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# N-linear connections and JN-linear connections on second order tangent bundle $T^2$

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Abstract.On second order tangent bundle $T^2M$ we define an N-linear connection which have nine coefficients in comparation with the JN-linear connection which have three coefficients, only. To work with an N-linear connection on $T^2M$ is an advantage for the physical applications to electrodinamics, elasticity, quantum field theories, etc.

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Let $M$ be a real $C^\infty$- manifold with $n$ dimensions and $(T^2M,\pi,M)$ its tangent bundle, [1] - [3]. The local coordinates on $3n$-dimensional manifolds $T^2M$ are denoted by $\left(x^i,y^{(1)i},y^{(2)i}\right)=\left(x,y^{(1)},y^{(2)}\right)=u,(i=1,2,...,n)$.

Let $\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^{(1)i}},\frac{\partial}{\partial y^{(2)i}}\right)$be the natural basis of the tangent space $TT^2M$ at the point $u\in T^2M$ and let us consider the natural 2−tangent structure on $T^2M, J:\chi\left(T^2M\right)\to\chi\left(T^2M\right)$ given by

$$J\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial y^{(1)i}},J\left(\frac{\partial}{\partial y^{(1)i}}\right)=\frac{\partial}{\partial y^{(2)i}},J\left(\frac{\partial}{\partial y^{(2)i}}\right)=0. \quad (1)$$

We denote with $N$ a nonlinear connection on $T^2M$ with the local coefficients $\left(\begin{array}{cc}\underset{(1)}{N^i}_j & \underset{(2)}{N^i}_j\end{array}\right)$ $(i,j=1,2,...,n)$ ,[3], [6]. Hence, the tangent space of $T^2M$ in the point $u\in T^2M$ is given by the direct sum of the linear vector spaces:

$$T_uT^2M=N_0\left(u\right)\oplus N_1\left(u\right)\oplus V_2\left(u\right),\forall u\in T^2M \quad (2)$$

An adapted basis to the direct decomposition (2) is given by $\left\{\frac{\delta}{\delta x^i},\frac{\delta}{\delta y^{(1)i}},\ \frac{\delta}{\delta y^{(2)i}}\right\}$, where

$$\begin{aligned}
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}}, \\
\frac{\delta}{\delta y^{(1)i}} &= \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}}, \\
\frac{\delta}{\delta y^{(2)i}} &= \frac{\partial}{\partial y^{(2)i}}.
\end{aligned} \tag{3}$$

The dual basis of (3) is $\left\{\delta x^i, \delta y^{(1)i}, \delta y^{(2)i}\right\}$, where

$$\begin{aligned}
\delta x^i &= dx^i, \\
\delta y^{(1)i} &= dy^{(1)i} + \underset{(1)}{N}{}^i{}_j dx^j, \\
\delta y^{(2)i} &= dy^{(2)i} + \underset{(1)}{N}{}^i{}_j dy^{(1)j} + \left(\underset{(2)}{N}{}^i{}_m + \underset{(1)}{N}{}^i{}_j \underset{(1)}{N}{}^m{}_j\right) dx^j.
\end{aligned} \tag{4}$$

Definition. *([2],[3]) A linear connection $D$ on $T^2M$, $D:\chi\left(T^2M\right)\times\chi\left(T^2M\right)\to\chi\left(T^2M\right)$ is called an N-linear connection on $T^2M$ if it preserves by parallelism the horizontal and vertical distributions $N_0, N_1$ and $V_2$ on $T^2M$.*

An $N$-linear connection $D$ on $T^2M$ is characterized by its coefficients, in the adapted basis (3), in the form:

$$\begin{aligned}
&D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \underset{(00)}{L}{}^i{}_{jk} \frac{\delta}{\delta x^i}, D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^{(1)j}} = \underset{(10)}{L}{}^i{}_{jk} \frac{\delta}{\delta y^{(1)i}}, D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} = \underset{(20)}{L}{}^i{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\
&D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\delta}{\delta x^j} = \underset{(01)}{C}{}^i{}_{jk} \frac{\delta}{\delta x^i}, D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\delta}{\delta y^{(1)j}} = \underset{(11)}{C}{}^i{}_{jk} \frac{\delta}{\delta y^{(1)i}}, D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} = \underset{(21)}{C}{}^i{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\
&D_{\frac{\delta}{\delta y^{(2)k}}} \frac{\delta}{\delta x^j} = \underset{(02)}{C}{}^i{}_{jk} \frac{\delta}{\delta x^i}, D_{\frac{\delta}{\delta y^{(2)k}}} \frac{\delta}{\delta y^{(1)j}} = \underset{(12)}{C}{}^i{}_{jk} \frac{\delta}{\delta y^{(1)i}}, D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} = \underset{(22)}{C}{}^i{}_{jk} \frac{\partial}{\partial y^{(2)i}}.
\end{aligned} \tag{5}$$

The system of **nine** functions

$$D\Gamma\left(N\right) = \left(\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk}\right), \tag{6}$$

are called the **coefficients** of the $N-$linear connection $D$.

The torsion tensor $T$ of an $N-$ linear connection $D\Gamma\left(N\right)$ is expressed, as usually, by $T\left(X,Y\right) = D_XY - D_YX - \left[X,Y\right]$ and, in the adapted basis (3), it have **fourteen** components: $\underset{(0\alpha)}{T}{}^i{}_{jk}, \underset{(\beta\alpha)}{P}{}^i{}_{jk}, \underset{(2\gamma)}{Q}{}^i{}_{jk}, \underset{(\beta\gamma)}{S}{}^i{}_{jk}$, $(\alpha = 0,1,2;\ \beta,\gamma =$

$1, 2;\ \underset{(21)}{S}{}^{i}_{jk} = 0)$ (see (7.2) , [2] , pg.41) . The curvature tensor $R$ of $D\Gamma(N)$ , in the adapted basis (3), have **eighteen** components: $\underset{(0\alpha)}{R}{}^{i}_{j\ kl}, \underset{(\beta\alpha)}{P}{}^{i}_{j\ kl}, \underset{(2\alpha)}{Q}{}^{i}_{j\ kl}, \underset{(\beta\alpha)}{S}{}^{i}_{j\ kl}$, $(\alpha = 0, 1, 2;\ \beta = 1, 2)$ ( see (7.11) ,[2],pg.43).

Generally, an N-linear connection $D\Gamma(N)$ on $T^2M$ is not compatible with the natural 2-tangent structure $J$ given by (1).

Definition.*An N-linear connection $D\Gamma(N)$ on $T^2M$ is called $JN-$ linear connection if it is absolut parallel with respect to $J$, i.e.:*

$$D_X J = 0,\ \forall X \in \chi\left(T^2M\right). \tag{7}$$

Theorem 1 (Gh. Atanasiu [2], pg. 39, [3], pg.25) *A $JN-$ linear connection on $T^2M$ is characterized by the coefficients $JD\Gamma(N)$ given by(6), where*

$$\begin{aligned}
\underset{(00)}{L}{}^{i}_{jk} &= \underset{(10)}{L}{}^{i}_{jk} = \underset{(20)}{L}{}^{i}_{jk}\left(= L^{i}_{jk}\right)\\
\underset{(01)}{C}{}^{i}_{jk} &= \underset{(11)}{C}{}^{i}_{jk} = \underset{(21)}{C}{}^{i}_{jk}\left(= \underset{(1)}{C}{}^{i}_{jk}\right)\\
\underset{(02)}{C}{}^{i}_{jk} &= \underset{(12)}{C}{}^{i}_{jk} = \underset{(22)}{C}{}^{i}_{jk}\left(= \underset{(2)}{C}{}^{i}_{jk}\right).
\end{aligned} \tag{8}$$

It results that a $JN-$ linear connection on $T^2M$ has **three** essentially coefficients $JD\Gamma(N) = \left(L^{i}_{jk}, \underset{(1)}{C}{}^{i}_{jk}, \underset{(2)}{C}{}^{i}_{jk}\right)$.

In the adapted basis (3), the torsion tensor $T$ of a $JN-$ linear connection on $T^2M$ have **thirteen** components $\left(\underset{(21)}{Q}{}^{i}_{jk} = \underset{(20)}{P}{}^{i}_{jk}\right)$ and the curvature tensor $R$ of $JD\Gamma(N)$ have **six** components $R^{i}_{j\ kl}\left(= \underset{(0\alpha)}{R}{}^{i}_{j\ kl}\right), \underset{(\beta)}{P}{}^{i}_{j\ kl}\left(= \underset{(\beta\alpha)}{P}{}^{i}_{j\ kl}\right)$, $Q^{i}_{j\ kl}\left(= \underset{(0\alpha)}{Q}{}^{i}_{j\ kl}\right), \underset{(\beta)}{S}{}^{i}_{j\ kl}\left(= \underset{(\beta\alpha)}{S}{}^{i}_{j\ kl}\right), (\forall\alpha = 0, 1, 2; \beta = 1, 2)$.

Of course, for the physical applications there exists an advantage to work with an N-linear connection (see [4] , [5] , [8]) in comparation with a $JN-$ linear connection (see [1] , [6] , [7]).

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