Acta Universitatis Apulensis No. 34/2013
ISSN: 1582-5329 pp. 81-97

# THE GROWTH ESTIMATE OF ITERATED ENTIRE FUNCTIONS

Ratan Kumar Dutta

Abstract. In this paper we study growth properties of iterated entire functions which improve some earlier results.

2010 *Mathematics Subject Classification*: 30D35.
*Keywords and phrases*: Entire functions, growth, iteration, order, lower order, p-th order, lower p-th order.

## 1 Introduction, Definitions and Notations

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane $C$, it is well known [1] that $\lim_{r\to\infty} \frac{\log T(r,fog)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{\log T(r,fog)}{T(r,g)} = 0$. Later on Singh [11] investigated some comparative growth of $logT(r, fog)$ and $T(r, f)$. Further in [11] he raised the problem of investing the comparative growth of $logT(r, fog)$ and $T(r, g)$. However some results on the comparative growth of $logT(r, fog)$ and $T(r, g)$ are proved in [6]. Also in [7] Lahiri and Datta made close investigation on comparative growth properties of $logT(r, fog)$ and $T(r, g)$ together with that of $\log\log T(r, fog)$ and $T(r, f^{(k)})$.

Recently Banerjee and Dutta [2] made close investigation on comparative growth properties of iterated entire functions. In this paper, we study growth of iterated entire functions to generalist some results of Banerjee and Dutta [2] in terms of p-th order and lower p-th order.

The following definitions are well known.

**Definition 1.1** *The order $\rho_f$ and the lower order $\lambda_f$ of a meromorphic function is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

*If $f$ is entire then*

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

**Definition 1.2** *The hyper order $\overline{\rho_f}$ and the hyper lower order $\overline{\lambda}_f$ of a meromorphic function is defined as*

$$\overline{\rho_f} = \lim\sup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

*and*

$$\overline{\lambda}_f = \lim\inf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

*If $f$ is entire then*

$$\overline{\rho_f} = \lim\sup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

*and*

$$\overline{\lambda}_f = \lim\inf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

**Notation 1.3** *[10] $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer $m$, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.*

**Definition 1.4** *The p-th order $\rho_f^p$ and the lower p-th order $\lambda_f^p$ of a meromorphic function $f$ is defined as*

$$\rho_f^p = \lim\sup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}$$

*and*

$$\lambda_f^p = \lim\inf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}.$$

*If $f$ is an entire function then*

$$\rho_f^p = \lim\sup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}$$

*and*

$$\lambda_f^p = \lim\inf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Clearly $\rho_f^p \leq \rho_f^{p-1}$and $\lambda_f^p \leq \lambda_f^{p-1}$ for all $p$ and when $p = 1$ then p-th order and lower p-th order coincides with classical order and lower order respectively.

**Definition 1.5** *The hyper p-th order $\overline{\rho_f^p}$ and the hyper lower p-th order $\overline{\lambda_f^p}$ of a meromorphic function $f$ is defined as*

$$\overline{\rho_f^p} = \lim\sup_{r\to\infty} \frac{\log^{[p+1]} T(r,f)}{\log r}$$

*and*

$$\overline{\lambda_f^p} = \lim_{r\to\infty}\inf \frac{\log^{[p+1]} T(r,f)}{\log r}.$$

*If $f$ is an entire function then*

$$\overline{\rho_f^p} = \lim_{r\to\infty}\sup \frac{\log^{[p+2]} M(r,f)}{\log r}$$

*and*

$$\overline{\lambda_f^p} = \lim_{r\to\infty}\inf \frac{\log^{[p+2]} M(r,f)}{\log r}.$$

Clearly $\overline{\rho_f^p} \le \overline{\rho_f^{p-1}}$ and $\overline{\lambda_f^p} \le \overline{\lambda_f^{p-1}}$ for all $p$ and when $p = 1$ then hyper p-th order and hyper lower p-th order coincides with hyper order and hyper lower order respectively.

**Definition 1.6** *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function $f$ if*

*(i) $\lambda_f(r)$ is nonnegative and continuous for $r \ge r_0$, say;*

*(ii) $\lambda_f(r)$ is differentiable for $r \ge r_0$ except possibly at isolated points at which $\lambda_f'(r-0)$ and $\lambda_f'(r+0)$ exist;*

*(iii)* $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$;

*(iv)* $\lim_{r\to\infty} r\lambda_f'(r)\log r = 0$*; and*

*(v)* $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

According to Lahiri and Banerjee [4], $f(z)$ and $g(z)$ be two entire functions then the iteration of $f$ with respect to $g$ is defined as follows:

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))\\
&\;\;\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\\
&\;\;\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\\
f_n(z) &= f(g(f........(f(z) \text{ or } g(z))........)),\\
&\quad \text{according as } n \text{ is odd or even,}\\
&= f(g_{n-1}(z)) = f(g(f_{n-2}(z))),
\end{aligned}$$

and so are

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ .... & \quad .... \\ .... & \quad .... \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Throughout the paper we assume $f, g$ etc. are non constant entire functions having respective p-th orders $\rho_f^p, \rho_g^p$ and respective lower p-th orders $\lambda_f^p, \lambda_g^p$. Also we do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [3], [12] and [13].

## 2 Lemmas

The following lemmas will be needed in the sequel.

**Lemma 2.1** *[3] Let $f(z)$ be an entire function. For $0 \leq r < R < \infty$, we have*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

**Lemma 2.2** *[1] If $f$ and $g$ are any two entire functions, for all sufficiently large values of $r$,*

$$M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, fog) \leq M(M(r, g), f)$$

**Lemma 2.3** *[9] Let $f(z)$ and $g(z)$ be two entire functions. Then we have*

$$T(r, f(g)) \geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

**Lemma 2.4** *[5] Let $f$ be an entire function. Then for $k > 2$,*

$$\liminf_{r \to \infty} \frac{\log^{[k-1]} M(r, f)}{\log^{[k-2]} T(r, f)} = 1.$$

**Lemma 2.5** *[7] Let $f$ be a meromorphic function. Then for $\delta (> 0)$ the function $r^{\lambda_f + \delta - \lambda_f(r)}$ is an increasing function of $r$.*

**Lemma 2.6** *[8] Let $f$ be an entire function of finite lower order. If there exist entire functions $a_i$ $(i = 1, 2, 3..........n;\ n \leq \infty)$ satisfying $T(r, a_i) = o\{T(r, f)\}$ and*

$$\sum_{i=1}^{n} \delta(a_i, f) = 1 \quad \textit{then} \quad \lim_{r\to\infty} \frac{T(r, f)}{\log M(r, f)} = \frac{1}{\pi}.$$

**Lemma 2.7** *Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0 < \rho_f^p < \infty$ and $0 < \rho_g^p < \infty$. Then for all sufficiently large $r$ and $\varepsilon > 0$,*

$$\log^{[(n-1)p]} T(r, f_n) \leq \begin{cases} (\rho_f^p + \varepsilon)\log M(r, g) + O(1) & \textit{when } n \textit{ is even} \\ (\rho_g^p + \varepsilon)\log M(r, f) + O(1) & \textit{when } n \textit{ is odd} \end{cases}$$

*where $p \geq 1$.*

**Proof.** First suppose that n is even. Then from Lemma 2.1 and second part of Lemma 2.2 also from definition of p-th order, it follows that for all sufficiently large values of $r$,

$$\begin{aligned} T(r, f_n) &\leq \log M(r, f_n) \\ &\leq \log M(M(r, g_{n-1}), f) \\ \text{i.e., } \log^{[p]} T(r, f_n) &\leq \log^{[p+1]} M(M(r, g_{n-1}), f) \\ &\leq \log[M(r, g_{n-1})]^{\rho_f^p+\varepsilon}. \\ \text{So, } \log^{[p+1]} T(r, f_n) &\leq \log^{[2]} M(r, g(f_{n-2})) + O(1). \end{aligned}$$

Taking repeated logarithms (p-1) times, we get

$$\begin{aligned} \log^{[2p]} T(r, f_n) &\leq \log^{[p+1]} M(M(r, f_{n-2}), g) + O(1) \\ &\leq \log[M(r, f_{n-2})]^{\rho_g^p+\varepsilon} + O(1) \\ \text{i.e., } \log^{[2p+1]} T(r, f_n) &\leq \log^{[2]} M(r, f_{n-2}) + O(1). \end{aligned}$$

Again taking repeated logarithms (p-1) times, we get

$$\log^{[3p]} T(r, f_n) \leq \log[M(r, g_{n-3})]^{\rho_f^p+\varepsilon} + O(1).$$

Finally, after taking repeated logarithms (n-4)p times more, we have for all sufficiently large values of $r$,

$$\begin{aligned} \log^{[(n-1)p]} T(r, f_n) &\leq \log[M(r, g)]^{\rho_f^p+\varepsilon} + O(1) \\ \text{i.e., } \log^{[(n-1)p]} T(r, f_n) &\leq (\rho_f^p + \varepsilon)\log M(r, g) + O(1). \end{aligned}$$

Similarly if $n$ is odd then for all sufficiently large values of $r$

$$\log^{[(n-1)p]} T(r, f_n) \leq (\rho_g^p + \varepsilon)\log M(r, f) + O(1).$$

This proves the lemma. ■

**Lemma 2.8** *Let $f(z)$ and $g(z)$ be two non constant entire functions such that $0 < \lambda_f^p < \infty$ and $0 < \lambda_g^p < \infty$. Then for any $\varepsilon$ $(0 < \varepsilon < min\{\lambda_f^p, \lambda_g^p\})$ and $p \geq 1$,*

$$\log^{[(n-1)p]} T(r, f_n) \geq \begin{cases} (\lambda_f^p - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g^p - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

*for all sufficiently large values of $r$.*

**Proof.** To prove this lemma we first consider $n$ is even. Then from Lemma 2.1 and Lemma 2.3 we get for $\varepsilon$ $(0 < \varepsilon < \min\{\lambda_f^p, \lambda_g^p\})$ and for all large values of $r$

$$\begin{aligned} T(r, f_n) &= T(r, f(g_{n-1})) \\ &\geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right). \\ \therefore \quad \log^{[p]} T(r, f_n) &\geq \log^{[p+1]} M\left(\frac{1}{8} M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right) + O(1) \\ &\geq \log\left[\frac{1}{8} M\left(\frac{r}{4}, g_{n-1}\right) + O(1)\right]^{\lambda_f^p - \varepsilon} + O(1) \\ &\geq \log\left[\frac{1}{9} M\left(\frac{r}{4}, g_{n-1}\right)\right]^{\lambda_f^p - \varepsilon} + O(1) \\ &\geq (\lambda_f^p - \varepsilon) \log M\left(\frac{r}{4}, g_{n-1}\right) + O(1) \\ &\geq (\lambda_f^p - \varepsilon) T\left(\frac{r}{4}, g_{n-1}\right) + O(1) \\ &\geq (\lambda_f^p - \varepsilon) \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1), \end{aligned}$$

$$\begin{aligned} \text{that is, } \log^{[2p]} T(r, f_n) &\geq \log^{[p+1]} M\left(\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1) \\ &\geq \log\left[\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1)\right]^{\lambda_g^p - \varepsilon} + O(1) \\ &\geq \log\left[\frac{1}{9} M\left(\frac{r}{4^2}, f_{n-2}\right)\right]^{\lambda_g^p - \varepsilon} + O(1). \\ \text{i.e.,} \quad \log^{[2p]} T(r, f_n) &\geq (\lambda_g^p - \varepsilon) \log M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1) \\ \cdots\cdots &\quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \\ \cdots\cdots &\quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \\ \text{Therefore, } \log^{[(n-2)p]} T(r, f_n) &\geq (\lambda_g^p - \varepsilon) \log M\left(\frac{r}{4^{n-2}}, f(g)\right) + O(1). \\ \text{So,} \quad \log^{[(n-1)p]} T(r, f_n) &\geq (\lambda_f^p - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1) \quad \text{when } n \text{ is even.} \end{aligned}$$

Similarly

$$\log^{[(n-1)p]} T(r, f_n) \geq (\lambda_g^p - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f\right) + O(1) \quad \text{when } n \text{ is odd.}$$

This proves the lemma. ■

# 3 Theorems

**Theorem 3.1** *Let $f$ and $g$ be two non-constant entire functions having finite lower orders. Then*

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{T(r, g)} \leq 3\rho_f^p 2^{\lambda_g},$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{T(r, g)} \geq \frac{\lambda_f^p}{(4^{n-1})^{\lambda_g}}$$

*when $n$ is even and*

$$(iii) \quad \liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{T(r, f)} \leq 3\rho_g^p 2^{\lambda_f},$$

$$(iv) \quad \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_n)}{T(r, f)} \geq \frac{\lambda_g^p}{(4^{n-1})^{\lambda_f}}$$

*when $n$ is odd.*

**Proof.** We may clearly assume $0 < \lambda_f^p \leq \rho_f^p < \infty$ and $0 < \lambda_g^p \leq \rho_g^p < \infty$. Now from Lemma 2.7 for arbitrary $\varepsilon > 0$

$$\log^{[(n-1)p]} T(r, f_n) \leq (\rho_f^p + \varepsilon) \log M(r, g) + O(1) \tag{3.1}$$

when $n$ is even.

Let $0 < \varepsilon <$min$\{1, \lambda_f^p, \lambda_g^p\}$. Since

$$\liminf_{r\to\infty} \frac{T(r, g)}{r^{\lambda_g(r)}} = 1,$$

there is a sequence of values of $r$ tending to infinity for which

$$T(r, g) < (1 + \varepsilon) r^{\lambda_g(r)} \tag{3.2}$$

and for all large values of $r$

$$T(r, g) > (1 - \varepsilon) r^{\lambda_g(r)}. \tag{3.3}$$

Thus for a sequence of values of $r$ tending to infinity we get for any $\delta(>0)$

$$\begin{aligned}\frac{\log M(r,g)}{T(r,g)} &\leq \frac{3T(2r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{1-\varepsilon}\frac{(2r)^{\lambda_g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}}\frac{1}{r^{\lambda_g(r)}} \\ &\leq \frac{3(1+\varepsilon)}{1-\varepsilon}2^{\lambda_g+\delta}\end{aligned}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of $r$.
Since $\varepsilon$, $\delta>0$ be arbitrary, we have

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \leq 3.2^{\lambda_g}. \tag{3.4}$$

Therefore from (3.1) and (3.4) we get

$$\liminf_{r\to\infty}\frac{\log^{[(n-1)p]}T(r,f_n)}{T(r,g)} \leq 3\rho_f^p 2^{\lambda_g}.$$

when $n$ is even.
Again for even $n$ we have from Lemma 2.8

$$\begin{aligned}\log^{[(n-1)p]}T(r,f_n) &\geq (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1) \\ &\geq (\lambda_f^p-\varepsilon)T\left(\frac{r}{4^{n-1}},g\right)+O(1) \\ &\geq (\lambda_f^p-\varepsilon)(1-\varepsilon)(1+O(1))\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}},\ \text{by (3.3).}\end{aligned}$$

Since $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of $r$, we have

$$\log^{[(n-1)p]}T(r,f_n) \geq (\lambda_f^p-\varepsilon)(1-\varepsilon)(1+O(1))\frac{r^{\lambda_g(r)}}{(4^{n-1})^{\lambda_g+\delta}}$$

for all large values of $r$.

So by (3.2) for a sequence of values of $r$ tending to infinity

$$\log^{[(n-1)p]}T(r,f_n) \geq (\lambda_f^p-\varepsilon)\frac{1-\varepsilon}{1+\varepsilon}(1+O(1))\frac{T(r,g)}{(4^{n-1})^{\lambda_g+\delta}}.$$

Since $\varepsilon$ and $\delta$ are arbitrary, it follows from the above that

$$\limsup_{r\to\infty}\frac{\log^{[(n-1)p]}T(r,f_n)}{T(r,g)} \geq \frac{\lambda_f^p}{(4^{n-1})^{\lambda_g}}.$$

Similarly for odd $n$ we get the second part of the theorem.
This proves the theorem. ■

**Theorem 3.2** *Let $f$ and $g$ be two non-constant entire functions such that $\lambda_f^p$ and $\lambda_g^p(>0)$ are finite. Also there exist entire functions $a_i$ $(i=1,2,3..........n;\ n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,g)\}$ as $r\to\infty$ and*

$$\sum_{i=1}^{n}\delta(a_i,g)=1.$$

*Then*

$$\frac{\pi\lambda_f^p}{(4^{n-1})^{\lambda_g}}\leq\limsup_{r\to\infty}\frac{\log^{[(n-1)p]}T(r,f_n)}{T(r,g)}\leq\pi\rho_f^p$$

*when $n$ is even.*

**Proof.** If $\lambda_f^p=0$ then the first inequality is obvious. Now we suppose that $\lambda_f^p>0$. For $0<\varepsilon<\min\{1,\lambda_f^p,\lambda_g^p\}$ we have from Lemma 2.8 for all large values of $r$

$$\begin{aligned}\frac{\log^{[(n-1)p]}\ T(r,f_n)}{T(r,g)} &\geq (\lambda_f^p-\varepsilon)\frac{\log M\left(\frac{r}{4^{n-1}},g\right)}{T(r,g)}+O(1)\quad\text{when } n \text{ is even}\\ &\geq (\lambda_f^p-\varepsilon)\frac{\log M\left(\frac{r}{4^{n-1}},g\right)}{T\left(\frac{r}{4^{n-1}},g\right)}\frac{T\left(\frac{r}{4^{n-1}},g\right)}{T(r,g)}+O(1). \qquad (3.5)\end{aligned}$$

Also from (3.2) and (3.3) we get for a sequence of values of $r\to\infty$ and for $\delta>0$

$$\begin{aligned}\frac{T\left(\frac{r}{4^{n-1}},g\right)}{T(r,g)} &> \frac{1-\varepsilon}{1+\varepsilon}\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}}\frac{1}{r^{\lambda_g(r)}}\\ &\geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{(4^{n-1})^{\lambda_g+\delta}}\end{aligned}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of $r$.
Since $\varepsilon,\ \delta>0$ be arbitrary, so using Lemma 2.6, we have from (3.5)

$$\limsup_{r\to\infty}\frac{\log^{[(n-1)p]}T(r,f_n)}{T(r,g)}\geq\frac{\pi\lambda_f^p}{(4^{n-1})^{\lambda_g}}.$$

If $\rho_f^p=\infty$, the second inequality is obvious. So we may assume $\rho_f^p<\infty$. Then the second inequality follows from Lemma 2.6 and Lemma 2.7.
This proves the theorem. ■

**Theorem 3.3** *Let $f$ and $g$ be two non-constant entire functions such that $\lambda_f^p(>0)$ and $\lambda_g^p$ are finite. Also there exist entire functions $a_i$ $(i=1,2,3..........n;\ n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,f)\}$ as $r\to\infty$ and*

$$\sum_{i=1}^{n}\delta(a_i,f)=1.$$

*Then*

$$\frac{\pi\lambda_g^p}{(4^{n-1})^{\lambda_f}} \leq \limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_n)}{T(r,f)} \leq \pi\rho_g^p$$

*when $n$ is odd.*

**Theorem 3.4** *Let $f$ and $g$ be two non-constant entire functions such that $0 < \lambda_f^p \leq \rho_f^p < \infty$ and $0 < \lambda_g^p \leq \rho_g^p < \infty$. Then for $k = 0, 1, 2, 3, .......$*

$$\frac{\overline{\lambda_g^p}}{\rho_g^p} \leq \liminf_{r\to\infty} \frac{\log^{[np+1]} T(r,f_n)}{\log^{[p]} T(r,g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[np+1]} T(r,f_n)}{\log^{[p]} T(r,g^{(k)})} \leq \frac{\overline{\rho_g^p}}{\lambda_g^p}$$

*when $n$ is even and*

$$\frac{\overline{\lambda_f^p}}{\rho_f^p} \leq \liminf_{r\to\infty} \frac{\log^{[np+1]} T(r,f_n)}{\log^{[p]} T(r,f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[np+1]} T(r,f_n)}{\log^{[p]} T(r,f^{(k)})} \leq \frac{\overline{\rho_f^p}}{\lambda_f^p}$$

*when $n$ is odd, where $f^{(k)}$ denote the k-th derivative of $f$.*

**Proof.** First suppose that $n$ is even. Then for given $\varepsilon(0 < \varepsilon < min\{\lambda_f^p, \lambda_g^p\})$ we get from Lemma 2.8 for all large values of $r$

$$\begin{aligned}
\log^{[(n-1)p]}\ T(r,f_n) &\geq (\lambda_f^p - \varepsilon)\log M\left(\frac{r}{4^{n-1}}, g\right) + O(1) \\
&\geq (\lambda_f^p - \varepsilon) T\left(\frac{r}{4^{n-1}}, g\right) + O(1) \\
\text{i.e., }\ \log^{[np]}\ T(r,f_n) &\geq \log^{[p]} T\left(\frac{r}{4^{n-1}}, g\right) + O(1). \\
\text{So, }\ \log^{[np+1]}\ T(r,f_n) &\geq \log^{[p+1]} T\left(\frac{r}{4^{n-1}}, g\right) + O(1).
\end{aligned}$$

So for all large values of $r$

$$\frac{\log^{[np+1]} T(r,f_n)}{\log^{[p]} T(r,g^{(k)})} \geq \frac{\log^{[p+1]} T\left(\frac{r}{4^{n-1}}, g\right)}{\log \frac{r}{4^{n-1}}} . \frac{\log \frac{r}{4^{n-1}}}{\log^{[p]} T(r,g^{(k)})} + o(1). \tag{3.6}$$

Since

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r,g^{(k)})}{\log r} = \rho_g^p,$$

so for all large values of $r$ and arbitrary $\varepsilon > 0$ we have

$$\log^{[p]} T(r,g^{(k)}) < (\rho_g^p + \varepsilon)\log r. \tag{3.7}$$

Since $\varepsilon > 0$ is arbitrary, so from (3.6) and (3.7) we have

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[np+1]} T(r, f_n)}{\log^{[p]} T(r, g^{(k)})} &\geq \liminf_{r\to\infty} \frac{\log^{[p+1]} T\left(\frac{r}{4^{n-1}}, g\right)}{\log \frac{r}{4^{n-1}}} \cdot \left(\frac{\log r - \log 4^{n-1}}{\rho_g^p \log\ r}\right) \\ &\geq \frac{\overline{\lambda_g^p}}{\rho_g^p}. \end{aligned} \tag{3.8}$$

Again from Lemma 2.7 we get for all large values of $r$

$$\begin{aligned} \log^{[(n-1)p]}\ T(r, f_n) &\leq (\rho_f^p + \varepsilon) \log M(r, g) + O(1) \\ \text{i.e. } \frac{\log^{[np+1]} T(r, f_n)}{\log^{[p]} T(r, g^{(k)})} &\leq \frac{\log^{[p+2]} M(r, g)}{\log^{[p]} T(r, g^{(k)})} + o(1). \end{aligned} \tag{3.9}$$

Since

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, g^{(k)})}{\log r} = \lambda_g^p,$$

so for all large values of $r$ and arbitrary $\varepsilon (0 < \varepsilon < \lambda_g^p)$ we have

$$\log^{[p]} T(r, g^{(k)}) > (\lambda_g^p - \varepsilon) \log r. \tag{3.10}$$

Since $\varepsilon > 0$ is arbitrary, so from (3.9) and (3.10) we have

$$\limsup_{r\to\infty} \frac{\log^{[np+1]} T(r, f_n)}{\log^{[p]} T(r, f^{(k)})} \leq \frac{\overline{\rho_g^p}}{\lambda_g^p}. \tag{3.11}$$

Combining (3.8) and (3.11) we obtain the first part of the theorem.

Similarly when $n$ is odd then we have the second part of the theorem. This proves the theorem. ■

**Theorem 3.5** *Let $f$ and $g$ be two non-constant entire functions such that $0 < \lambda_f^p \leq \rho_f^p < \infty$ and $0 < \lambda_g^p \leq \rho_g^p < \infty$. Then*

$$(i) \quad \frac{\lambda_g^p}{\rho_g^p} \leq \liminf_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, g)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, g)} \leq \frac{\rho_g^p}{\lambda_g^p}$$

*when $n$ is even and*

$$(ii) \quad \frac{\lambda_f^p}{\rho_f^p} \leq \liminf_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, f)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, f)} \leq \frac{\rho_f^p}{\lambda_f^p}$$

*when $n$ is odd.*

**Proof.** First suppose that $n$ is even. Then for given $\varepsilon(0<\varepsilon<min\{\lambda_f^p,\lambda_g^p\})$ we get from Lemma 2.7 and Lemma 2.8 for all large values of $r$

$$\begin{aligned}
\log^{[(n-1)p]}\ T(r,f_n) &\leq (\rho_f^p+\varepsilon)\log M(r,g)+O(1)\\
\text{i.e.}\quad \log^{[np]}\ T(r,f_n) &\leq \log^{[p+1]} M(r,g)+O(1)\\
\text{i.e.}\quad \frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} &\leq \frac{\log^{[p+1]} M(r,g)}{\log^{[p]} T(r,g)}+o(1) \qquad (3.12)\\
\text{i.e.}\ \liminf_{r\to\infty}\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} &\leq 1 \quad \text{[by Lemma 2.4]}. \qquad (3.13)
\end{aligned}$$

Also,

$$\begin{aligned}
\log^{[(n-1)p]}\ T(r,f_n) &\geq (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\
\text{i.e.}\quad \log^{[np]}\ T(r,f_n) &\geq \log^{[p+1]} M\left(\frac{r}{4^{n-1}},g\right)+O(1).
\end{aligned}$$

So

$$\begin{aligned}
\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} &\geq \frac{\log^{[p]} T\left(\frac{r}{4^{n-1}},g\right)}{\log\frac{r}{4^{n-1}}}.\left(\frac{\log r-\log 4^{n-1}}{\rho_g^p\log\ r}\right)+o(1)\\
\text{i.e.}\quad \liminf_{r\to\infty}\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} &\geq \frac{\lambda_g^p}{\rho_g^p}. \qquad (3.14)
\end{aligned}$$

Also from (3.12), we get for all large values of $r$,

$$\begin{aligned}
\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} &\leq \frac{\log^{[p+1]} M(r,g)}{\log r}\frac{\log r}{\log^{[p]} T(r,g)}+o(1)\\
\therefore \limsup_{r\to\infty}\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} &\leq \frac{\rho_g^p}{\lambda_g^p}. \qquad (3.15)
\end{aligned}$$

Again from Lemma 2.8,

$$\begin{aligned}
\log^{[(n-1)p]}\ T(r,f_n) &\geq (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\
\text{i.e.}\quad \log^{[np]}\ T(r,f_n) &\geq \log^{[p+1]} M\left(\frac{r}{4^{n-1}},g\right)+O(1). \qquad (3.16)
\end{aligned}$$

From (3.3) we obtain for all large values of $r$ and for $\delta>0$ and $\varepsilon(0<\varepsilon<1)$

$$\begin{aligned}
\log M\left(\frac{r}{4^{n-1}},g\right) &> (1-\varepsilon)\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4^{n-1}}\right)}}\\
&\geq \frac{1-\varepsilon}{(4^{n-1})^{\lambda_g+\delta}}r^{\lambda_g(r)}
\end{aligned}$$

because $r^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of $r$.

So by (3.2) we get for a sequence of values of $r$ tending to infinity

$$\begin{aligned} \log M\left(\frac{r}{4^{n-1}}, g\right) &\geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{(4^{n-1})^{\lambda_g+\delta}}T(r,g) \\ \text{i.e. } \log^{[2]} M\left(\frac{r}{4^{n-1}}, g\right) &\geq \log T(r,g) + O(1). \end{aligned}$$

Taking repeated logarithms (p-1) times, we get

$$\log^{[p+1]} M\left(\frac{r}{4^{n-1}}, g\right) \geq \log^{[p]} T(r,g) + O(1). \tag{3.17}$$

Now from (3.16) and (3.17)

$$\limsup_{r\to\infty} \frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,f)} \geq 1. \tag{3.18}$$

So the theorem follows from (3.13), (3.14), (3.15) and (3.18) when $n$ is even. Similarly when n is odd we get (ii). ■

**Corollary 3.6** *Using the hypothesis of Theorem 3.5 if $f$ and $g$ are of regular growth then*

$$\lim_{r\to\infty} \frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} = \lim_{r\to\infty} \frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,f)} = 1.$$

**Remark 3.7** *The conditions $\lambda_f^p, \lambda_g^p > 0$ and $\rho_{f,}^p\ \rho_g^p < \infty$ are necessary for Theorem 3.5 and Corollary 3.6, which are shown by the following examples.*

**Example 3.8** *Let $f = z, g = \exp^{[p]} z$. Then $\lambda_f^p = \rho_f^p = 0$ and $0 < \lambda_g^p = \rho_g^p < \infty$. Now when $n$ is even then*

$$f_n = \exp^{[\frac{np}{2}]} z.$$

*Therefore,*

$$T(r,f_n) \leq \log M(r,f_n) = \exp^{[\frac{np}{2}-1]} r.$$

*So,*

$$\begin{aligned} \log^{[np]} T(r,f_n) &\leq \log^{[np]}(\exp^{[\frac{np}{2}-1]} r) \\ &= \log^{[np-\frac{np}{2}+1]} r \\ &= \log^{[\frac{np}{2}+1]} r. \end{aligned}$$

*Also when $n$ is odd*

$$f_n = \exp^{[(\frac{n-1}{2})p]} z.$$

*Therefore,*

$$T(r,f_n) \le \log M(r,f_n) = \exp^{[(\frac{n-1}{2})p-1]} r.$$

*So,*

$$\begin{aligned} \log^{[np]} T(r,f_n) &\le \log^{[np]}(\exp^{[(\frac{n-1}{2})p-1]} r) \\ &= \log^{[np-(\frac{n-1}{2})p+1]} r \\ &= \log^{[(\frac{n+1}{2})p+1]} r \end{aligned}$$

*Now*

$$\log^{[p]} T(r,f) = \log^{[p+1]} r$$

*and*

$$\begin{aligned} 3T(2r,g) &\ge \log M(r,g) = \exp^{[p-1]} r \\ \text{i.e.} \quad \log^{[p]} T(r,g) &\ge \log r + O(1). \end{aligned}$$

*Therefore when $n$ is even*

$$\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,g)} \le \frac{\log^{[\frac{np}{2}+1]} r}{\log r + O(1)} \to 0 \quad as \quad r \to \infty,$$

*and when $n$ is odd*

$$\frac{\log^{[np]} T(r,f_n)}{\log^{[p]} T(r,f)} \le \frac{\log^{[(\frac{n+1}{2})p+1]} r}{\log^{[p+1]} r} \to 0 \quad as \quad r \to \infty.$$

**Example 3.9** *Let $f = \exp^{[2p]} z, g = \exp^{[p]} z$. Then $\lambda_f^p = \rho_f^p = \infty, \quad \lambda_g^p = \rho_g^p = 1$.*

*Now when $n$ is even*

$$f_n = \exp^{[\frac{3np}{2}]} z.$$

*Therefore*

$$\begin{aligned} 3T(2r,f_n) &\ge \log M(r,f_n) = \exp^{[\frac{3np}{2}-1]} r \\ \text{i.e.} \quad T(r,f_n) &\ge \frac{1}{3} \exp^{[\frac{3np}{2}-1]} \frac{r}{2} \\ \therefore \quad \log^{[np]} T(r,f_n) &\ge \log^{[np]}(\exp^{[\frac{3np}{2}-1]} \frac{r}{2}) + o(1) \\ &= \exp^{[\frac{np}{2}-1]} \frac{r}{2} + o(1). \end{aligned}$$

*Also when $n$ is odd*

$$f_n = \exp^{[(\frac{3n+1}{2})p]} z.$$

*Therefore*

$$\begin{aligned} 3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[(\frac{3n+1}{2})p-1]} r \\ i.e. \quad T(r, f_n) &\geq \frac{1}{3}\exp^{[(\frac{3n+1}{2})p-1]} \frac{r}{2} \\ \therefore \quad \log^{[np]} T(r, f_n) &\geq \log^{[np]}(\exp^{[(\frac{3n+1}{2})p-1]} \frac{r}{2}) + o(1) \\ &= \exp^{[(\frac{n+1}{2})p-1]} \frac{r}{2} + o(1). \end{aligned}$$

*Also*

$$\log^{[p]} T(r, f) \leq \exp^{[p-1]} r \ \text{ and } \ \log^{[p]} T(r, g) \leq \log r.$$

*Therefore when $n$ is even*

$$\frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, g)} \geq \frac{\exp^{[\frac{np}{2}-1]} \frac{r}{2} + o(1)}{\log r} \to \infty \quad as \quad r \to \infty,$$

*and when $n$ is odd*

$$\frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, f)} \geq \frac{\exp^{[(\frac{n+1}{2})p-1]} \frac{r}{2} + o(1)}{\exp^{[p-1]} r} \to \infty \quad as \quad r \to \infty.$$

**Theorem 3.10** *Let $f$ and $g$ be two entire functions such that $0 < \lambda_f^p \leq \rho_f^p < \infty$ and $0 < \lambda_g^p \leq \rho_g^p < \infty$. Then for $k = 0, 1, 2, 3, ......$*

$$(i) \quad \frac{\lambda_g^p}{\rho_f^p} \leq \liminf_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, f^{(k)})} \leq \frac{\rho_g^p}{\lambda_f^p}$$

*when $n$ is even.*

$$(ii) \quad \frac{\lambda_f^p}{\rho_g^p} \leq \liminf_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[np]} T(r, f_n)}{\log^{[p]} T(r, g^{(k)})} \leq \frac{\rho_f^p}{\lambda_g^p}$$

*when $n$ is odd.*

**Proof.** First suppose that $n$ is even. Then for given $\varepsilon(0 < \varepsilon < min\{\lambda_f^p, \lambda_g^p\})$ we have from Lemma 2.7 for all large values of $r$,

$$\begin{aligned} \log^{[(n-1)p]} \ T(r, f_n) &\leq (\rho_f^p + \varepsilon)\log M(r, g) + O(1) \\ \text{i.e.} \quad \log^{[np]} \ T(r, f_n) &\leq \log^{[p+1]} M(r, g) + O(1). \end{aligned}$$

Also we know that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, g^{(k)})}{\log r} = \lambda_g^p.$$

Now

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{[np]}T(r,f_n)}{\log^{[p]}T(r,f^{(k)})} &\le \limsup_{r\to\infty}\frac{\log^{[p+1]}M(r,g)}{\log^{[p]}T(r,f^{(k)})}\\ &\le \limsup_{r\to\infty}\left[\frac{\log^{[p+1]}M(r,g)}{\log r}.\frac{\log r}{\log^{[p]}T(r,f^{(k)})}\right]\\ &= \frac{\rho_g^p}{\lambda_f^p} \qquad (3.19)\end{aligned}$$

Again from Lemma 2.8 we have for all large values of $r$,

$$\begin{aligned}\log^{[(n-1)p]}\ T(r,f_n) &\ge (\lambda_f^p-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ \text{i.e.,}\quad \log^{[np]}\ T(r,f_n) &\ge \log^{[p+1]}M\left(\frac{r}{4^{n-1}},g\right)+O(1)\\ &\ge (\lambda_g^p-\varepsilon)\log r+O(1).\end{aligned}$$

Also

$$\log^{[p]}T(r,f^{(k)})<(\rho_f^p+\varepsilon)\log r.$$

Therefore,

$$\frac{\log^{[np]}T(r,f_n)}{\log^{[p]}T(r,f^{(k)})}\ge\frac{(\lambda_g^p-\varepsilon)\log\ r+O(1)}{(\rho_f^p+\varepsilon)\log r}.$$

Since $\varepsilon>0$ is arbitrary we get

$$\liminf_{r\to\infty}\frac{\log^{[np]}T(r,f_n)}{\log^{[p]}T(r,f^{(k)})}\ge\frac{\lambda_g^p}{\rho_f^p}. \qquad (3.20)$$

Therefore from (3.19) and (3.20) we have the result for even $n$.

Similarly for odd $n$ we have (ii).

This proves the theorem. ■

## References


[1] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, (1970), 75-92.

[2] D. Banerjee and R. K. Dutta, *The growth of iterated entire functions*, Bulliten of Mathematical Analysis and Applications, Volume **3(3)** (2011), 35-49.

[3] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[4] B. K. Lahiri and D. Banerjee, *Relative fix points of entire functions*, J. Indian Acad. Math., **19(1)** (1997), 87-97.

[5] I. Lahiri, *Generalised proximate order of meromorphic functions*, Matematnykn Bechnk, **41** (1989), 9-16.

[6] I. Lahiri, *Growth of composite integral functions*, Indian J. Pure and Appl. Math., **20(9)** (1989), 899-907.

[7] I. Lahiri and S. K. Datta, *On the growth of composite entire and meromorphic functions*, Indian J. Pure and Appl. Math., **35(4)** (2004), 525-543.

[8] Q. Lin and C. Dai, *On a conjecture of Shah concerning small functions*, Kexue Tong (English Ed.), **31(4)** (1986), 220-224.

[9] K. Niino and C. C. Yang, *Some growth relationships on factors of two composite entire functions*, Factorization Theory of Meromorphic Functions and Related Topics, Marcel Dekker Inc. (New York and Basel), (1982), 95-99.

[10] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., **69** (1963), 411-414.

[11] A. P. Singh, *Growth of composite entire functions*, Kodai Math. J., **8** (1985), 99-102.

[12] G. Valiron, *Lectures on the general theory of Integral functions, Chelsea Publishing Company*, 1949.

[13] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers and Science Press, Beijing, 2003.

Ratan Kumar Dutta
Department of Mathematics
Patulia High School
Patulia, Kolkata-119
West bengal
India
E-mail: *ratan_3128@yahoo.com*