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SOME STRONG DIFFERENTIAL SUBORDINATIONS USING A NEW GENERALIZED MULTIPLIER TRANSFORMATION

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Abstract. The object of this paper is to obtain some strong subordination results regarding a new class $S_n^m(\alpha,\beta,\rho)$ defined by using a new generalized multiplier transformation.

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1. Introduction and Preliminaries

Denote by U the open unit disc of the complex plane $U = \{z \in C : |z| < 1\}$ and by $\overline{U}$ the closed unit disc of the complex plane $\overline{U} = \{z \in C : |z| \leq 1\}$. Let $H(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$. Let

$$A_\zeta^*(n) = \{f \in H(U \times \overline{U}) : f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + ..., \ z \in U, \zeta \in \overline{U}\}, \quad (1.1)$$

where $n \in N$ and $a_k(\zeta)$ are analytic functions in $\overline{U}$ for $k \geq n+1$. For $n = 1$, $A_\zeta^*(1) = A_\zeta^*$, with $a_k(\zeta)$ analytic functions in $\overline{U}$ for $k \geq 2$. Let

$$H^*[a,n,\zeta] = \{f \in H(U \times \overline{U}) : f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + ..., \ z \in U, \zeta \in \overline{U}\},$$

where $a \in C$, $n \in N$ and $a_k(\zeta)$ are analytic functions in $\overline{U}$, $k \geq n$.

In [1], J. A. Antonino and S. Romaguera have introduced the concept of strong differential subordination, generalising the notion of differential subordination, which was developed further by G. I. Oros [7] and by G. I. Oros & Gh. Oros [8].

Definition 1.1 [8] Let $f(z,\zeta)$, $g(z,\zeta)$ be analytic functions in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $g(z,\zeta)$, written $f(z,\zeta) \prec\prec g(z,\zeta)$, if there exists an analytic function w in U, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z,\zeta) = g(w(z),\zeta)$ for all $\zeta \in \overline{U}$.

Remark 1.2 ([8]) i) If $g(z,\zeta)$ is analytic in $U\times\overline{U}$ and univalent in U for all $\zeta\in\overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta)=g(0,\zeta)$, for all $\zeta\in\overline{U}$, and $f(U\times\overline{U})\subseteq g(U\times\overline{U})$. ii) If $f(z,\zeta)\equiv f(z)$and $g(z,\zeta)\equiv g(z)$, the strong subordination becomes the usual subordination.

We extend the new generalized multiplier transformation introduced and investigated in [9] and studied further in [10, 11, 12], to the new class of analytic functions $A^*_\zeta(n)$.

Definition 1.3 For $f\in A^*_\zeta(n),\ n\in N,\ m\in N_0=N\cup\{0\},\ \beta\geq 0$ andα a real number with $\alpha+\beta>0$, a new generalized multiplier operator $I^m_{\alpha,\beta}$ for $z\in U,\ \zeta\in\overline{U}$ is defined by

$$\begin{aligned} I^0_{\alpha,\beta}f(z,\zeta) &= f(z,\zeta),\\ I^1_{\alpha,\beta}f(z,\zeta) &= \frac{\alpha f(z,\zeta)+\beta z f'_z(z,\zeta)}{\alpha+\beta},\\ &\vdots\\ I^m_{\alpha,\beta}f(z,\zeta) &= I_{\alpha,\beta}\left(I^{m-1}_{\alpha,\beta}f(z,\zeta)\right). \end{aligned}$$

Remark 1.4 We observe that $I^m_{\alpha,\beta}:A^*_\zeta(n)\longrightarrow A^*_\zeta(n)$, is a linear operator and for $f(z,\zeta)$ given by (1.1), we have

$$I^m_{\alpha,\beta}f(z,\zeta)=z+\sum_{k=n+1}^{\infty}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m a_k(\zeta)z^k. \tag{1.2}$$

It follows from (1.2) that

$$I^m_{\alpha,0}f(z,\zeta)=f(z,\zeta), \tag{1.3}$$

$$(\alpha+\beta)I^{m+1}_{\alpha,\beta}f(z,\zeta)=\alpha I^m_{\alpha,\beta}f(z,\zeta)+\beta z\left(I^m_{\alpha,\beta}f(z,\zeta)\right)'_z. \tag{1.4}$$

Remark 1.5 i) $I^m_{l+1-\beta,\beta}f(z,\zeta)=I^m(l,\beta)f(z,\zeta),\ l>-1,\ \beta\geq 0$ (See Alina A Lupas [2] and Alina A Lupas et.al. [3]) (Considered for $l\geq 0,\ \beta\geq 0$).

ii) $I^m_{1-\beta,\beta}f(z,\zeta)=I^m(\beta)f(z,\zeta)=z+\sum_{k=n+1}^{\infty}(1+(k-1)\beta)^m a_k(\zeta)z^k,\ \beta\geq 0$. $I^m(0)f(z,\zeta)=I^mf(z,\zeta)$ was considered in [13].

iii) $I^m_{\alpha,1}f(z,\zeta)=D^m_\alpha f(z,\zeta)=z+\sum_{k=n+1}^{\infty}\left(\frac{\alpha+k}{\alpha+1}\right)^m a_k(\zeta)z^k,\ \alpha>-1.$

We need the following lemmas, which are extension of lemmas in [5] and [6], to the new class of analytic functions $A^*_\zeta(n)$.

Lemma 1.6 *([4]) Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta \in \overline{U}$ and let γ be a complex number with $\gamma \neq 0$ and $Re(\gamma) \geq 0$. If $p \in H^*[a,n,\zeta]$ and*

$$p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta) \prec\prec h(z,\zeta),\ z \in U,\ \zeta \in \overline{U},$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta),\ z \in U,\ \zeta \in \overline{U},$$

where

$$q(z,\zeta)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t,\zeta).t^{(\gamma/n)-1}dt,\ z \in U,\ \zeta \in \overline{U},$$

is convex and is the dominant.

Lemma 1.7 *([4]) Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and for $z \in U,\ \zeta \in \overline{U}$, let $h(z,\zeta)=q(z,\zeta)+n\rho zq'_z(z,\zeta)$, where $\rho > 0$ and n is a positive integer.*
If $p(z,\zeta)=q(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\cdots,\ z \in U,\ \zeta \in \overline{U}$ is holomorphic in $U \times \overline{U}$ and $p(z,\zeta)+\rho zp'_z(z,\zeta) \prec\prec h(z,\zeta),\ z \in U,\ \zeta \in \overline{U}$, then $p(z,\zeta) \prec\prec q(z,\zeta),\ z \in U,\ \zeta \in \overline{U}$ and this result is sharp.

In this present investigation, by making use of strong differential subordination properties, we consider certain suitable classes of admissible functions and investigate some strong differential subordination results of analytic functions associated with a new generalized multiplier operator $I^m_{\alpha,\beta}$.

2. Main Results

Definition 2.1 Let $n \in N,\ m \in N_0 = N \cup \{0\},\ \rho \in [0,1),\ \beta \geq 0,\ \alpha$ a real number with $\alpha+\beta > 0$. A function $f \in A^*_\zeta(n)$ is said to be in the class $S^m_n(\alpha,\beta,\rho)$, if the following condition is satisfied:

$$\text{Re}\left[I^m_{\alpha,\beta}f(z,\zeta)\right]'_z > \rho,\ z \in U,\ \zeta \in \overline{U}.$$

Theorem 2.2 *If* $n \in N,\ m \in N_0 = N \cup \{0\},\ \rho \in [0,1),\ \beta \geq 0,\ \alpha$ *a real number with* $\alpha + \beta > 0$, *then*

$$S_n^{m+1}(\alpha,\beta,\rho) \subset S_n^m(\alpha,\beta,\delta)$$

where

$$\delta = \delta_n(\alpha,\beta,\rho) = (2\rho - \zeta) + 2(\zeta - \rho)\left(\frac{\alpha+\beta}{n\beta}\right) B\left(\frac{\alpha+\beta}{n\beta}\right),$$

with

$$B(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt.$$

Proof. Let $f \in S_n^{m+1}(\alpha,\beta,\rho)$. By using the property (1.4) we get

$$I_{\alpha,\beta}^{m+1} f(z,\zeta) = \frac{\alpha I_{\alpha,\beta}^m f(z,\zeta) + \beta z \left(I_{\alpha,\beta}^m f(z,\zeta)\right)_z'}{\alpha+\beta},\ z \in U,\ \zeta \in \overline{U}. \tag{2.1}$$

Differentiating (2.1) with respect to z, we obtain

$$\left(I_{\alpha,\beta}^{m+1} f(z,\zeta)\right)_z' = p(z,\zeta) + (\beta/(\alpha+\beta)) z p_z'(z,\zeta),\ z \in U,\ \zeta \in \overline{U}, \tag{2.2}$$

where

$$p(z,\zeta) = \left(I_{\alpha,\beta}^m f(z,\zeta)\right)_z',\ z \in U,\ \zeta \in \overline{U}, \tag{2.3}$$

with $p(0,\zeta) = 1$ for all $\zeta \in \overline{U}$.
Since $f \in S_n^{m+1}(\alpha,\beta,\rho)$, by using Definition 2.1, we have from (2.2)

$$\mathrm{Re}\left(p(z,\zeta) + (\beta/(\alpha+\beta)) z p_z'(z,\zeta)\right) > \rho,\ z \in U,\ \zeta \in \overline{U}$$

which is equivalent to

$$p(z,\zeta) + (\beta/(\alpha+\beta)) z p_z'(z,\zeta) \prec\prec \frac{\zeta + (2\rho - \zeta)z}{1+z} \equiv h(z,\zeta).$$

Using Lemma 1.6, with $\gamma = (\alpha+\beta)/\beta$, we have $p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta)$, where

$$q(z,\zeta) = \frac{(\alpha+\beta)/\beta}{n z^{(\alpha+\beta)/n\beta}} \int_0^z \left(\frac{\zeta + (2\rho-\zeta)t}{1+t}\right) t^{\left(\frac{\alpha+\beta}{n\beta}\right)-1} dt.$$

The function $q(z,\zeta)$ is convex and is the best dominant. With $p(z,\zeta) \prec\prec q(z,\zeta)$ and $q(z,\zeta)$ being convex, and the fact that the image of $U \times \overline{U}$ is symmetric with respect to the real axis, it results that, $\mathrm{Re}\left(\left(I_{\alpha+\beta}^m f(z,\zeta)\right)_z'\right) > q(1,\zeta) = \delta$, where

$$\delta = \delta_n(\alpha,\beta,\rho) = (2\rho - \zeta) + 2(\zeta - \rho)\left(\frac{\alpha+\beta}{n\beta}\right) B\left(\frac{\alpha+\beta}{n\beta}\right),$$

with

$$B\left(\frac{\alpha+\beta}{n\beta}\right)=\int_0^1 \frac{t^{\frac{\alpha+\beta}{n\beta}-1}}{1+t}dt.$$

Hence $f \in S_n^m(\alpha,\beta,\delta)$ and the proof of the Theorem is complete.

Theorem 2.3 *Let $q(z,\zeta)$ be a convex function with $q(0,\zeta)=1$, for all $\zeta\in\overline{U}$, and let $h(z,\zeta)$ be a function such that*

$$h(z,\zeta)=q(z,\zeta)+(n\beta/(\alpha+\beta))\,zq'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.$$

If $f\in A^*_\zeta(n)$ and satisfies the strong differential subordination

$$\left(I^{m+1}_{\alpha,\beta}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \tag{2.4}$$

then $\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)'_z \prec\prec q(z,\zeta)$ and the result is sharp.

Proof. From (2.1), (2.2), (2.3) and (2.4), we obtain

$$p(z,\zeta)+(\beta/(\alpha+\beta))\,zp'_z(z,\zeta)\prec\prec q(z,\zeta)+(n\beta/(\alpha+\beta))\,zq'_z(z,\zeta)\equiv h(z,\zeta).$$

Then, by using Lemma 1.7 we get $p(z,\zeta)\prec\prec q(z,\zeta)$ or $\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)'_z \prec\prec q(z,\zeta)$ and the result is sharp.

Theorem 2.4 *Let $q(z,\zeta)$ be a convex function with $q(0.\zeta)=1$, for all $\zeta\in\overline{U}$, and the function $h(z,\zeta)$, given by*

$$h(z,\zeta)=q(z,\zeta)+nzq'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.$$

If $f(z,\zeta)\in A^*_\zeta(n)$ satisfies the strong differential subordination $\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U}$, then $\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)/z\prec q(z,\zeta)$ and the result is sharp.

Proof. If we let $p(z,\zeta)=\left(\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)/z\right),\ z\in U,\ \zeta\in\overline{U}$, then we obtain

$$\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)'_z=p(z,\zeta)+zp'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.$$

So the strong subordination $\left(I^{m}_{\alpha,\beta}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U}$, becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec q(z,\zeta)+nzq'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

and hence from Lemma 1.7, we have $\left(\left(I^m_{\alpha,\beta} f(z,\zeta)\right)/z\right) \prec\prec q(z,\zeta)$. The result is sharp.

Remark 2.5 i) Taking $\alpha = l + 1 - \beta$ in Theorem 2.2, Theorem 2.3 and Theorem 2.4, we obtain corresponding results for the operator $I^m(l,\beta),\ l > -1,\ \beta \geq 0$.
ii) Letting $\alpha = 1 - \beta$ in Theorem 2.2, Theorem 2.3 and Theorem 2.4, we obtain corresponding results for the operator $I^m(\beta),\ \beta \geq 0$.
iii) Putting $\beta = 1$ in Theorem 2.2, Theorem 2.3 and Theorem 2.4, we get corresponding results for the operator $D^m_\alpha,\ \alpha > -1$.

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