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THE EXPONENTIAL MAP AND THE EUCLIDEAN ISOMETRIES

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Abstract. In the first section the basic properties of the exponential map of a Lie group are reviewed. The second section contains the Tarence Tao proof to the property that every compact connected Lie group is exponential. A direct specific proof to this property in the case of the special orthogonal group $SO(n)$, $n = 2$ and $n = 3$ is also presented. In the last section this property is used to describe the Euclidean isometries of the space $\mathbb{R}^n$, when $n = 2$ and $n = 3$.

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The exponential map of a Lie group

Let G be a Lie group with its Lie algebra $\mathfrak{g}$. It is well known that the exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma_X(1)$, where $X \in \mathfrak{g}$ and γ_X is the one-parameter subgroup of G induced by X. Recall the following properties of the exponential map:

1) For any $t \in \mathbb{R}$ and for any $X \in \mathfrak{g}$ we have $\gamma_X(t) = \exp(tX)$;

2) For any $s, t \in \mathbb{R}$ and for any $X \in \mathfrak{g}$, we have $\exp(sX)\exp(tX) = \exp(s+t)X$;

3) For any $t \in \mathbb{R}$ and for any $X \in \mathfrak{g}$, we have $\exp(-tX) = (\exp tX)^{-1}$;

4) $\exp : \mathfrak{g} \to G$ is a smooth mapping, it is a local diffeomorphism at $0 \in \mathfrak{g}$ and $\exp(0) = e$, where e is the unity element of the group G;

5) The image $\exp(\mathfrak{g})$ of the exponential map generates the connected component G_e of the unity $e \in G$;

6) If $f : G_1 \to G_2$ if a morphism of Lie groups and $f_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is the induced morphism of Lie algebras by f, then $f \circ \exp_1 = \exp_2 \circ f_*$.

As we can note from the previous property 5), the following two problems are of special importance:

Problem 1. *Find conditions on the group G such that the exponential map is surjective.*

Problem 2. *Determine the image $E(G)$ of the exponential map.*

J. Dixmier has proposed first time the Problem 2 for resoluble Lie groups. Concerning Problem 1, only in few special situations we have $G = E(G)$, i.e. the surjectivity of the exponential map. A Lie group satisfying this property is called *exponential*. A monograph devoted to the study of such Lie groups is [7].

2.Every compact connected Lie group is exponential

The standard proof of this property ([1], [3]) is to use the Cartan conjugacy, then the surjectivity of the exponential map for a torus and the fact that every element of the compact connected Lie group is obtained in a maximal torus.

We shall present the recent Tarence Tao [6] idea to prove the stated general property by connecting on a Lie group the Riemannian exponential map of a manifold with the Lie exponential map. Let G be a compact connected Lie group endowed with a bi-invariant Riemannian metric. Because G is connected and compact, it is complete, hence we can apply the Hopf-Rinow theorem to conclude that any two points are connected by at least one geodesic. That is the Riemannian exponential map $\exp_R : \mathfrak{g} \to G$, is surjective. But, on the other hand, one can check that the Lie exponential map $\exp : \mathfrak{g} \to G$ and the Riemannian exponential map $\exp_R : \mathfrak{g} \to G$, agree. This property can be seen by observing that the group structure naturally defines a connection on the tangent bundle which is both torsion-free and it preserves the bi-invariant metric, hence it must agree with the Levi-Civita metric

It is well-known that the Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group $SO(n)$ consists in all skew-symmetric matrices in $M_n(\mathbb{R})$, and the Lie bracket is the standard commutator of matrices defined by $[A, B] = AB - BA$.

The exponential map $\exp : \mathfrak{so}\,(n) \to SO(n)$ is defined by

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

According to the well-known Hamilton-Cayley theorem, it follows that every power $X^k, k \geq n$, is a linear combination of $X^0, X^1, \cdots, X^{n-1}$, hence we can write

$$\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k,$$

where the real coefficients $a_0(X), \cdots, a_{n-1}(X)$ depend only on the matrix X. From this formula, it follows that $\exp(X)$ is a polynomial of X. The problem to find a resonable formula for $\exp(X)$ is reduced to the determination of the coefficients $a_0(X), \cdots, a_{n-1}(X)$. We will call this general question, the *Rodrigues problem.* The general problem involving power series of matrices is stated and studied in the paper [2].

When $n = 2$, a skew-symmetric matrix $B \in \mathfrak{so}(2)$ can be written as $B = \theta J$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and from the Hamilton-Cayley relation $J^2 = -I_2$ and the series expansion of $\sin\theta$ and $\cos\theta$ it is easy to show that:

$$e^B = e^{\theta J} = (\cos\theta)I_2 + (\sin\theta)J = (\cos\theta)I_2 + \frac{\sin\theta}{\theta}B. \tag{1}$$

When $n = 3$, a real skew-symmetric matrix $B \in \mathfrak{so}(3)$ is of the form:

$$B = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

and letting $\theta = \sqrt{a^2 + b^2 + c^2} = \frac{1}{2}||B||$ with $||B||$ the Frobenius norm of matrices, we have the well-known formula due to Rodrigues:

$$e^B = I_3 + \frac{\sin\theta}{\theta}B + \frac{1-\cos\theta}{\theta^2}B^2 \tag{2}$$

with $e^B = I_3$ when $B = 0$.

It turns out that it is more convenient to normalize B, that is, to write $B = \theta B_1$ (where $B_1 = B/\theta$, assuming that $\theta \neq 0$), in which case the formula becomes:

$$e^{\theta B_1} = I_3 + (\sin\theta)B_1 + (1-\cos\theta)B_1^2 \tag{3}$$

Clearly the special orthogonal group $SO(n)$ is compact and connected. Now, we present a direct specific proof for the property that $SO(n)$ is exponential, in the cases $n = 2$ and $n = 3$.

If $n = 2$ then, according to the formula (1), the equation $\exp(B) = R$, where $R \in SO(2)$,

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

is equivalent to

$$(\cos\theta)I_2 + \frac{\sin\theta}{\theta}B = R.$$

Considering the trace in both sides of this equality we get $2\cos\theta = tr(R)$, hence we can find θ satisfying this relation since clearly we have $-2 \leq tr(R) \leq 2$.

It follows that

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2),$$

and then we are done.

Also, when $n = 3$, given $R \in \mathbf{SO}(3)$, we can find $\cos\theta$ because $tr(R) = 1 + 2\cos\theta$ and we can find B_1 by observing that:

$$\frac{1}{2}(R - R^\top) = (\sin\theta)B_1.$$

Actually, the above formula cannot be used when $\theta = 0$ or $\theta = \pi$, as $\sin\theta = 0$ in these cases. When $\theta = 0$, we have $R = I_3$ and $B_1 = 0$, and when $\theta = \pi$, we need to find B_1 such that:

$$B_1^2 = \frac{1}{2}(R - I_3).$$

As B_1 is a skew-symmetric 3×3 matrix, this amounts to solving some simple equations with three unknowns. Again, the problem is completely solved.

A general proof of the surjectivity of $\exp : \mathfrak{so}(n) \to SO(n)$, when $n \geq 4$, is presented in details in [5].

3.The Euclidean isometries preserving the orientation

Consider the Euclidean space $\mathbb{R}^n$ with the well-known Euclidean norm $\|\cdot\|$. An *isometry* of $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \to \mathbb{R}^n$, preserving the distances, that is for every $x, y \in \mathbb{R}^n$ the following relation holds

$$\|f(x) - f(y)\| = \|x - y\|. \tag{4}$$

According to the Ulam theorem, every isometry of $\mathbb{R}^n$ with $f(0) = 0$, is a linear map of the form $f(x) = Rx$, with $R \in O(n)$, the orthogonal group. If $\det R = 1$, that is $R \in SO(n)$, then the isometry f *preserves the orientation*. Otherwise, we say that f *reverses the orientation*. The problem to describe geometrically the Euclidean isometries is reduced in this way to the interpretation of the matrices in $O(n)$ or $SO(n)$.

Using the surjectivity of the exponential map, $\exp : \mathfrak{so}(n) \to SO(n)$, we can describe the isometries of $\mathbb{R}^n$ preserving the orientation.

When $n = 2$, from the previous alternative proof, we have $R \in SO(2)$ if and only if R is a rotation matrix, i.e.

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where the rotation angle θ is defined by the equation $2\cos\theta = tr(R)$.

When $n = 3$, then $R \in SO(3)$ if and only if

$$R = I_3 + \frac{\sin\theta}{\theta}B + \frac{1-\cos\theta}{\theta^2}B^2, \tag{5}$$

where the angle θ is defined by the equation $1 + 2\cos\theta = tr(R)$, that is $\theta = \arccos\frac{tr(R)-1}{2}$, if $\theta \neq 0$. Hence, when $\theta \neq \pi$, B is the skew-symmetric matrix uniquely defined by the equation

$$B = \frac{\theta}{2\sin\theta}(R - R^\top).$$

If $\theta = 0$, then $R = I_3$, and the isometry f is the identity map of $\mathbb{R}^3$. If $\theta = \pi$, then we can find the matrix B_1 as in the discussion in the previous section.

Assuming that

$$B = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

then formula (5) expresses a rotation in $\mathbb{R}^3$ of axis defined by the vector $\overrightarrow{v}(a,b,c)$ and angle θ.

If the isometry of $\mathbb{R}^3$ reverses the orientation, then $\det R = -1$. Because $\det(-R) = (-1)^3\det R = (-1)(-1) = 1$, it follows that the isometry g of $\mathbb{R}^3$, defined by $g(x) = (-R)x$, preserves the orientation. In this case we obtain the representation formula

$$R = -I_3 - \frac{\sin\theta}{\theta}B - \frac{1-\cos\theta}{\theta^2}B^2,$$

with the same geometric interpretation. In this way all isometries of the space $\mathbb{R}^3$ are completely described.

Remark No 1. In the paper [4] the following description of the matrices $R \in SO(n)$ for $n \geq 4$ is given : If $\{e^{i\theta_1}, e^{-i\theta_1}, \cdots, e^{i\theta_p}, e^{-i\theta_p}\}$ is the set of distinct eigenvalues of R different from 1, where $2p \leq n$ and $0 < \theta_i \leq \pi$, then there are p skew-symmetric matrices $B_1, \cdots, B_p$ such that $B_iB_j = B_jB_i = O_n, i \neq j$, $B_i^3 = -B_i$, for all i, j with $1 \leq i, j \leq p$, and

$$R = I_n + \sum_{i=1}^{n}[(\sin\theta_i)B_i + (1-\cos\theta_i)B_i^2].$$

This result gives an implicit description of the Euclidean isometries of the space $\mathbb{R}^n$ when $n \geq 4$, in terms of $2p$ parameters $\theta_1, \cdots, \theta_p, B_1, \cdots, B_p$.

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References

[1] Andrica, D.,Casu, I.N., *Lie Groups, Exponential Map, and Geometric Mechanics* (Romanian), Cluj University Press, 2008.

[2] Andrica, D., Rohan, R-A. *The image of the exponential map and some applications*, Proc. 8th Joint Conference on Mathematics and Computer Science MaCS, Komarno, Slovakia, July 14-17, 2010, H.F.Pop and A.Bege, Eds., Novadat, Györ, pp.3-14.

[3] Bröcker, T., tom Dieck, T., *Representations of compact Lie groups*, Springer-Verlag, GTM, vol. 98, New York, 1985.

[4] Gallier, J., Xu, D.,*Computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices*, International Journal of Robotics and Automation, Vol.17, No.4, 2002, 2-11.

[5] Gallier, J., *Notes on Differential Geometry and Lie Groups*, University of Pennsylvannia, 2012.

[6] Tao, T., *Two small facts about Lie groups*, 25 June, 2011.

[7] Wüstner, M., *Lie groups with surjective exponential function*, Shaker Verlag, Berichte aus der Mathematik, Aachen 2001.

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