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HARMONIC UNIFORMLY $\beta-$STARLIKE FUNCTIONS DEFINED BY CONVOLUTION AND INTEGRAL CONVOLUTION

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Abstract. In this paper, we introduce and study a subclass of harmonic univalent functions defined by convolution and integral convolution. Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combination are determined for functions in this family. Consequently, many of our results are either extensions or new approaches to those corresponding to previously known results.

Keywords and Phrases: Harmonic functions, analytic functions, univalent functions, starlike domain, convex domain, convolution.

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1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the family of functions $f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = \sum_{k=1}^{\infty} b_k z^k \;\; (|b_1| < 1), \tag{1}$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (|b_1| < 1). \tag{2}$$

We note that the family $\mathcal{S_H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero ($g \equiv 0$).

Also, we denote by $T\mathcal{S_H}$ the subfamily of $\mathcal{S_H}$ consisting of harmonic functions of the form $f = h + \overline{g}$ such that h and g are of the form:

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{3}$$

In [3] Clunie and Sheil-Small, investigated the class $\mathcal{S_H}$ as well as its geometric subclasses and its properties. Since then, there have been several studies related to the class $\mathcal{S_H}$ and its subclasses. Following Clunie and Sheil-Small [3], Frasin [7, 8], Jahangiri [10, 11], Silverman [13], Silverman and Silvia [14], Dixit et al. [4, 5, 6] and others have investigated various subclasses of $\mathcal{S_H}$ and its properties.

Recently, Rosy et al. [12], defined a subclass $\mathcal{G_H}(\gamma) \subset \mathcal{S_H}$ consisting of harmonic univalent functions $f(z)$ satisfying the following condition

$$Re\left\{(1+e^{i\alpha})\frac{zf'(z)}{z'f(z)} - e^{i\alpha}\right\} \geq \gamma, \ \ 0 \leq \gamma < 1, \ \ \alpha \in R, \tag{4}$$

were $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta}))$, $0 \leq r < 1$ and θ is real. They proved that if $f = h + \overline{g}$ given by (2) and if

$$\sum_{k=2}^{\infty} \frac{2k-1-\gamma}{1-\gamma}|a_k| + \sum_{k=1}^{\infty} \frac{2k+1+\gamma}{1-\gamma}|b_k| \leq 1, \ \ 0 \leq \gamma < 1, \tag{5}$$

then f is a Goodman-Ronning type harmonic univalent function in $\mathcal{U}$. This condition is proved to be also necessary if h and g are of the form (3).

The convolution of two power series

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k, \ \text{and} \ \Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \tag{6}$$

is defined by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \lambda_k \mu_k z^k, \tag{7}$$

where $\lambda_k \geq 0$ and $\mu_k \geq 0$. Also the integral convolution is defined by

$$(\Phi \diamond \Psi)(z) = z + \sum_{k=2}^{\infty} \frac{\lambda_k \mu_k}{k} z^k. \tag{8}$$

Motivated by the works of Ahuja [1], Dixit et al. [4, 5], Frasin and Murugusundaramoorthy [8] and Rosy et al. [12], we consider the subclass $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, t)$ of functions of the form (2) satisfying the condition

$$\Re\left((1+\beta e^{i\alpha})\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}} - \beta e^{i\alpha}\right) > \gamma, \tag{9}$$

where $\beta \geq 0$, $0 \leq \gamma < 1$, $\alpha \in R$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \leq t \leq 1$, $\Phi(z)$ and $\Psi(z)$ are of the form (6). We further let $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$ denote the subclass of $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi, \beta, \gamma, t)$ consisting of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ such that h and g are of the form (3).

We note that by specializing the functions Φ, Ψ and parameters β, γ and t we obtain well-known harmonic univalent functions as well as many new ones. For example, $\mathcal{G}_{\mathcal{H}}(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}; 1, \gamma, 1) = \mathcal{HC}(\gamma)$ studied by Kim et al. [9] and $\mathcal{G}_{\mathcal{H}}(z + \sum\limits_{k=2}^{\infty} k^{n+1} z^k, z + \sum\limits_{k=2}^{\infty} k^{n+1} z^k; 1, \gamma, 1) = RS(\gamma)$ studied by Yalcin et al. [15]. Also, $\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, \gamma, 1) = G_{\mathcal{H}}(\gamma)$ studied by Rosy et al. [12], $\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \beta, \gamma; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\beta, \gamma; t)$ was considered by Ahuja et al. [1]. Further, the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; 0, \gamma, 1) = \overline{\mathcal{HS}}(\Phi, \Psi; \gamma)$ studied by Dixit et al. [4], $\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, \gamma, 1) = S^*_H(\gamma)$ and $\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}; 0, \gamma, 1) = K(\gamma)$ were introduced and studied by Jahangiri [11]. For $\gamma = 0$ the classes $S^*_{\mathcal{H}}(\gamma)$ and $K(\gamma)$ were studied by Silverman and Silvia [14], for $\gamma = 0$ and $b_1 = 0$ see [2, 13].

In this paper, we give a sufficient condition for $f = h + \bar{g}$ given by (2) to be in $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, t)$ and it is shown that this condition is also necessary for functions in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$. We also obtain extreme points, distortion bounds, convolution and convex combination properties. Further, we obtain the closure property of this class under integral operator. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $\mathcal{S}_{\mathcal{H}}$ as listed previously in this section.

2. COEFFICIENT BOUNDS

Our first theorem gives a sufficient condition for functions in $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, t)$.

Theorem 1 *Let $f = h + \bar{g}$ be so that h and g are given by (1). If*

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta) - t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_k| + \sum_{k=1}^{\infty} \frac{[k(1+\beta) + t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_k| \leq 1, \tag{10}$$

where $\beta \geq 0$, $0 \leq \gamma < 1$, $0 \leq t \leq 1$, $k^2(1-\gamma) \leq [k(1+\beta) - t(\beta+\gamma)]\lambda_k$ and $k^2(1-\gamma) \leq [k(1+\beta) + t(\beta+\gamma)]\mu_k$ for $k \geq 2$. Then $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, t)$.

Proof. To prove that $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, t)$, we only need to show that if (10) holds, then the required condition (9) is satisfied. For (9), we can write

$$\Re\left\{(1+\beta e^{i\alpha})\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}} - \beta e^{i\alpha}\right\} = \Re\frac{A(z)}{B(z)} > \gamma.$$

Using the fact that $\Re\{\omega\} \geq \gamma$ if and only if $|1-\gamma+\omega| \geq |1+\gamma-\omega|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0, \tag{11}$$

where

$$A(z) = (1+\beta e^{i\alpha})[h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}] - \beta e^{i\alpha}[h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}]$$

and

$$B(z) = h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}.$$

Substituting for $A(z)$ and $B(z)$ in (11) and making use of (10), we obtain

$$\begin{aligned}
&|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \\
&\quad = \Big|(1+\beta e^{i\alpha})[h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}] - \beta e^{i\alpha}[h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}] \\
&\qquad\quad + (1-\gamma)\left(h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}\right)\Big| \\
&\qquad - \Big|(1+\beta e^{i\alpha})[h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}] - \beta e^{i\alpha}[h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}] \\
&\qquad\quad - (1+\gamma)\left(h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}\right)\Big|
\end{aligned}$$

$$\begin{aligned}
&\geq 2(1-\gamma)|z| - \sum_{k=2}^{\infty} 2\left[\frac{k(1+\beta)-t(\beta+\gamma)}{k}\right]\lambda_k|a_k||z|^k \\
&\quad - \sum_{k=1}^{\infty} 2\left[\frac{k(1+\beta)+t(\beta+\gamma)}{k}\right]\mu_k|b_k||z|^k \\
&= 2(1-\gamma)|z|\left\{1 - \sum_{k=2}^{\infty}\left[\frac{k(1+\beta)-t(\beta+\gamma)}{k(1-\gamma)}\right]\lambda_k|a_k||z|^{k-1}\right. \\
&\quad \left. - \sum_{k=1}^{\infty}\left[\frac{k(1+\beta)+t(\beta+\gamma)}{k(1-\gamma)}\right]\mu_k|b_k||z|^{k-1}\right\} \\
&> 2(1-\gamma)\left\{1 - \sum_{k=2}^{\infty}\left[\frac{k(1+\beta)-t(\beta+\gamma)}{k(1-\gamma)}\right]\lambda_k|a_k| - \sum_{k=1}^{\infty}\left[\frac{k(1+\beta)+t(\beta+\gamma)}{k(1-\gamma)}\right]\mu_k|b_k|\right\} \\
&\geq 0
\end{aligned}$$

which implies that $f \in \mathcal{G}_{\mathcal{H}}(\Phi,\Psi;\beta,\gamma,t)$.

The coefficient bound (10) is sharp for the harmonic function

$$f(z) = z + \sum_{k=2}^{\infty}\frac{k(1-\gamma)}{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}x_k z^k + \sum_{k=1}^{\infty}\frac{k(1-\gamma)}{[k(1+\beta)+t(\beta+\gamma)]\mu_k}\overline{y_k z^k},$$

where $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, shows that the coefficient bound given by (10) is sharp.

Next, we show that the above sufficient condition is also necessary for functions in the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$.

Theorem 2. *Let $f = h + \bar{g}$ be so that h and g are given by (3). Then $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ if and only if*

$$\sum_{k=2}^{\infty}\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_k| + \sum_{k=1}^{\infty}\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_k| \leq 1, \qquad (12)$$

where $\beta \geq 0$, $0 \leq \gamma < 1$, $0 \leq t \leq 1$, $k^2(1-\gamma) \leq [k(1+\beta)-t(\beta+\gamma)]\lambda_k$ and $k^2(1-\gamma) \leq [k(1+\beta)+t(\beta+\gamma)]\mu_k$ for $k \geq 2$.

Proof. Since $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t) \subset \mathcal{G}_{\mathcal{H}}(\Phi,\Psi;\beta,\gamma,t)$, we only need to prove the only if part of the theorem. To this end, for functions f of the form (3), we notice that the condition (9) is equivalent to

$$\Re\left\{\frac{(1+\beta e^{i\alpha})[h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}] - \beta e^{i\alpha}[h_t(z)\diamond\Phi(z) + \overline{g_t(z)\diamond\Psi(z)}]}{h_t(z)\diamond\Phi(z) + \overline{g_t(z)\diamond\Psi(z)}} - \gamma\right\} \geq 0.$$

Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum\limits_{k=2}^{\infty}(k-\gamma t)\frac{\lambda_k}{k}|a_k|r^{k-1}-\sum\limits_{k=1}^{\infty}(k+\gamma t)\frac{\mu_k}{k}|b_k|r^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}}\right\}$$
$$-\Re\left\{\beta e^{i\alpha}\frac{\sum\limits_{k=2}^{\infty}(k-t)\frac{\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}(k+t)\frac{\mu_k}{k}|b_k|r^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}}\right\}\geq 0.$$

Since $Re(-e^{i\alpha}) \geq -|e^{i\alpha}| = -1$, the above inequality reduces to

$$\frac{(1-\gamma)-\sum\limits_{k=2}^{\infty}[k(1+\beta)-t(\beta+\gamma)]\frac{\lambda_k}{k}|a_k|r^{k-1}-\sum\limits_{k=1}^{\infty}[k(1+\beta)+t(\beta+\gamma)]\frac{\mu_k}{k}|b_k|r^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}}\geq 0. \tag{13}$$

If the condition (12) does not hold then the numerator in (13) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (13) is negative. This contradicts the condition for $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$. Hence the proof is complete.

3. Extreme points and distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$.

Theorem 3. *Let f be given by (3). Then $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_k(z)), \tag{14}$$

where $h_1(z) = z$, $h_k(z) = z - \frac{k(1-\gamma)}{[k(1+\beta)-t(\beta+\gamma)]\lambda_k} z^k$ $(k = 2, 3, ...)$, $g_k(z) = z + \frac{k(1-\gamma)}{[k(1+\beta)+t(\beta+\gamma)]\mu_k} \overline{z}^k$ $(k = 1, 2, 3, ...)$, $\sum\limits_{k=1}^{\infty}(X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular, the extreme points of $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (14), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\left(X_k h_k(z)+Y_k g_k(z)\right) \\ &= \sum_{k=1}^{\infty}\left(X_k+Y_k\right)z-\sum_{k=2}^{\infty}\frac{k(1-\gamma)}{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}X_k z^k \\ &\qquad +\sum_{k=1}^{\infty}\frac{k(1-\gamma)}{[k(1+\beta)+t(\beta+\gamma)]\mu_k}Y_k\overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}\left(\frac{k(1-\gamma)}{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}\right)X_k \\ &\quad+\sum_{k=1}^{\infty}\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}\left(\frac{k(1-\gamma)}{[k(1+\beta)+t(\beta+\gamma)]\mu_k}\right)Y_k \\ &=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1 \end{aligned}$$

and so $f\in clco\ \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. Conversely, suppose that $f\in clco\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. Setting

$$X_k=\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_k|,\ \ k=2,3,\ldots,$$

and

$$Y_k=\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_k|,\ \ k=1,2,\ldots,$$

where $\sum\limits_{k=1}^{\infty}(X_k+Y_k)=1$. Then note that by Theorem , $0\leq X_k\leq 1$ $(k=2,3,\ldots)$ and $0\leq Y_k\leq 1$ $(k=1,2,3,\ldots)$. We define $X_1=1-\sum\limits_{k=2}^{\infty}X_k-\sum\limits_{k=1}^{\infty}Y_k$ and by Theorem , $X_1\geq 0$. Consequently, we obtain $f(z)=\sum\limits_{k=1}^{\infty}\left(X_k h_k(z)+Y_k g_k(z)\right)$.

Using Theorem , it is easily seen that $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ is convex and closed, so $clco\ \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)=\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. In other words, the statement of Theorem is really for $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$.

The following theorem gives the distortion bounds for functions in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ which yields a covering result for this class.

Theorem 4. *Let $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ and $A\leq[k(1+\beta)-t(\beta+\gamma)]\frac{\lambda_k}{k}$, $A\leq[k(1+\beta)+$*

$t(\beta+\gamma)]\frac{\mu_k}{k}$ *for* $k \geq 2$, *where* $A = \min\left\{[2(1+\beta)-t(\beta+\gamma)]\frac{\lambda_2}{2},\ [2(1+\beta)+t(\beta+\gamma)]\frac{\mu_2}{2}\right\}$ *then*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\gamma}{A} - \frac{(1+\beta)+t(\beta+\gamma)}{A}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\gamma}{A} - \frac{(1+\beta)+t(\beta+\gamma)}{A}|b_1|\right)r^2.$$

Proof.

Let $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$. Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&= (1+|b_1|)r + \frac{1-\gamma}{A}r^2\sum_{k=2}^{\infty}\left(\frac{A}{1-\gamma}|a_k| + \frac{A}{1-\gamma}|b_k|\right) \\
&\leq (1+|b_1|)r + \frac{1-\gamma}{A}r^2\sum_{k=2}^{\infty}\left(\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_k|\right. \\
&\qquad \left. + \frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_k|\right) \\
&\leq (1+|b_1|)r + \frac{1-\gamma}{A}\left(1 - \frac{(1+\beta)+t(\beta+\gamma)}{(1-\gamma)}|b_1|\right)r^2 \\
&= (1+|b_1|)r + \left(\frac{1-\gamma}{A} - \frac{(1+\beta)+t(\beta+\gamma)}{A}|b_1|\right)r^2
\end{aligned}$$

and similarly,

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\gamma}{A} - \frac{(1+\beta)+t(\beta+\gamma)}{A}|b_1|\right)r^2.$$

The upper and lower bounds given in Theorem are respectively attained for the following functions.

$$f(z) = z + |b_1|\bar{z} + \left(\frac{1-\gamma}{A} - \frac{(1+\beta)+t(\beta+\gamma)}{A}|b_1|\right)\bar{z}^2$$

and

$$f(z)=(1-|b_1|)z-\left(\frac{1-\gamma}{A}-\frac{(1+\beta)+t(\beta+\gamma)}{A}|b_1|\right)z^2.$$

The following covering result follows from the left hand inequality in Theorem .

Corollary Let f of the form (3) be so that $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ and $A\leq[k(1+\beta)-t(\beta+\gamma)]\frac{\lambda_k}{k}$, $A\leq[k(1+\beta)+t(\beta+\gamma)]\frac{\mu_k}{k}$ for $k\geq 2$, where $A=\min\left\{[2(1+\beta)-t(\beta+\gamma)]\frac{\lambda_2}{2},\ [2(1+\beta)+t(\beta+\gamma)]\frac{\mu_2}{2}\right\}$. Then

$$\left\{\omega:|\omega|<\frac{A+1-\gamma}{A}+\frac{A-1+\gamma}{A}|b_1|\right\}\subset f(\mathcal{U}).$$

4. CONVOLUTION AND CONVEX COMBINATIONS

In this section we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|z^k+\sum\limits_{k=1}^{\infty}|b_k|\overline{z}^k$ and $F(z)=z-\sum\limits_{k=2}^{\infty}|A_k|z^k+\sum\limits_{k=1}^{\infty}|B_k|\overline{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f*F)(z)=f(z)*F(z)=z-\sum_{k=2}^{\infty}|a_k||A_k|z^k+\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k. \tag{15}$$

Using the definition, we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ is closed under convolution.

Theorem 5. *For $0\leq\gamma<1$, let $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ and $F\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. Then $f*F\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$.*

Proof. Let $f(z)=z-\sum\limits_{k=2}^{\infty}|a_k|z^k+\sum\limits_{k=1}^{\infty}|b_k|\overline{z}^k$ and $F(z)=z-\sum\limits_{k=2}^{\infty}|A_k|z^k+\sum\limits_{k=1}^{\infty}|B_k|\overline{z}^k$

be in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. Then the convolution $f*F$ is given by (15). We wish to show that the coefficient of $f*F$ satisfy the required condition given in Theorem . For $F\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$, we note that $|A_k|\leq 1$ and $|B_k|\leq 1$. Now for the convolution function $f*F$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_k||A_k|+\sum_{k=1}^{\infty}\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_k||B_k|\\
&\leq\sum_{k=2}^{\infty}\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_k|+\sum_{k=1}^{\infty}\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_k|\\
&\leq 1,
\end{aligned}$$

since $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. Therefore $f*F\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$. Next, we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ is closed under convex combination of its members.

Theorem 6. *The class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$ is closed under convex combination.*

Proof. For $i=1,2,3,\dots$ let $f_i(z)\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,t)$, where f_i is given by

$$f_i(z)=z-\sum_{k=2}^{\infty}|a_{ik}|z^k+\sum_{k=1}^{\infty}|b_{ik}|\overline{z}^k.$$

For $\sum\limits_{i=1}^{\infty}t_i=1, 0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{ik}|\right)z^k+\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{ik}|\right)\overline{z}^k.$$

Since,

$$\sum_{k=2}^{\infty}\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}|a_{ik}|+\sum_{k=1}^{\infty}\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}|b_{ik}|\leq 1,$$

from the above equation we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[k(1+\beta)-t(\beta+\gamma)]\lambda_k}{k(1-\gamma)}\sum_{i=1}^{\infty}t_i|a_{ik}|+\sum_{k=1}^{\infty}\frac{[k(1+\beta)+t(\beta+\gamma)]\mu_k}{k(1-\gamma)}\sum_{i=1}^{\infty}t_i|b_{ik}|\\
&=\sum_{i=1}^{\infty}t_i\left\{\sum_{k=2}^{\infty}\frac{[(k-t)(1+\beta)+k(1-\gamma)t]}{k(1-\gamma)}|a_{ik}|+\sum_{k=1}^{\infty}\frac{[(k+t)(1+\beta)-k(1-\gamma)t]}{k(1-\gamma)}|b_{ik}|\right\}\\
&\leq\sum_{i=1}^{\infty}t_i=1.
\end{aligned}$$

This is the condition required by (12) and so $\sum\limits_{i=1}^{\infty} t_i f_i(z) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$.

5. CLASS PRESERVING INTEGRAL OPERATOR

Finally, we consider the closure property of the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c[f(z)]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int\limits_0^z \xi^{c-1} f(\xi) d\xi \quad (c > -1).$$

Theorem 7. *Let $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$, then $\mathcal{L}_c[f(z)] \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$.*

Proof. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\begin{aligned}
\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c} \int\limits_0^z \xi^{c-1} h(\xi) d\xi + \overline{\frac{c+1}{z^c} \int\limits_0^z \xi^{c-1} g(\xi) d\xi} \\
&= \frac{c+1}{z^c} \int\limits_0^z \xi^{c-1} \left(\xi - \sum_{k=2}^{\infty} |a_k| \xi^k \right) d\xi + \overline{\frac{c+1}{z^c} \int\limits_0^z \xi^{c-1} \left(\sum_{k=1}^{\infty} |b_k| \xi^k \right) d\xi} \\
&= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,
\end{aligned}$$

where $A_k = \frac{c+1}{c+k} |a_k|$ and $B_k = \frac{c+1}{c+k} |b_k|$. Hence

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[k(1+\beta) - t(\beta+\gamma)]\lambda_k}{k(1-\gamma)} \left(\frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{[k(1+\beta) + t(\beta+\gamma)]\mu_k}{k(1-\gamma)} \left(\frac{c+1}{c+k} |b_k| \right) \\
&\quad \leq \sum_{k=2}^{\infty} \frac{[k(1+\beta) - t(\beta+\gamma)]\lambda_k}{k(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \frac{[k(1+\beta) + t(\beta+\gamma)]\mu_k}{k(1-\gamma)} |b_k| \\
&\quad \leq 1,
\end{aligned}$$

since $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$, therefore by Theorem , $\mathcal{L}_c(f(z)) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$.

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