ON SOME APPLICATIONS OF THE FINITE ELEMENT METHOD IN THE CALCULUS OF THE GEOMETRICAL BODIES

by
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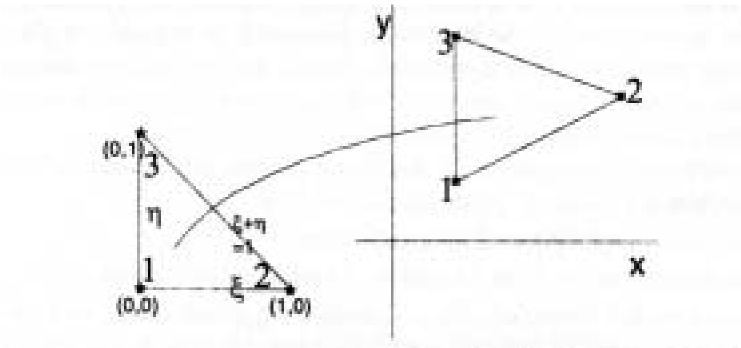
Abstract. Many papers and results are dedicated to the calculus of the divert types of the volumes and areas of geometrical bodies. If it could not be found the exacts approximations for the above mentinated volumes and areas many solutions was given using the numerical approximations.So, the most known approximations are given by the cubature formulas in the case of volumes and the usual geometrical formulas in the case of areas. Between these we shall mention the Cavaliei-Simpson and the product formulas of the trapezium type. The purpose of this paper is to present an another modality to compute the volume an area of a plane surface about of which we shall make the supposition that are known just a limited set of points and who are, usually, random distributed. This modality is based on the finite element method. This will be, also, the main advantage comparatively at the above mentionated formulas which require a given number of points who are usually in a specified position in the domain The method who will be presented has the advantage that .the used points can be random distributed in the domain are not necessary in a specified number.

We shall consider for every given point $(x_i, y_i, z_i), i = 0,...,M$ the corresponding point in the xoy plane $(x_i, y_i), i = 0,...,M$. For every triangle having the apexes (x_1, y_1), (x_2, y_2) and (x_3, y_3) we have the next application between the real triangle and the standard triangle of apexes $(0,0),(1,0),(0,1)$ so that $(0,0) \to (x_1, y_1),(1,0) \to (x_2, y_2)$, $(0,1) \to (x_3, y_3)$:

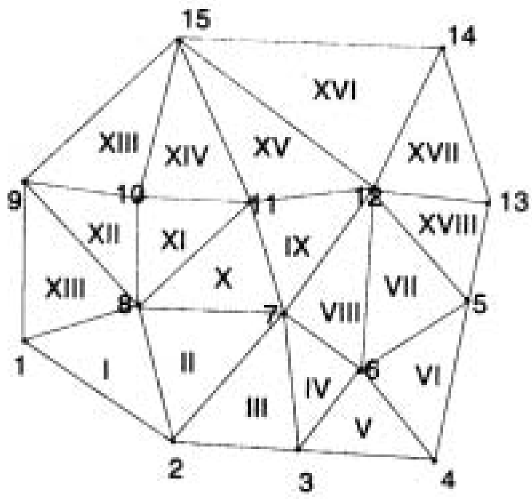
$$\phi(\xi, \eta) = (x = x_1(1-\xi-\eta) + x_2\xi + x_3\eta, y = y_1(1-\xi-\eta) + y_2\xi + y_3\eta) \quad (1)$$



În order to control the domain we need the next:

1. A vector containing the projections on the Ox axis denoting by $x[i], i = 1,...., \text{no. of knots}$ or $x_i, i = 1,..., \text{no. of knots}$;
2. A vector containing the projections on the Oy axis denoting by $y[i], i = 1,...., \text{no. of knots}$ or $y_i, i = 1,..., \text{no. of knots}$;
3. A matrix who give for every triangle the knots belonging to its sides denoting by $me[e, i_e], e = 1,..., ne, i_e = 1,..., 3$, where ne means the number of elements;

Finally, we have to divide the domain into triangular elements and to denote all these triangular elements. The next picture is an example of all above mentioned:



In this case, the relation given by (1) can be written more general as follows:

$$\phi(e,\xi,\eta) = \begin{pmatrix} x = x_{me[e,1]}(1-\xi-\eta) + x_{me[e,2]}\xi + x_{me[e,3]}\eta \\ y = y_{me[e,1]}(1-\xi-\eta) + y_{me[e,2]}\xi + y_{me[e,3]}\eta \end{pmatrix} \qquad (2)$$

The next step consist in inverting all the $\phi(e,\xi,\eta)$ functions. As a result we shall obtain ξ, η as functions of x and y.

Because the number of knots used is the case of the real triangle and also in the case of the standard triangle is equal to three and represents the number of apexes of all the above mentioned triangles we shall name this kind of elements as a *linear element*. In order to work easily with this kind of element we shall introduce the functions denoted by $N_i, i=1,2,3$ and their derivatives by the next table:

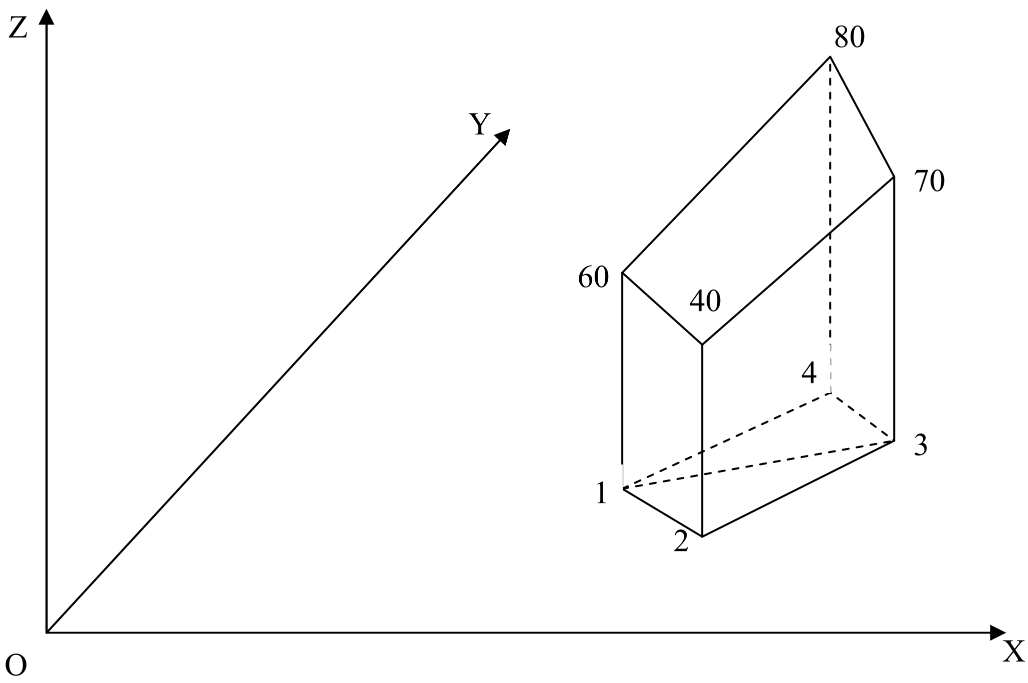
Point	$N_i(\xi,\eta)$	$\frac{\partial N_i(\xi,\eta)}{\partial \xi}$	$\frac{\partial N_i(\xi,\eta)}{\partial \eta}$
1	$1-\xi-\eta$	-1	-1
2	ξ	1	0
3	η	0	1

In order to create for an element the corresponding interpolation function we shall use the next formula:

$$f(x,y)=\sum_{k=1}^{3} u_k N_k(x,y) \qquad \textbf{(3)}$$

where $u_i, i=1,2,3$ have the same values with $z_i, i=1,2,3$ for the considered element. The main property of this function is that f has the values on every common frontier of every two enclosed triangles.

We shall consider the next example having the next associated picture:



The projection on the xOy plane is given by four points numbered from 1 to 4 and having the coordinates (3,2), (4,1), (6,6) and (7,3). In other words, we have given four points in the space having the next coordinates: (3,2,60), (4,1,40), (7,3,70) and (6,6,80). The vectors and the matrix are the next:

1. $x_1 = 3, x_2 = 4, x_3 = 7, x_4 = 6;$
2. $y_1 = 2, y_2 = 1, y_3 = 6, y_4 = 3;$
3. $me[1,1] = 1, me[1,2] = 2, me[1,3] = 3, me[2,1] = 1, me[2,2] = 3, me[2,3] = 4.$

Let us consider the first element as being made with the knots no. 1,2,3 and the second with the knots 1,3,4. Using the relations (1), (2), (3) we shall obtain for $\phi_1, \phi_2, \phi_1^{-1}, \phi_2^{-1}, f_1, f_2$ the next values:

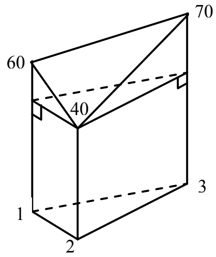
$$\phi_1(\xi,\eta) = \begin{pmatrix} x = \xi + 4\eta + 3 \\ y = -\xi + \eta + 2 \end{pmatrix}$$
$$\phi_2(\xi,\eta) = \begin{pmatrix} x = 4\xi + 3\eta + 3 \\ y = \xi + 4\eta + 2 \end{pmatrix} \tag{4}$$

$$\phi_1^{-1}(x,y) = \left(\frac{x - 4y + 5}{5}, \frac{x + y - 5}{5} \right)$$
$$\phi_2^{-1}(x,y) = \left(\frac{4x - 3y - 6}{13}, \frac{-x + 4y - 5}{13} \right) \tag{5}$$

$$f_1(x,y) = u_1 N_1(x,y) + u_2 N_2(x,y) + u_3 N_3(x,y) = 2x + 18y + 30$$
$$f_2(x,y) = u_1 N_1(x,y) + u_3 N_2(x,y) + u_4 N_3(x,y) = \frac{20x + 50y + 620}{13} \tag{6}$$

We can easily verify that f_1 and f_2 has the same values on the common frontier given by the segment who unite the points no. 1 and 3. In other words, f_1 and f_2 has the same values on the common frontier of the first and second triangle.

In order to calculate the volume of the created geometrical body let us divide this into two bodies. For the first geometrical body we have the next figure:



and, using the usual geometrical formulas, we find that

$V_1 = 161.4709$

For the second geometrical body we find

$V_2 = 487.5003.$

The volume of the entire geometrical body is given as the sum of the two above mentioned volumes and it has the next value:

V=648.9712.

Using the usual geometrical formulas can be easily calculated the area of the entire geometrical body.

References:

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