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INTUITIONISTIC FUZZY CONGRUENCE RELATIONS ON RESIDUATED LATTICES

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Abstract. In this paper, the concept of intuitionistic fuzzy congruence relation on a residuated lattice is introduced and its properties is studied. The relationship between intuitionistic fuzzy filters and intuitionistic fuzzy congruence relations on a residuated lattice is obtain. Then the intuitionistic fuzzy congruence relation corresponding to a given intuitionistic fuzzy filter on residuated lattices is defined and some of its properties is obtained. The quotient algebra induced by this relation is defined. It is proved that this is also a residuated lattice.

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1. Introduction

M. Ward and R.P. Dilworth [16] introduced the concept of residuated lattices as generalization of ideal lattices of rings. The residuated lattice plays the role of semantics for a multiple- valued logic is called residuated logic. Residuated logic is a generalization of intuitionistic logic. Therefore it is weaker than classical logic. Important examples of residuated lattices related to logic are Boolean algebra corresponding to basic logic, BL-algebras corresponding to Hajec's basic logic, MV-algebras corresponding to Lukasiewicz many valued logic. The residuated lattices have been widely studied (See [4], [14] and [15]).

The concept of fuzzy sets was introduced by Zadeh [17] in 1965. As a generalization of fuzzy set, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1] in 1985. After that the notion of intuitionistic fuzzy sets was applied to group, BCK-algebras [8], pseudo MV-algebras [7]. In particular, the notion of intuitionistic fuzzy congruence relation on a lattice, semigroup are defined and some of their properties are investigated (See [11], [12] and [13]).

In this paper, we apply the notion of intuitionistic fuzzy sets to a residuated lattice. In section 2, we recall some definitions and theorems which will be needed in this paper. In section 3, we define the notion of intuitionistic fuzzy congruence relation on a residuated lattice and study its properties. In section 4, we show that there is a one to one correspondence between the set of all intuitionistic fuzzy congruence relations and the set of all intuitionistic fuzzy filters (which are normal fuzzy set) of a residuated lattice. Then we prove that the quotient algebra induced by an intuitionistic fuzzy filter is a residuated lattice and investigate some related results.

2. Preliminaries

In this section, we review some basic concepts and results which are needed in the later sections.

Definition 2.1. ([4], [15]) *A residuated lattice is an algebraic structure* $(L, \wedge, \vee, \rightarrow, *, 0, 1)$ *such that*

(1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice with the least element 0 and the greatest element 1,*

(2) $(L, *, 1)$ *is a commutative monoid where* 1 *is a unit element,*

(3) $x * y \leq z$ *iff* $x \leq y \rightarrow z$, *for all* $x, y, z \in L$.

We denote the residuated lattice $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ *by* L.

Proposition 2.2. ([5], [14]) *Let* L *be a residuated lattice. Then we have the following properties: for all* $x, y, z \in L$,
(1) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(2) $x * y \leq x \wedge y$, $x * (x \rightarrow y) \leq x \wedge y$,
(3) $x * (y \vee z) = (x * y) \vee (x * z)$,
(4) $x \rightarrow y \leq (x * z) \rightarrow (y * z)$,
(5) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$,
(6) $x * (y \wedge z) \leq (x * y) \wedge (x * z)$.

Definition 2.3. ([17]) *Let* X *be a non-empty subset. A fuzzy set in* X *is a mapping* $\mu : X \longrightarrow [0, 1]$.

Definition 2.4. *A fuzzy equivalence relation* R *on a non-empty set* X *is a fuzzy subset of* $X \times X$ *satisfying the following conditions:*
$(R1)$ $R(x, x) = \bigvee\{R(y, z) : y, z \in X\}$ *(reflexive),*
$(R2)$ $R(x, y) = R(y, x)$ *(symmetric),*
$(R3)$ $R(x, z) \geq R(x, y) \wedge R(y, z)$ *(transitive),*
for all $x, y, z \in X$.

Definition 2.5. ([9]) *A fuzzy equivalence relation* θ *on a residuated lattice* L *is called a fuzzy congruence relation on* L *if*

(C1) $\theta(x * y, z * w) \geq \theta(x, z) \wedge \theta(y, w)$,
(C2) $\theta(x \rightarrow y, z \rightarrow w) \geq \theta(x, z) \wedge \theta(y, w)$,
(C3) $\theta(x \wedge y, z \wedge w) \geq \theta(x, z) \wedge \theta(y, w)$,
(C4) $\theta(x \vee y, z \vee w) \geq \theta(x, z) \wedge \theta(y, w)$,
for all $x, y, z, w \in L$.

For sets X, Y and Z, $f = (f_1, f_2) : X \rightarrow Y \times Z$ is called a complex mapping if $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Z$ are mappings. Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 2.6.([1]) *Let* X *be a non-empty set. A complex mapping* $A = (\mu_A, \nu_A) : X \rightarrow I \times I$ *is called an intuitionistic fuzzy set (in short, IFS) in* X *if* $\mu_A + \nu_A = 1$ *for each* $x \in X$*, where the mapping* $\mu_A : X \rightarrow I$ *and* $\nu_A : X \rightarrow I$ *denote the degree of membership (namely* $\mu_A(x)$*) and the degree of non-membership (namely* $\nu_A(x)$*) of each* $x \in X$ *to* A *respectively. In particular,* $0_\sim$ *and* $1_\sim$ *denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in a set* X *defined by* $0_\sim(x) = (0, 1)$ *and* $1_\sim(x) = (1, 0)$ *for each* $x \in X$ *respectively.*

Definition 2.7.([2]) *Let* X *be a non-empty set and let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be IFS in* X*. Then*
(1) $A \subseteq B$ *iff* $\mu_A \leq \mu_B$ *and* $\nu_B \leq \nu_A$.
(2) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$.
(3) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(4) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(5) $[]A = (\mu_A, 1 - \mu_A)$, $<> A = (1 - \nu_A, \nu_A)$.

Definition 2.8. ([10]) *Let* $A = (\mu_A, \nu_A)$ *be an IFS of a residuated lattice* L. $A = (\mu_A, \nu_A)$ *is called intuitionistic fuzzy filter of* L *if it satisfies the following conditions: for any* $x, y \in L$
(IF1) $\mu_A(x) \leq \mu_A(1)$ *and* $\nu_A(x) \geq \nu_A(1)$,
(IF2) $\mu_A(y) \geq \mu_A(x) \wedge \mu_A(x \rightarrow y)$,
(IF3) $\nu_A(y) \leq \nu_A(x) \vee \nu_A(x \rightarrow y)$.

Theorem 2.9. ([10]) *Let* $A = (\mu_A, \nu_A)$ *be an IFS of a residuated lattice* L*. Then* $A = (\mu_A, \nu_A)$ *is an intuitionistic fuzzy filter of* L *if and only if*
(IF4) $x \leq y$ *implies* $\mu_A(x) \leq \mu_A(y)$, $\nu_A(x) \geq \nu_A(y)$,
(IF5) $\mu_A(x * y) \geq \mu_A(x) \wedge \mu_A(y)$,
(IF6) $\nu_A(x * y) \leq \nu_A(x) \vee \nu_A(y)$,
for all $x, y \in L$.

Definition 2.10. ([6]) *Let* X *be a non-empty set. Then a complex mapping* $R = (\mu_R, \nu_R) : X \times X \rightarrow I \times I$ *is called an intuitionistic fuzzy relation (in short, IFR) on* X *if* $\mu_R(x, y) + \nu_R(x, y) \leq 1$ *for each* $(x, y) \in X \times X$*. We will denote the set of all IFR on a set* X *as IFR(X).*

Definition 2.11.([6]) *Let X be a set and $R, Q \in IFR(X)$. Then the composition of R and Q, $Q \circ R$, is defined as follows: for any $x, y \in X$,*
$\mu_{Q\circ R}(x,y) = \bigvee_{z\in X}[\mu_R(x,z) \wedge \mu_Q(z,y)]$ *and* $\nu_{Q\circ R}(x,y) = \bigwedge_{z\in X}[\nu_R(x,z) \vee \nu_Q(z,y)]$.

Definition 2.12. ([6]) *Let X be a set and $R \in IFR(X)$. Then R is called an intuitionistic fuzzy equivalence relation (in short, IFE) on X if it satisfies the following conditions:*
(i) R is intuitionistic fuzzy reflexive, i.e., $R(x,x) = (1,0)$ for each $x \in X$,
(ii) R is intuitionistic fuzzy symmetric, i.e., $R(x,y) = R(y,x)$, for any $x, y \in X$,
(iii) R is intuitionistic fuzzy transitive, i.e., $R \circ R \subseteq R$.

3. Intuitionistic fuzzy congruence relation

Definition 3.1. *Let $R = (\mu_R, \nu_R)$ be an IFE on a residuated lattice L. Then R is called an intuitionistic fuzzy congruence relation (in short IFC) if it satisfies the following conditions: for any $x, y, z, w \in L$*
(IFC1) $\mu_R(x*z, y*w) \geq \mu_R(x,y) \wedge \mu_R(z,w)$, $\nu_R(x*z, y*w) \leq \nu_R(x,y) \vee \nu_R(z,w)$,
(IFC2) $\mu_R(x \to z, y \to w) \geq \mu_R(x,y) \wedge \mu_R(z,w)$, $\nu_R(x \to z, y \to w) \leq \nu_R(x,y) \vee \nu_R(z,w)$,
(IFC3) $\mu_R(x\wedge z, y\wedge w) \geq \mu_R(x,y) \wedge \mu_R(z,w)$, $\nu_R(x\wedge z, y\wedge w) \leq \nu_R(x,y) \vee \nu_R(z,w)$,
(IFC4) $\mu_R(x\vee z, y\vee w) \geq \mu_R(x,y) \wedge \mu_R(z,w)$, $\nu_R(x\vee z, y\vee w) \leq \nu_R(x,y) \vee \nu_R(z,w)$.

Theorem 3.2. *Let $R = (\mu_R, \nu_R)$ be an IFE on a residuated lattice L. Then R is an IFC if and only if it satisfies the following conditions:*
(IFC5) $\mu_R(x*z, y*z) \geq \mu_R(x,y)$, $\nu_R(x*z, y*z) \leq \nu_R(x,y)$,
(IFC6) $\mu_R(x \to z, y \to z) \geq \mu_R(x,y)$, $\nu_R(x \to z, y \to z) \leq \nu_R(x,y)$,
(IFC7) $\mu_R(z \to x, z \to y) \geq \mu_R(x,y)$, $\nu_R(z \to x, z \to y) \leq \nu_R(x,y)$,
(IFC8) $\mu_R(x \wedge z, y \wedge z) \geq \mu_R(x,y)$, $\nu_R(x \wedge z, y \wedge z) \leq \nu_R(x,y)$,
(IFC9) $\mu_R(x \vee z, y \vee z) \geq \mu_R(x,y)$, $\nu_R(x \vee z, y \vee z) \leq \nu_R(x,y)$,
for all $x, y, z \in L$.

Proof. Let $R = (\mu_R, \nu_R)$ be an IFC on L. We have $\mu_R(z,z) = 1$ and $\nu_R(z,z) = 0$. Suppose that $\star \in \{*, \to, \wedge, \vee\}$. By Definition 3.1, $\mu_R(x \star z, y \star z) \geq \mu_R(x,y) \wedge \mu_R(z,z) = \mu_R(x,y)$ and $\nu_R(x\star z, y \star z) \leq \nu_R(x,y) \vee \nu_R(z,z) = \nu_R(x,y)$. Hence it satisfies conditions (IFC5)-(IFC9).

Conversely, since $R = (\mu_R, \nu_R)$ is an IFE, then

$$\begin{aligned}\mu_R(x\star z, y\star w) &\geq \bigvee_{t\in L}[\mu_R(x\star z, t) \wedge \mu_R(t, y\star w)] \\ &\geq \mu_R(x\star z, y\star z) \wedge \mu_R(y\star z, y\star w)] \\ &\geq \mu_R(x,y) \wedge \mu_R(z,w)\end{aligned}$$

and

$$\begin{aligned}\nu_R(x\star z,y\star w) &\leq \bigwedge_{t\in L}[\mu_R(x\star z,t)\vee\mu_R(t,y\star w)]\\ &\leq \mu_R(x\star z,y\star z)\vee\mu_R(y\star z,y\star w)]\\ &\leq \mu_R(x,y)\vee\mu_R(z,w).\end{aligned}$$

Therefore it satisfies conditions (IFC1)- (IFC4) and it is an intuitionistic fuzzy congruence relation.

Example 3.3. Let $L=\{0,a,b,1\}$ with $0<a,b<1$ and elements a,b are incomparable. Define

$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then L become a residuated lattice. Define fuzzy sets
$\mu_R(x,x)=\mu_R(a,1)=1$, $\mu_R(0,x)=\mu_R(a,b)=\mu_R(b,1)=t$
$\nu_R(x,x)=\nu_R(b,1)=0$, $\nu_R(0,x)=\nu_R(a,1)=\mu_R(b,1)=s$
where $s+t\leq 1$ and $x\in L$. Then $R=(\mu_R,\nu_R)$ is an IFC on L.

In the following theorems, we will show that intuitionistic fuzzy congruence relation on a a residuated lattice is a generalization of fuzzy congruence relation.

Theorem 3.4. *Let $R=(\mu_R,\nu_R)$ be an IFE on a residuated lattice L. Then $R=(\mu_R,\nu_R)$ is an IFC on L if and only if the fuzzy sets μ_R and $\bar{\nu_R}$ are fuzzy congruence relation on L, where $\bar{\nu_R}=1-\nu_R$.*

Proof. Let $R=(\mu_R,\nu_R)$ be an IFC on L. It is clear that μ_R is a fuzzy congruence relation on L. Let $x,y,z\in L$, and $\star\in\{*,\to,\wedge,\vee\}$. we have

$$\bar{\nu_R}(x\star z,y\star z)=1-\nu_R(x\star z,y\star z)\geq 1-\nu_R(x,y)=\bar{\nu_R}(x,y),$$

Hence $\bar{\nu_A}$ is a fuzzy congruence relation on L.

Conversely, suppose that μ_A and $\bar{\nu_A}$ are fuzzy congruence relation in L. For each $x,y\in L$ and $\star\in\{*,\to,\wedge,\vee\}$. we have

$$1-\nu_R(x\star z,y\star z)=\bar{\nu_R}(x\star z,y\star z)\geq\bar{\nu_R}(x,y)\geq 1-\nu_R(x,y)$$

Then R is an intuitionistic fuzzy congruence relation on L.

Corollary 3.5. *Let $R=(\mu_R,\nu_R)$ be an IFE on a residuated lattice L. Then $R=(\mu_R,\nu_R)$ is an IFC on L if and only if $[]R=(\mu_R,\bar{\mu_R})$ and $<>R=(\bar{\nu_R},\nu_R)$ are intuitionistic fuzzy congruence relations on L.*

Proof. Let $R = (\mu_R, \nu_R)$ be an IFC on L. By Theorem 3.4, $\mu_R = \bar{\bar{\mu}}_R$ and $\bar{\nu}_R$ are fuzzy congruence relations on L and $[]R = (\mu_R, \bar{\mu}_R)$ and $<> R = (\bar{\nu}_R, \nu_R)$ are intuitionistic fuzzy congruence relations on L.

Conversely, let $[]R = (\mu_R, \bar{\mu}_R)$ and $<> R = (\bar{\nu}_R, \nu_R)$ be intuitionistic fuzzy congruence relations on L. Then μ_R and $\bar{\nu}_R$ are fuzzy congruence relations on L. By Theorem 3.4, we obtain that $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy congruence relation on L.

For any $t \in [0, 1]$ and a fuzzy set μ in a non-empty set X, the set

$$U(\mu, \alpha) = \{x \in X : \mu(x) \geq \alpha\},$$

is called an upper $\alpha-$ level of μ and the set

$$L(\mu, \alpha) = \{x \in X : \mu(x) \leq \alpha\},$$

is called a lower $\alpha-$ level of μ.

In the following theorems, we will study the relationship between congruence relations and intuitionistic fuzzy congruence relations on a residuated lattice.

Theorem 3.6. *Let $R = (\mu_R, \nu_R)$ be an IFR on a residuated lattice L. Then $R = (\mu_R, \nu_R)$ is an IFC on L if and only if for all $\alpha, \beta \in [0, 1]$, the sets $U(\mu_R, \alpha)$ and $L(\nu_R, \beta)$ are congruence relations on L.*

Proof. Suppose that $R = (\mu_R, \nu_R)$ is an IFC on L and $\alpha, \beta \in [0, 1]$.
(i) First, we will show that $U(\mu_R, \alpha)$ is an equivalence relation on L.
• Since $\mu_R(x, x) = 1 \geq \alpha$, then $(x, x) \in U(\mu_R, \alpha)$. Hence $U(\mu_R, \alpha)$ is reflexive.
• It is clear that $U(\mu_R, \alpha)$ is symmetric.
• Let $(x, y), (y, z) \in U(\mu_R, \alpha)$. Then $\mu_R(x, y), \mu_R(y, z) \geq \alpha$. Since $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy equivalence relation on L, we obtain that

$$\alpha \leq \mu_R(x, z) \wedge \mu_R(z, y) \leq \bigvee_{t \in L} [\mu_R(x, t) \wedge \mu_R(t, y)] = \mu_{R \circ R}(x, y) \leq \mu_R(x, y),$$

Therefore $U(\mu_R, \alpha)$ is transitive. Hence $U(\mu_R, \alpha)$ is an equivalence relation on L.

Suppose that $\star \in \{*, \rightarrow, \wedge, \vee\}$ and $(x, y), (z, w) \in U(\mu_R, \alpha)$. Then $\mu_R(x, y), \mu_R(z, w) \geq \alpha$. By Definition 3.1, we have

$$\alpha \leq \mu_R(x, y) \wedge \mu_R(z, w) \leq \mu_R(x \star y, z \star w),$$

that is $(x \star z, y \star w) \in U(\mu_R, \alpha)$. Hence $U(\mu_R, \alpha)$ is a congruence relation on L.

(ii) Now, we will prove that $L(\nu_R, \beta)$ is a congruence relation on L.
• Since $\nu_R(x, x) = 0 \leq \beta$, then $(x, x) \in L(\nu_R, \beta)$. Therefore $L(\nu_R, \beta)$ is reflexive.
• It is obvious that $U(\mu_R, \alpha)$ is symmetric.
• Let $(x, y), (y, z) \in L(\nu_R, \beta)$. Then $\nu_R(x, y), \nu_R(y, z) \leq \beta$. Since $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy equivalence relation on L, we get that

$$\beta \geq \nu_R(x, z) \vee \nu_R(z, y) \geq \bigwedge_{z \in L} [\nu_R(x, t) \vee \nu_R(t, y)] = \nu_{R \circ R}(x, y) \geq \nu_R(x, y),$$

Therefore $L(\nu_R, \beta)$ is transitive. Hence $L(\nu_R, \beta)$ is an equivalence relation on L. Let $\star \in \{*, \rightarrow, \wedge, \vee\}$ and $(x, y), (z, w) \in L(\nu_R, \beta)$. Then $\nu_R(x, y), \nu_R(z, w) \leq \beta$. We get that

$$\beta \geq \nu_R(x, y) \vee \nu_R(z, w) \geq \nu_R(x \star y, z \star w),$$

i. e, $(x \star z, y \star w) \in L(\nu_R, \beta)$. Hence $L(\nu_R, \beta)$ is a congruence relation on L.

Conversely, Suppose that for all $\alpha, \beta \in [0, 1]$, the sets $U(\mu_A, \alpha)$ and $L(\nu_A, \beta)$ are congruence relations on L. We will prove that $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy equivalence relation on L.
• Since $U(\mu_R, 1)$ and $L(\nu_R, 0)$ are reflexive, then $R(x, x) = (1, 0)$ for each $x \in L$.
• It is clear that R is intuitionistic fuzzy symmetric.
• Suppose that $x, y, z \in L$. Let $\mu_R(x, z) = s_1$ and $\mu_R(z, y) = t_1$. Put $\alpha = s_1 \wedge t_1$. Then $\mu_R(x, z), \mu_R(z, y) \geq \alpha$. Hence $(x, z), (z, y) \in U(\mu_A, \alpha)$. Since $U(\mu_A, \alpha)$ is transitive, we obtain that $(x, y) \in U(\mu_A, \alpha)$, that is

$$\mu_R(x, y) \geq \alpha = s_1 \wedge t_1 = \mu_R(x, z) \wedge \mu_R(z, y).$$

Since $z \in L$ is arbitrary, we get that

$$\mu_R(x, y) \geq \bigvee_{z \in L} \mu_R(x, z) \wedge \mu_R(z, y).$$

Now, let $\nu_R(x, z) = s_2$ and $\nu_R(z, y) = t_2$. Put $\beta = s_2 \vee t_2$. Then $\nu_R(x, z), \nu_R(z, y) \leq \beta$. Hence $(x, z), (z, y) \in L(\nu_R, \beta)$. Since $L(\nu_R, \beta)$ is transitive, then $(x, y) \in L(\nu_R, \beta)$, that is

$$\nu_R(x, y) \leq \beta = s_2 \vee t_2 = \nu_R(x, z) \vee \nu_R(z, y).$$

Since $z \in L$ is arbitrary, we get that

$$\nu_R(x, y) \leq \bigwedge_{z \in L} \nu_R(x, z) \vee \nu_R(z, y).$$

Therefore $R \circ R \subseteq R$ and then R is intuitionistic fuzzy equivalence relation.

• Let $\star \in \{*, \to, \wedge, \vee\}$. Suppose that $\mu_R(x, y) = s_3$ and $\mu_R(z, w) = t_3$. Put $\alpha = s_3 \wedge t_3$. Then $\mu_R(x, y), \mu_R(z, w) \geq \alpha$. Hence $(x, y), (z, w) \in U(\mu_A, \alpha)$. Since $U(\mu_A, \alpha)$ is a congruence relation, we obtain that $(x \star z, y \star w) \in U(\mu_A, \alpha)$, that is

$$\mu_R(x \star z, y \star w) \geq \alpha = s_3 \wedge t_3 = \mu_R(x, y) \wedge \mu_R(z, w).$$

Suppose that $\nu_R(x, y) = s_4$ and $\nu_R(z, w) = t_4$. Put $\beta = s_4 \vee t_4$. Then $\nu_R(x, y), \nu_R(z, w) \leq \beta$. Hence $(x, y), (z, w) \in L(\nu_R, \beta)$. By assumption, $L(\nu_R, \beta)$ is a congruence relation, we obtain that $(x \star z, y \star w) \in L(\nu_R, \beta)$, that is

$$\nu_R(x \star z, y \star w) \leq \beta = s_4 \vee t_4 = \nu_R(x, y) \vee \nu_R(z, w).$$

Hence $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy congruence relation on L.

Corollary 3.7. *Let R be a relation on a residuated lattice L. Then R is a congruence relation on L if and only if (χ_R, χ_{R^C}) is an IFC on L.*

Proof. Let R be a relation on a residuated lattice L. Since χ_R and χ_{R^C} are fuzzy sets in L such that $\chi_R(x, y) + \chi_{R^C}(x, y) = 1$ for all $x, y \in L$, then (χ_R, χ_{R^C}) is an IFR. We have $U(\chi_R, \alpha) = R$, for all $0 < \alpha \leq 1$ and $U(\chi_R, \alpha) = L \times L$ for $\alpha = 0$. Also, $L(\chi_{R^C}, \beta) = R$, for all $0 \leq \beta < 1$ and $L(\chi_{R^C}, \beta) = L \times L$ for $\beta = 1$. By the above theorem (χ_R, χ_{F^R}) is an intuitionistic congruence relation on L if and only if R is a congruence relation on L.

4. IFC induced by IFF

Definition 4.1. *Let $R = (\mu_R, \nu_R)$ be an IFC on a residuated lattice L. Then the fuzzy subset $A_R = (\mu_{A_R}, \nu_{A_R})$ which is defined by*

$$\mu_{A_R}(x) = \mu_R(x, 1) \text{ and } \nu_{A_R}(x) = \nu_R(x, 1),$$

is called the intuitionistic fuzzy subset induced by R.

Theorem 4.2. *Let $R = (\mu_R, \nu_R)$ be an IFC on a residuated lattice L. Then A_R is an intuitionistic fuzzy filter of L.*

Proof. Let $x, y \in L$ be arbitrary. Then

(IF1)
$$\mu_{A_R}(1) = \mu_R(1, 1) = \mu_R(x \to 1, 1 \to 1) \geq \mu_R(x, 1) = \mu_{A_R}(x),$$
$$\nu_{A_R}(1) = \nu_R(1, 1) = \nu_R(x \to 1, 1 \to 1) \leq \nu_R(x, 1) = \nu_{A_R}(x).$$

(IF2)

$$\mu_{A_R}(y) = \mu_R(y,1) = \mu_R(y \vee (x * (x \to y)), y \vee 1) \geq \mu_R(x * (x \to y), 1 * 1)$$
$$\geq \mu_R(x,1) \wedge \mu_R(x \to y, 1) = \mu_{A_R}(x) \wedge \mu_{A_R}(x \to y),$$

(IF3)

$$\nu_{A_R}(y) = \nu_R(y,1) = \nu_R(y \vee (x * (x \to y)), y \vee 1) \leq \nu_R(x * (x \to y), 1 * 1)$$
$$\leq \nu_R(x,1) \vee \nu_R(x \to y, 1) = \nu_{A_R}(x) \vee \nu_{A_R}(x \to y).$$

Definition 4.3. *Let $A = (\mu_A, \nu_A)$ be an IFF of a residuated lattice L. The intuitionistic fuzzy relation $R_A = (\mu_{R_A}, \nu_{R_A})$ on L which is defined by*

$$\mu_{R_A}(x,y) = \mu_A(x \to y) \wedge \mu_A(y \to x),$$
$$\nu_{R_A}(x,y) = \nu_A(x \to y) \vee \nu_A(y \to x),$$

is called the intuitionistic fuzzy relation induced by A.

Lemma 4.4. *Let $A = (\mu_A, \nu_A)$ be an IFF of a residuated lattice L. Then*
(1) $\mu_A(x \to y) \leq \mu_A[(x * z) \to (y * z)]$, $\nu_A(x \to y) \geq \nu_A[(x * z) \to (y * z)]$,
(2) $\mu_A(x \to y) \leq \mu_A((y \to z) \to (x \to z))$, $\nu_A(x \to y) \geq \nu_A((y \to z) \to (x \to z))$,
(3) $\mu_A(x \to y) \leq \mu_A((x \wedge z) \to (y \wedge z))$, $\nu_A(x \to y) \geq \nu_A((x \wedge z) \to (y \wedge z))$,
(4) $\mu_A(x \to y) \leq \mu_A((x \vee z) \to (y \vee z))$, $\nu_A(x \to y) \geq \nu_A((x \vee z) \to (y \vee z))$,
for all $x, y, z \in L$.

Proof.(1) and (2) follow from Proposition 2. 2 parts (4), (5) and Theorem 2.9 part (1).
(3) Since $(x \wedge z) * (x \to y) \leq (x * (x \to y)) \wedge (z * (y \to x)) \leq y \wedge z$, then $(x \to y) \leq (x \wedge z) \to (y \wedge z)$. Hence (3) hold by Theorem 2.9 part (1).
(4) By proposition 2.2 part (3), $(x \vee z) * (x \to y) = (x * (x \to y)) \vee (z * (x \to y)) \leq y \vee z$. Then $x \to z \leq (x \vee z) \to (y \vee z)$. So (4) obtain by Theorem 2.9 part (1).

Theorem 4.5. *Let $A = (\mu_A, \nu_A)$ be an IFF of a residuated lattice L. Then $R_A = (\mu_{R_A}, \nu_{R_A})$ is an IFC on L.*

Proof. It follows from Lemma 4.4.

Theorem 4.6. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy filter on a residuated lattice L. Then $A_{R_A} = A$.*

Proof. Let $x \in L$. Since A is an intuitionistic fuzzy filter of L, by (IF1)

$$\mu_{A_{R_A}}(x) = \mu_{R_A}(x,1) = \mu_A(x \to 1) \wedge \mu_A(1 \to x) = \mu_A(x),$$
$$\nu_{A_{R_A}}(x) = \nu_{R_A}(x,1) = \nu_A(x \to 1) \vee \nu_A(1 \to x) = \nu_A(x).$$

Hence $A_{R_A} = A$.

Theorem 4.7. *Let $R = (\mu_R, \nu_R)$ be an IFC on a residuated lattice L. Then $R_{A_R} = R$.*

Proof. Let $x, y \in L$. Then

$$\begin{aligned}\mu_{R_{A_R}}(x,y) &= \mu_{A_R}(x\to y)\wedge\mu_{A_R}(y\to x)\\ &= \mu_R(x\to y,1)\wedge\mu_R(y\to x,1)\\ &= \mu_R(x\to y,y\to y)\wedge\mu_R(y\to x,x\to x)\\ &\geq \mu_R(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{R_{A_R}}(x,y) &= \nu_{A_R}(x\to y)\vee\nu_{A_R}(y\to x)\\ &= \nu_R(x\to y,1)\vee\nu_R(y\to x,1)\\ &= \nu_R(x\to y,y\to y)\vee\nu_R(y\to x,x\to x)\\ &\leq \nu_R(x,y)\end{aligned}$$

Therefore $R_{A_R}\supseteq R$. Conversely, we have

$$\begin{aligned}\mu_R(x,y) &\geq \mu_R(x,x\vee y)\wedge\mu_R(x\vee y,y)\\ &= \mu_R(x\vee(y*(y\to x)),x\vee y)\wedge\mu_R(x\vee y,y\vee(x*(x\to y)))\\ &\geq \mu_R(y*(y\to x),y)\wedge\mu_R(x,x*(x\to y))\\ &\geq \mu_R(y*(y\to x),y*1)\wedge\mu_R(x*1,x*(x\to y))\\ &\geq \mu_R(y\to x,1)\wedge\mu_R(1,x\to y)\\ &= \mu_{A_R}(y\to x)\wedge\mu_{A_R}(x\to y)=\mu_{R_{A_R}}(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_R(x,y) &\leq \nu_R(x,x\vee y)\vee\nu_R(x\vee y,y)\}\\ &= \nu_R(x\vee(y*(y\to x)),x\vee y)\vee\nu_R(x\vee y,y\vee(x*(x\to y)))\\ &\leq \nu_R(y*(y\to x),y)\vee\nu_R(x,x*(x\to y))\\ &\leq \nu_R(y*(y\to x),y*1)\vee\nu_R(x*1,x*(x\to y))\\ &\leq \nu_R(x\to y,1)\vee\mu_R(1,y\to x)\\ &= \nu_{A_R}(x\to y)\vee\nu_{A_R}(y\to x)=\nu_{R_{A_R}}(x,y)\end{aligned}$$

Therefore $R_{A_R}\subseteq R$.

Corollary 4.8. *(Correspondence theorem) There is a bijection between the set of all intuitionistic fuzzy congruence relations and the set of all intuitionistic fuzzy filters $A=(\mu_A,\mu_A)$ of a residuated lattice L such that $\mu_A(1)=1$ and $\nu_A(1)=0$.*

Proof. Denote the set of all intuitionistic fuzzy congruence relations on L by IFC(L)and the set of all intuitionistic fuzzy filters such that $\mu_A(1)=1$ and $\nu_A(1)=0$ by $IFF(L)$. Define $\psi : IFC(L)\to IFF(L)$ by $\psi(R)=A_R$ and $\varphi : IFF(L)\to IFC(L)$ by $\varphi(A)=R_A$. By Theorems 4.3 and 4.4, ψ and φ are well defined. By Theorems 4.6 and 4.7, φ and ψ are inverse of each other.

Definition 4.9. *Let R be an intuitionistic fuzzy congruence relation on a residuated lattice L and $a\in L$. Define the complex mapping $R_a : L\to I\times I$ as follows:*

$$R_a(x) = R(a, x), \textit{ for all } x \in L$$

Then R_a is an IFS and it is called an intuitionistic fuzzy equivalence class of R containing a.

Proposition 4.10. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy filter of a residuated lattice L and R_A be the IFC induced by A. Then the following hold:*
(1) $(R_A)_a = (R_A)_b$ if and only if $\mu_A(a \to b) = \mu_A(b \to a) = \mu_A(1)$ and $\nu_A(a \to b) = \nu_A(b \to a) = \nu_A(1)$,
(2) $(R_A)_a = (R_A)_1$ if and only if $\mu_A(a) = \mu_A(1)$ and $\nu_A(a) = \nu_A(1)$.

Proof. Let $(R_A)_a = (R_A)_b$. We have $(R_A)_a(a) = (R_A)_b(a)$ and obtain that

$$\mu_A(a \to a) \wedge \mu_A(a \to a) = \mu_{R_A}(a, a) = \mu_{R_A}(a, b) = \mu_A(b \to a) \wedge \mu_A(a \to b),$$
$$\nu_A(a \to a) \vee \nu_A(a \to a) = \nu_{R_A}(a, a) = \nu_{R_A}(a, b) = \nu_A(b \to a) \vee \nu_A(a \to b),$$

It follows that

$$\mu_A(b \to a) = \mu_A(a \to b) = \mu_A(1),$$
$$\nu_A(b \to a) = \nu_A(a \to b) = \nu_A(1).$$

Conversely, suppose that $\mu_A(b \to a) = \mu_A(a \to b) = \mu_A(1)$ and $\nu_A(b \to a) = \nu_A(a \to b) = \nu_A(1)$. By Theorem 2.2 part (5),

$$\mu_A(x \to a) \wedge \mu_A(a \to b) \le \mu_A((x \to a) * (a \to b)) \le \mu_A(x \to b)$$
$$\mu_A(x \to b) \wedge \mu_A(b \to a) \le \mu_A((x \to b) * (b \to a)) \le \mu_A(x \to a)$$
$$\nu_A(x \to a) \vee \nu_A(a \to b) \ge \nu_A((x \to a) * (a \to b)) \ge \nu_A(x \to b)$$
$$\nu_A(x \to b) \vee \nu_A(b \to a) \ge \nu_A((x \to b) * (b \to a)) \ge \nu_A(x \to a)$$

By using assumption, we have

$$\mu_A(x \to a) \le \mu_A(x \to b) \quad \text{and} \quad \mu_A(x \to b) \le \mu_A(x \to a),$$
$$\nu_A(x \to a) \ge \nu_A(x \to b) \quad \text{and} \quad \nu_A(x \to b) \ge \nu_A(x \to a)$$

Therefore $\mu_A(x \to b) = \mu_A(x \to a)$ and $\nu_A(x \to b) = \nu_A(x \to a)$. Similarly, we can show that $\mu_A(b \to x) = \mu_A(a \to x)$ and $\nu_A(b \to x) = \nu_A(a \to x)$. Thus $(R_A)_a(x) = (R_A)_b(x)$ for all $x \in L$.
(2) It follows from part (1).

Theorem 4.11. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy filter of a residuated lattice L. Define*

$$a \equiv_A b \quad \textit{if and only if} \quad (R_A)_a = (R_A)_b.$$

Then $\equiv_A$ is a congruence relation on L.

Proof. The proof follows from Proposition 4. 10.

Definition 4.12. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy filter of a residuated lattice L and R_A be the IFC induced by A. The set $\{(R_A)_a : a \in L\}$ is called the intuitionistic fuzzy quotient set of L by R_A and denoted by L/R_A. On this set, we define*

$$(R_A)_a * (R_A)_b = (R_A)_{a*b} \quad , \quad (R_A)_a \to (R_A)_b = (R_A)_{a\to b}$$
$$(R_A)_a \wedge (R_A)_b = (R_A)_{a\wedge b} \quad , \quad (R_A)_a \vee (R_A)_b = (R_A)_{a\vee b}.$$

Theorem 4.13. *$A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy filter of a residuated lattice L. Then $L/R_A = (L/R_A, \wedge, \vee, \to, *, 0_\sim, 1_\sim)$ is a residuated lattice.*

Proof. We have $(R_A)_a = (R_A)_b$ and $(R_A)_c = (R_A)_d$ if and only if $a \equiv_A b$ and $c \equiv_A d$. Since $\equiv_A$ is the congruence relation on L by Theorem 4.11, then all above operations are well defined. It is easy to show that $(L/R_A, \wedge, \vee, 0_\sim, 1_\sim)$ is a bounded lattice, $*$ is commutative, associative and has $1_\sim$ as an identity. The operation $\vee$ defines a relation $\leq$ on L/R_A by

$$(R_A)_a \leq (R_A)_b \quad \text{if and only if} \quad (R_A)_{a\vee b} = (R_A)_b \quad \text{for all } a, b \in L.$$

This relation is a partial order on L/R_A. Using Proposition 4.10, we see that $(R_A)_a \leq (R_A)_b$ if and only if $a \to b \in U(\mu_A, \mu_A(1))$ and $a \to b \in L(\nu_A, \nu_A(1))$for all $a, b \in L$. Now, we will show that $(R_A)_a \leq (R_A)_b \to (R_A)_c$ if and only if $(R_A)_a * (R_A)_b \leq (R_A)_c$ for all $a, b, c \in L$. We have

$$(R_A)_a \leq (R_A)_b \to (R_A)_c$$

$$\iff \quad (R_A)_a \leq (R_A)_{b\to c} \quad \text{by Definition 4.12}$$

$$\iff \quad (a \to (b \to c)) \in U(\mu_A, \mu_A(1)) \text{ and } (a \to (b \to c)) \in L(\nu_A, \nu_A(1))$$

$$\iff \quad ((a * b) \to c) \in U(\nu_A, \nu_A(1)) \text{ and } ((a * b) \to c) \in L(\nu_A, \nu_A(1))$$

$$\iff \quad (R_A)_{a*b} \leq (R_A)_c$$

$$\iff \quad (R_A)_a * (R_A)_b \leq (R_A)_c \quad \text{by Definition 4.12.}$$

This completes the proof.

Theorem 4.14. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy filter of a residuated lattice L and L/R_A be the corresponding quotient algebra. Then the map $h : L \to$*

L/R_A defined by $h(a) = (R_A)_a$ for all $a \in L$ is a surjective homomorphism and $\ker(h) = U(\nu_A, \nu_A(1)) \cap L(\nu_A, \nu_A(1))$*, where* $\ker(h) = \{x \in L : h(x) = (R_A)_1\}$*. Moreover, L/R_A is isomorphic to the commutative residuated lattice $L/\equiv_A$.*

Proof. It follows from Definition 4.12 and Theorem 4.13, that h is surjective homomorphism. By Proposition 4. 10 part (2), we have $x \in \ker(h)$ if and only if $(R_A)_x = h(x) = (R_A)_1$ if and only if if and only if $\mu_A(x) = \mu_A(1)$ and $\nu_A(x) = \nu_A(1)$ if and only if $x \in U(\nu_A, \nu_A(1)) \cap L(\nu_A, \nu_A(1))$.
By part (1) and (2) of Proposition 4.10, L/R_A is isomorphic to the commutative residuated lattice $L/\equiv_A$.

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