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SYMMETRIES AND THE DIFFERENTIAL FORM FOR A NONLINEAR DIFFUSION EQUATION WITH CONVECTION TERM

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Abstract. In this paper, The Differential Form Method is used to obtain the determined equations of nonlinear diffusion equation with convection term. Later on, the potential symmetries and Lie point symmetries have been discussed for the problem by considering the four special cases of the problem. Finally, group invariant solutions have been obtained.

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1. Introduction

In a pioneer work, Harrison and Estabrook [9], introduced the method of writing differential equations or system of differential equations in terms of differential forms and finding their symmetries. Later on, Papachristou and Harrison [14-16], generalized the method to vector valued or Lie algebra-valued differential forms and used in the two-dimensional Dirac equation and the Yang-Mills free field equations in Minkowski space-time. Waller [17] used 1-form and contraction in nonlinear diffusion equations arising in plasma physics. Edeled developed the theory of differential forms, in [3-6], he explore the use of differential forms in physics. In [4, 5], he considered a method of characteristics in any number of dimensions using isovector treatments. Web et al. [19] consider nonlinear Shrödinger equations for a type of MHD waves, using the differential form method. In a paper [18], he also analyzes a nonlinear magnetic potential equation with conservation laws, with the Liouville equation as a special case. A generalized nonlinear Shrödinger equation with attention to both symmetries and Bäcklund transformations were considered by Harnad and Winternitz [8]. Pakdemirli et al. [12, 13] considered boundary layer equations for non-Newtonian fluids, including arbitrary shear stress, power law fluid, and other models. Ozer and Suhubi [11] considered nonvacuum Maxwell equations with nonlinear constitutive relations.

Recently, Davison and Kara [2] treated Burgers equation to obtain potential and approximate symmetries using differential form method. In the present study, to obtain the determined equations of nonlinear diffusion equation with convection term, it is assumed that when the differential forms are zero then their Lie derivatives are also zero. Later on, the symmetries of four special cases of the problem have been considered. Finally, the group invariant solutions are obtained for all the cases of the problem. This method save considerable work in complicated cases, specially in cases where not all forms in the Ideal are of the same rank.

2. Determined equations of Diffusion equation with convection term

Consider the nonlinear diffusion equation with convection term in the following form:

$$u_t = (k(u)u_x)_x + q(u) \tag{1}$$

where, $k(u)$ and $q(u)$ are arbitrary smooth functions. Equation (1) is used to model a wide range of phenomena in physics, engineering, chemistry, etc.
For the case $k(u) = 1$ and $q(u) = 0$, Eq.(1) reduces to classical Heat equation

$$u_t = u_{xx} \tag{2}$$

For the case $q(u) = 0$, Eq. (1) reduces to the standard nonlinear heat equation

$$u_t = (k(u)u_x)_x \tag{3}$$

Lie Symmetries of Eq. (3) were completely described by Ovsyannikov [10]. For constructing the differential forms of Eq. (1), we consider the following Auxiliary system:

$$v = ku_x,$$
$$u_t = v_x + q \tag{4}$$

We introduce the following 2-forms:

$$\alpha = k\,du\,dt - v\,dx\,dt = ku_x\,dx\,dt - v\,dx\,dt$$

$$\beta = du\,dx + dv\,dt + q\,dx\,dt = u_t\,dt\,dx + v_x\,dx\,dt + q\,dx\,dt$$

which gives the system (4) when annulled. Here we drop the wedge product $\wedge$ to save writing. Consider the symmetry of Eq. (1) in the form:

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}$$

First, taking the Lie derivative of α, as

$$\mathcal{L}_\alpha = X\rfloor(d\alpha) + d(X\rfloor\alpha)$$

$$= X\rfloor(-dv\,dx\,dt) + d(k\,\phi\,dt - k\,\tau\,du - v\,\xi\,dt + v\,\tau\,dx)$$

$$= (-\eta + k\,\phi_x - v\,\xi_x - v\,\tau_t)\,dx\,dt + (\phi\,k_u + k\,\phi_u + k\,\tau_t - v\,\xi_u)\,du\,dt + (k\,\phi_v - v\,\xi_v)\,dv\,dt$$

$$+(k\,\tau_x + v\,\tau_u)\,du\,dx + v\,\tau_v\,dv\,dx - k\,\tau_v\,dv\,du$$

Since, when $\alpha = \beta = 0$, we have $du\,dt = \frac{v}{k}\,dx\,dt$ and $du\,dx = -dv\,dt - q\,dx\,dt$. Therefore

$$\mathcal{L}_X\alpha\ |_{\alpha\,=\beta=0} = \left(-\eta + k\,\phi_x - v\,\xi_x + v\,\phi_u - \frac{v^2}{k}\xi_u + \frac{v}{k}\phi\,k_u - k\,q\,\tau_x - v\,q\,\tau_u\right)\,dx\,dt$$

$$+\,(k\,\phi_v - v\,\xi_v - k\tau_x - v\,\tau_u)\ dv\,dt + v\,\tau_v\,dv\,dx - k\,\tau_v\,dv\,du$$

Also, when $\alpha = \beta = 0$, we have $\mathcal{L}_X\alpha\ |_{\alpha\,=\beta=0} = 0$ and split the coefficients of $dx\,dt$, $dv\,dt$, etc. to obtain

$$dx\,dt:\ -\eta + k\,\phi_x - v\,\xi_x + v\,\phi_u - \frac{v^2}{k}\xi_u + \frac{v}{k}\phi\,k_u - k\,q\,\tau_x - v\,q\,\tau_u = 0 \tag{5}$$

$$dv\,dt:\ k\,\phi_v - v\,\xi_v - k\,\tau_x - v\,\tau_u = 0 \tag{6}$$

$$dv\,dx:\ v\,\tau_v = o \tag{7}$$

$$dv\,du:\ -k\,\tau_v = o \tag{8}$$

Now, taking the Lie derivative of β, as

$$\mathcal{L}_X\beta = X\rfloor(d\beta) + d(X\rfloor\beta)$$

$$= X\rfloor(q_u\,du\,dx\,dt) + d(\phi\,dx - \xi\,du) + \eta\,dt - \tau\,dv + q\,\xi\,dt - q\,\tau\,dx)$$

$$= (\phi q_u - \phi_t + \eta_x + q\xi_x + q\tau_t)dx\,dt + (\xi_t + \eta_u + q\,\xi_u)du\,dt + (\eta_t + \tau_t + q\,\xi_v)dv\,dt$$

$$+(\phi_u + \xi_x - q\,\tau_u)du\,dx + (\phi_v + \tau_x - q\,\tau_v)dv\,dx + (\xi_v - \tau_u)dv\,dv$$

For $\alpha = \beta = 0$, we obtain

$$\mathcal{L}_X\alpha\ |_{\alpha\,=\beta=0} = \left[\phi\,q_u - \phi_t + \eta_x + q\,\tau_t + \frac{v}{k}\,(\xi_t + \eta_u + q\,\xi_u) - q\,\phi_u + q^2\,\tau_u\right]dx\,dt$$

$$+\,(\eta_t + \tau_t + q\,\xi_v - \phi_u - \xi_x + q\,\tau_u)\,dv\,dt + (\phi_v + \tau_x - q\,\tau_v)\,dv\,dx + (\xi_v - \tau_u)\,du\,dv$$

Again, when $\alpha = \beta = 0$, we have $\mathcal{L}_X\beta \mid_{\alpha\ =\beta=0} = 0$ and split the coefficients of $dx\,dt$, $dv\,dt$, etc. to obtain

$$dx\,dt:\ \phi\,q_u - \phi_t + \eta_x + q\,\tau_t + \frac{v}{k}\left(\xi_t + \eta_u + q\,\xi_u\right) - q\,\phi_u + q^2\,\tau_u = 0 \tag{9}$$

$$dv\,dt:\ \eta_t + \tau_t + q\,\xi_v - \phi_u - \xi_x + q\,\tau_u = 0 \tag{10}$$

$$dv\,dx:\ \phi_v + \tau_x - q\,\tau_v = 0 \tag{11}$$

$$du\,dv:\ \xi_v - \tau_u = 0 \tag{12}$$

From Eq. (7) and (8), we observe that $\tau_v = 0$, which with Eq. (11) gives $\phi_v = -\tau_x$ and this together with Eq. (12), after combined with Eq. (6) gives

$$k\,\tau_x + v\,\tau_u = 0$$

Separating coefficients of v gives $\tau_x = \tau_u = 0$, so that $\tau = \tau(t)$.
Next, Eq. (5) and (9) can be put in the following form

$$\eta = k\,\phi_x - v\,\xi_x + v\,\phi_u - \frac{v^2}{k}\xi_u + \frac{v}{k}\phi\,k_u \tag{13}$$

$$k\left(\phi\,q_u - \phi_t + q\,\tau_t - q\,\phi_u\right) + v\,\xi_t + v\,q\,\xi_u + k\,\eta_x + v\,\eta_u = 0 \tag{14}$$

Putting Eq. (13) in (14), we get

$$k\left(\phi\,q_u - \phi_t + q\,\tau_t - q\,\phi_u\right) + v\,\xi_t + v\,q\,\xi_u + k\left[k\,\phi_{xx} - v\,\xi_{xx} + v\,\phi_{ux} - \frac{v^2}{k}\xi_{ux} + \frac{v}{k}\phi_x\,k_u\right]$$
$$+v\left[k_u\,\phi_x + k\phi_{xu} - v\,\xi_{xu} + v\,\phi_{uu} + \frac{v^2}{k^2}\xi_u\,k_u - \frac{v^2}{k}\xi_{uu} - \frac{v}{k^2}\phi\,k_u^2 + \frac{v}{k}\phi_u\,k_u + \frac{v}{k}\phi\,k_{uu}\right] = 0 \tag{15}$$

Collecting all the terms of Eq. (15) in power of v and setting their coefficients equal to zero, we obtain

$$\phi\,q_u - \phi_t + q\,\tau_t - q\,\phi_u + k_{\,xx} = 0 \tag{16}$$

$$-k\,\xi_{xx} + k\,\phi_{ux} + \phi_x k_u + k_u\phi_x + k\,\phi_{xu} + \xi_t + q\,\xi_u = 0 \tag{17}$$

$$-\xi_{uX} - \xi_{xu} + \phi_{uu} - \frac{1}{k^2}\phi k_u^2 + \frac{1}{k}\phi_u k_u + \frac{1}{k}\phi k_{uu} = 0 \tag{18}$$

$$\frac{1}{k^2}\xi_u k_u - \frac{1}{k}\xi_{uu} = 0 \tag{19}$$

By separating the coefficients of q in Eq. (17), we obtain $\xi_u = 0$.
Finally, substituting Eq. (13) in (10) and solving together with the above equations we write all the determined equations in the following simple and compact form

$$\phi k_u + k(\tau_t - 2\xi_x) = 0 \tag{20}$$

$$\phi_t + q(\phi_u - \tau_t) - \phi q_u - k\phi_{xx} = 0 \tag{21}$$

$$\xi_t + 2k_u\phi_x - k\xi_{xx} + 2k\phi_{xu} = 0 \tag{22}$$

$$\phi_u k_u + \phi k_{uu} + k\phi_{uu} + k_u(\tau_t - 2\xi_x) = 0 \tag{23}$$

where, $\tau = \tau(t), \xi = \xi(x,t), \phi = \phi(t,x,u)$ and $\eta = k\phi_x - v\xi_x + v\phi_u - \frac{v^2}{k}\xi_u + \frac{v}{k}\phi k_u$.

3. Some Particular Cases

Next, consider the following four cases:
Case I: $k(u) = e^u, q(u) = e^{bu}$
For this case, the solutions of determined equations (20)-(23), is obtained in the form

$$\tau = C_1 + btC_3 \tag{24}$$

$$\xi = C_2 + \frac{(b-1)}{2}xC_3 \tag{25}$$

$$\phi = -C_3 \tag{26}$$

$$\eta = -\frac{(b+1)}{2}vC_3 \tag{27}$$

Thus, we have the symmetry generators

$$X_1 = \partial_t$$

$$X_2 = \partial_x$$

$$X_3 = bt\partial_t + \frac{(b-1)}{2}x\partial_x - \partial_u - \frac{(b+1)}{2}v\partial_v$$

The symmetry X_3 is the only genuine potential symmetry of the nonlinear diffusion equation as it is the only potential symmetry for which one or more $\xi, \tau$and ϕ depend on the auxiliary variable v.
In the absence of the auxiliary variable v, i.e., for the case $v = 0$ the symmetry generators called the Lie point symmetry generators. The commutation relation between the Lie point symmetry generators or vector fields is given by the following table:

	X_1	X_2	X_3
X_1	0	0	bX_1
X_2	0	0	$\frac{(b-1)}{2}X_2$
X_3	$-bX_1$	$-\frac{(b-1)}{2}X_2$	0

Table-1

The one-parameter groups G_i $(i = 1, 2, 3)$ generated by the X_i are given by using $exp(\epsilon X_i)(x, t, u)$ as follows

$$G_1 : (x, t + \epsilon, u), G_2 : (x + \epsilon, t, u), G_3 : (xe^{\epsilon(b-1)/2}, te^{\epsilon b}, u - \epsilon)$$

Since each group G_i is a symmetry group. The solution of equation corresponding to its different symmetry groups G_i (i =1,2,3) are obtained by using $\tilde{u} = g \cdot u = g \cdot f(x, t)$ as follows

$$u^{(1)} = f(x, t - \epsilon),\ u^{(2)} = f(x - \epsilon, t),\ u^{(3)} = f(xe^{-\epsilon(b-1)/2}, te^{-\epsilon b}) - \epsilon$$

Case II: $k(u) = u^a,\ q(u) = u^n$, where $a, n \neq 0$
For this case, the solutions of determined equations (20)-(23), is obtained in the form

$$\tau = C_1 + 2(n - 1)tC_3 \tag{28}$$

$$\xi = C_2 + (n - a - 1)xC_3 \tag{29}$$

$$\phi = -2uC_3 \tag{30}$$

$$\eta = -(n + a + 1)vC_3 \tag{31}$$

Thus, we have the symmetry generators

$$X_1 = \partial_t, X_2 = \partial_x$$

$$X_3 = 2(n - 1)\partial_t + (n - a - 1)x\partial_x - 2u\partial_u - (n + a + 1)v\partial_v$$

The symmetry X_3 is the only genuine potential symmetry of the nonlinear diffusion equation as it is the only potential symmetry for which one or more $\xi, \tau$and ϕ depend on the auxiliary variable v.
In the absence of the auxiliary variable v, i.e., for the case $v = 0$ the symmetry generators called the Lie point symmetry generators. The commutation relation between the Lie point symmetry generators or vector fields is given by the following table:

	X_1	X_2	X_3
X_1	0	0	$2(n-1)X_1$
X_2	0	0	$(n-a-1)X_2$
X_3	$-2(n-1)X_1$	$-(n-a-1)X_2$	0

Table-2

The one-parameter groups G_i $(i = 1, 2, 3)$ generated by the X_i are given by using $exp(\epsilon X_i)(x, t, u)$ as follows

$$G_1 : (x, t + \epsilon, u), G_2 : (x + \epsilon, t, u), G_3 : (xe^{\epsilon(n-a-1)}, te^{2\epsilon(n-1)}, ue^{-2\epsilon})$$

Since each group G_i is a symmetry group. The solution of equation corresponding to its different symmetry groups G_i (i =1,2,3) are obtained by using $\tilde{u} = g{\cdot}u = g{\cdot}f(x, t)$ as follows

$$u^{(1)} = f(x, t - \epsilon),\ u^{(2)} = f(x - \epsilon, t),\ u^{(3)} = e^{-2\epsilon} f(xe^{-\epsilon(n-a-1)}, te^{-2\epsilon(n-1)})$$

Case III: $k(u) = 1,\ q(u) = e^u$,
For this case, the solutions of determined equations (20)-(23), is obtained in the form

$$\tau = C_1 + 2tC_3 \tag{32}$$

$$\xi = C_2 + xC_3 \tag{33}$$

$$\phi = -2C_3 \tag{34}$$

$$\eta = -vC_3 \tag{35}$$

Thus, we have the symmetry generators

$$X_1 = \partial_t; X_2 = \partial_x$$

$$X_3 = 2t\partial_t + x\partial_x - 2\partial_u - v\partial_v$$

The symmetry X_3 is the only genuine potential symmetry of the nonlinear diffusion equation as it is the only potential symmetry for which one or more $\xi, \tau$and ϕ depend on the auxiliary variable v.
In the absence of the auxiliary variable v, i.e., for the case $v = 0$ the symmetry generators called the Lie point symmetry generators. The commutation relation between the Lie point symmetry generators or vector fields is given by the following table:

	X_1	X_2	X_3
X_1	0	0	$2X_1$
X_2	0	0	X_2
X_3	$-2X_1$	$-X_2$	0

Table-3

The one-parameter groups G_i $(i = 1, 2, 3)$ generated by the X_i are given by using $exp(\epsilon X_i)(x, t, u)$ as follows

$$G_1 : (x, t + \epsilon, u), G_2 : (x + \epsilon, t, u), G_3 : (xe^{\epsilon}, te^{2\epsilon}, u - 2\epsilon)$$

Since each group G_i is a symmetry group. The solution of equation corresponding to its different symmetry groups G_i (i =1,2,3) are obtained by using $\tilde{u} = g{\cdot}u = g{\cdot}f(x, t)$ as follows

$$u^{(1)} = f(x, t - \epsilon),\ u^{(2)} = f(x - \epsilon, t),\ u^{(3)} = f(xe^{-\epsilon}, te^{-2\epsilon}) - 2\epsilon$$

Case IV: $k(u) = 1,\ q(u) = u^n$, where $n \neq 0$
For this case, the solutions of determined equations (20)-(23), is obtained in the form

$$\tau = C_1 + 2(n - 1)tC_3 \tag{36}$$

$$\xi = C_2 + (n - 1)xC_3 \tag{37}$$

$$\phi = -2uC_3 \tag{38}$$

$$\eta = -(n + 1)vC_3 \tag{39}$$

Thus, we have the symmetry generators

$$X_1 = \partial_t; X_2 = \partial_x$$

$$X_3 = 2(n - 1)\partial_t + (n - 1)x\partial_x - 2u\partial_u - (n + 1)v\partial_v$$

The symmetry X_3 is the only genuine potential symmetry of the nonlinear diffusion equation as it is the only potential symmetry for which one or more $\xi, \tau$and ϕ depend on the auxiliary variable v.

In the absence of the auxiliary variable v, i.e., for the case $v = 0$ the symmetry generators called the Lie point symmetry generators. The commutation relation between the Lie point symmetry generators or vector fields is given by the following table:

	X_1	X_2	X_3
X_1	0	0	bX_1
X_2	0	0	$\frac{(b-1)}{2}X_2$
X_3	$-bX_1$	$-\frac{(b-1)}{2}X_2$	0

Table-4

The one-parameter groups G_i $(i = 1, 2, 3)$ generated by the X_i are given by using $exp(\epsilon X_i)(x, t, u)$ as follows

$$G_1 : (x, t + \epsilon, u), G_2 : (x + \epsilon, t, u), G_3 : (xe^{\epsilon(n-1)}, te^{2\epsilon(n-1)}, ue^{-2\epsilon})$$

Since each group G_i is a symmetry group. The solution of equation corresponding to its different symmetry groups G_i (i =1,2,3) are obtained by using $\tilde{u} = g{\cdot}u = g{\cdot}f(x, t)$ as follows

$$u^{(1)} = f(x, t - \epsilon),\ u^{(2)} = f(x - \epsilon, t),\ u^{(3)} = e^{-2\epsilon} f(xe^{-\epsilon(n-1)}, e^{-2\epsilon(n-1)}t)$$

4. Conclusions

The Differential form method is easy to apply. One can simply write all the differential equations as a set of first order equations and then the differential forms can be written by inspection. The proposed method has been successfully applied to analyzing the nonlinear diffusion equation with convection term. Potential and Lie point symmetries have been obtained for the nonlinear diffusion equation with convection term. Further, using Lie point symmetry groups, the solutions of the problem have been obtained. The method is also easy to apply for symbolic computation for Lie point symmetry, cf. Edelen [7]. A useful computer program liesymm, can be found in MAPLE, based on a paper by Carminati et al. [1], is use the proposed method and also easy to apply for symbolic computation. Thus, it is possible that the proposed method can be extended to solve a large class of problems in nonlinear differential equations.

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