{D}} \rho(\mathbf{x}) f(\mathbf{x}) \mathbf{dx} ,$$

and the crude Monte Carlo method must be interpreted in an appropriate manner. For example, if the integration region is the unit hypercube $\mathbf{D}_n = [0,1]^n$ and $\rho(\mathbf{x}) = 1,\ \mathbf{x} \in \mathbf{D}_n$, then the crude Monte Carlo estimator is

$$\overline{I}_N^c[f] = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{X}_i), \qquad (3)$$

where the sampling variables $\mathbf{X}_i, i = \overline{1,N}$, are independent uniformly distributed on $\mathbf{D}_n$.

3. CONTROL VARIATES METHOD

Control variates is a technique for reducing of variation, and it consists in the split of the integral (1) into two parts,

$$I[\rho; f] = \int_{IR^n} \rho(\mathbf{x}) h(\mathbf{x}) \mathbf{dx} + \int_{IR^n} \rho(\mathbf{x}) [f(\mathbf{x}) - h(\mathbf{x})] \mathbf{dx}, \qquad (4)$$

which are integrated separately, the first by mathematical theory and the second by Monte Carlo method. The function h must be simply enough to be integrate theoretically, and mimics f to absorb most of its variation.

This method was applied in [7] considering the function h given by the multivariate Bernstein polynomial, on the hypercube $\mathbf{D}_n$, having the degree m_k in the variable x_k:

$$\begin{aligned} B_{\mathbf{m}}(f, \mathbf{x}) &= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} \mathbf{p}_{\mathbf{m},\mathbf{k}}(\mathbf{x}) f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) \\ &= \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} p_{m_1,k_1}(x_1) \cdots p_{m_n,k_n}(x_n) f\left(\frac{k_1}{m_1}, \ldots, \frac{k_n}{m_n}\right). \end{aligned} \qquad (5)$$

We have used the following notations $\mathbf{m}=(m_1,...,m_n)$, $\mathbf{k}=(k_1,...,k_n)$, $\mathbf{x}=(x_1,...,x_n)$, $\mathbf{0}=(0,...,0)\in IR^n$, $\frac{\mathbf{k}}{\mathbf{m}}=\left(\frac{k_1}{m_1},\cdots,\frac{k_n}{m_n}\right)$, and respectively $p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k}$, $x\in[0,1], m\in IN, k=\overline{0,m}$, for the Bernstein basis.

In this manner the estimate of the integral $I[f]=\int_{\mathbf{D}_n} f(\mathbf{x})\mathbf{dx}$ is

$$\overline{I}_N^v[f]=\frac{1}{\prod_{i=1}^n(m_i+1)}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)+\frac{1}{N}\sum_{i=1}^{N}\left[f(\mathbf{x}_i)-B_{\mathbf{m}}(f;\mathbf{x}_i)\right] \tag{6}$$

where $\mathbf{x}_i, i=\overline{1,N}$, are independent uniformly distributed points in the hypercube $\mathbf{D}_n$.

4. WEIGHTED UNIFORM SAMPLING METHOD

This method was given in [4], reconsidered in [6], and recently in [2] it was compared with other methods for reducing of the variance.

Let us consider the integral

$$I[f]=\int_{\mathbf{D}} f(\mathbf{x})\mathbf{dx}=V\int_{\mathbf{D}}\frac{1}{V}f(\mathbf{x})\mathbf{dx},$$

where the bounded $\mathbf{D}\subset IR^n$ region has the volume V.

The crude Monte Carlo estimator for $I[f]$ is

$$\overline{I}_N^c[f]=\frac{V}{N}\sum_{i=1}^{N} f(\mathbf{X}_i),$$

with the sampling variables $\mathbf{X}_i, i=\overline{1,N}$, independent uniformly in the region $\mathbf{D}$, i.e. these have the common density probability function

$$\rho(\mathbf{x})=\begin{cases}\dfrac{1}{V}, & \text{if } \mathbf{x}\in\mathbf{D},\\ 0, & \text{if } \mathbf{x}\notin\mathbf{D}.\end{cases}$$

The method of weighted uniform sampling consists in the considering of function $g:\mathbf{D}\to IR$ such that

$$\int_{\mathbf{D}} g(\mathbf{x})\mathbf{dx}=1,$$

and the corresponding sampling function

$$\overline{I}_N^w[f]=\overline{I}_N^w[g;f]=\left(\sum_{i=1}^{N} f(\mathbf{X}_i)\right)\Big/\left(\sum_{i=1}^{N} g(\mathbf{X}_i)\right)$$

where $\mathbf{X}_i, i=\overline{1,N}$, are the same above sampling variables.

In [1] was considered the function *g*, from the weighted uniform sampling method, given by the multivariate Bernstein polynomial corresponding to the integrand *f* defined by (5), i.e.

$$g(\mathbf{x})=K\cdot B_{\mathbf{m}}(f;\mathbf{x}),$$

where the constant *K* is such that

$$\int_{\mathbf{D}_n} g(\mathbf{x})\mathbf{dx}=1.$$

From this condition we have that

$$g(\mathbf{x})=\frac{\prod_{i=1}^{n}(m_i+1)}{\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} f\left(\dfrac{\mathbf{k}}{\mathbf{m}}\right)}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}}\mathbf{p}_{\mathbf{m},\mathbf{k}}(\mathbf{x})f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)$$

$$=\frac{\prod_{i=1}^{n}(m_i+1)}{\sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n} f\left(\dfrac{k_1}{m_1},\ldots,\dfrac{k_n}{m_n}\right)}\times$$

$$\times\sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n} p_{m_1,k_1}(x_1)\cdots p_{m_n,k_n}(x_n)f\left(\frac{k_1}{m_1},\ldots,\frac{k_n}{m_n}\right).$$

Finally, the random numerical integration formula is given by

$$\overline{I}_N^w[f] = \frac{\sum_{i=1}^N f(\mathbf{x}_i)}{\sum_{i=1}^N g(\mathbf{x}_i)} = \frac{\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)}{\prod_{i=1}^n (m_i+1)} \cdot \frac{\sum_{i=1}^N f(\mathbf{x}_i)}{\sum_{i=1}^N \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} \mathbf{p}_{\mathbf{m},\mathbf{k}}(\mathbf{x}_i) f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)}. \tag{7}$$

The random points $\mathbf{x}_i, i = \overline{1, N}$, are independent uniformly distributed in the hypercube $\mathbf{D}_n$.

5. COMBINED SCHEME

The reducing technique of variance by proposed combined scheme consists in the split of integral from the control variates method (4), with the integration domain $\mathbf{D}_n$, and $h = B_{\mathbf{m}}(f)$, i.e.

$$I[f] = \frac{1}{\prod_{i=1}^n (m_i+1)} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) + \int_{\mathbf{D}_n} e(\mathbf{x}) \mathbf{dx}, \tag{8}$$

where $e(\mathbf{x}) = f(\mathbf{x}) - B_{\mathbf{m}}(f;\mathbf{x})$, then to apply the weighted uniform sampling method for the integral from the right side of (8). In a such way, the function *g*, from weighted uniform sampling method, is given by multivariate Bernstein polynomial, namely

$$g(\mathbf{x}) = K \cdot B_{\mathbf{m}}(e;\mathbf{x}),$$

where the constant *K* is such that

$$\int_{\mathbf{D}_n} g(\mathbf{x}) \mathbf{dx} = 1.$$

From this condition we have that

$$g(\mathbf{x}) = \frac{\prod_{i=1}^{n}(m_i+1)}{\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} e\left(\frac{\mathbf{k}}{\mathbf{m}}\right)} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} \mathbf{p}_{\mathbf{m},\mathbf{k}}(\mathbf{x}) e\left(\frac{\mathbf{k}}{\mathbf{m}}\right)$$

$$= \frac{\prod_{i=1}^{n}(m_i+1)}{\sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n} e\left(\frac{k_1}{m_1},\dots,\frac{k_n}{m_n}\right)} \times$$

$$\times \sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n} p_{m_1,k_1}(x_1)\cdots p_{m_n,k_n}(x_n) e\left(\frac{k_1}{m_1},\dots,\frac{k_n}{m_n}\right).$$

Finally, the random numerical integration formula is given by

$$\bar{I}_N^{vw}[f] = \frac{1}{\prod_{i=1}^{n}(m_i+1)} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) + \frac{\left[\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} e\left(\frac{\mathbf{k}}{\mathbf{m}}\right)\right]\left[\sum_{i=1}^{N} e(\mathbf{x}_i)\right]}{\left[\prod_{i=1}^{n}(m_i+1)\right]\left[\sum_{i=1}^{N}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} \mathbf{p}_{\mathbf{m},\mathbf{k}}(\mathbf{x}_i) e\left(\frac{\mathbf{k}}{\mathbf{m}}\right)\right]}. \quad (9)$$

The random points $\mathbf{x}_i, i = \overline{1, N}$, are independent uniformly distributed in the hypercube $\mathbf{D}_n$.

6. NUMERICAL EXPERIMENTS

Numerical examples are considered in the unidimensional $(n = 1)$ and bidimensional $(n = 2)$ cases for the estimator (9) with the integrand f given by $f(x) = \frac{1}{1+x}$ and $f(x, y) = \frac{1}{1+x+y}$ respectively.

The numerical results contained in the following two tables compare the estimates obtained by the combined scheme (9), weighted uniform sampling technique (7), control variates (6), and the crude Monte Carlo method (3).

Each table contains: the sampling volume *N*, the degree $m\left(\text{or } m_1, m_2\right)$ of the Bernstein polynomials, the estimate given by the combined scheme $\overline{I}_N^{vw}\left[f\right]$, the ratio values of the error estimates Err^c, Err^v, Err^w and Err^{vw} . We also remark that the estimations from each row of tables represent the mean values in one hundred of samplings.

$$I\left[f\right] = \log 2 = 0.69314718...$$

N	m	$\overline{I}_N^{vw}\left[g; f\right]$	Err^c/Err^v	Err^c/Err^w	Err^c/Err^{vw}
50	2	0.6932623	11	25	18
100	2	0.6933181	7	10	12
300	2	0.6932220	5	7	13
500	2	0.6931902	6	9	20
50	3	0.6932174	15	27	29
100	3	0.6932408	9	13	23
300	3	0.6931841	7	9	27
500	3	0.6931683	9	13	41
50	5	0.6931797	22	36	62
100	5	0.6931856	14	19	55
300	5	0.6931608	11	14	73
500	5	0.6931549	15	20	113
50	7	0.6931656	29	46	110
100	7	0.6931676	19	25	104
300	7	0.6931541	16	20	144
500	7	0.6931511	21	28	223

$$I\left[f\right] = \log\frac{27}{16} = 0.5232481...$$

N	m_1	m_2	$\overline{I}_N^{vw}\left[f\right]$	Err^c/Err^v	Err^c/Err^w	Err^c/Err^{vw}
50	2	2	0.5233825	5	8	18
100	2	2	0.5233434	6	9	17
300	2	2	0.5232675	5	7	37
50	2	3	0.5233648	6	8	21

100	2	3	0.5233352	7	11	18
300	2	3	0.5232573	7	9	79
50	2	5	0.5233474	7	9	25
100	2	5	0.5233276	9	12	20
300	2	5	0.5232523	8	12	175
50	3	3	0.5233172	8	11	35
100	3	3	0.5232994	9	13	31
300	3	3	0.5232556	8	11	97
50	3	5	0.5233049	9	12	43
100	3	5	0.5232942	11	15	34
300	3	5	0.5232508	11	14	269
50	4	5	0.5232850	11	15	66
100	4	5	0.5232777	13	18	53
300	4	5	0.5232502	12	17	355
50	5	5	0.5232748	13	17	92
100	5	5	0.5232690	15	20	76
300	5	5	0.5232500	14	18	389

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