Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 187-202

DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH OPERATORS

N.Magesh, N.B.Gatti and S.Mayilvaganan

Abstract. In the present paper, we obtain sandwich results involving Hadamard product for certain normalized analytic functions associated with Dziok-Srivastava operator in the open unit disk. Our results extend corresponding previously known results.

2000 *Mathematics Subject Classification*: Primary 30C45; Secondary 30C80.

1. Introduction

Let H be the class of analytic functions in $\mathcal{U} := \{z : |z| < 1\}$ and $H[a, n]$ be the subclass of H consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let A be the subclass of H consisting of functions of the form

$$f(z) = z + a_2 z^2 + \dots . \tag{1}$$

Let $p, h \in H$ and let $\phi(r, s, t; z) : \mathcal{C}^3 \times \mathcal{U} \to \mathcal{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{2}$$

then p is a solution of the differential superordination (2). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant. Recently Miller and Mocanu[11] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{3}$$

For $\alpha_j \in \mathcal{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathcal{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in N_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in N := \{1, 2, 3\ldots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [6] (see also [7, 21]) $\mathcal{H}_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} \mathcal{H}_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{4}$$

For brevity, we write

$$\mathcal{H}_m^l[\alpha_1]f(z) := \mathcal{H}_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z).$$

It is easy to verify from (4) that

$$z(\mathcal{H}_m^l[\alpha_1]f(z))' = \alpha_1 \mathcal{H}_m^l[\alpha_1 + 1]f(z) - (\alpha_1 - 1)\mathcal{H}_m^l[\alpha_1]f(z). \tag{5}$$

The linear operator $\mathcal{H}_m^l[\alpha_1]$ was earlier defined by Dziok-Srivastava [7], which contains such well-known operators as the Hohlov linear operator, Saitohs generalized linear operator, the Carlson- Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized version, the Bernardi- Libera -Livingston operator and the Srivastava - Owa fractional derivative operator. One may refer the papers [6] and [7] for further details and references of these operators.

Using the results of Miller and Mocanu[11], Bulboacă [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see [4]). Recently many authors [1], [8], [12] to [15] and [17] to [20] have used the results of Bulboacă [5] and obtained certain sufficient conditions applying first order differential subordinations and superordinations.

For our present investigation, we shall need the following definition and lemmas.

Definition 1.1. [11, p.817, Definition 2] *Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathcal{U}} - \mathcal{E}(f)$, where*

$$\mathcal{E}(f) = \{\zeta \in \partial\mathcal{U} : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} - \mathcal{E}(f)$.

Lemma 1.1. [10, p.132, Theorem 3.4h] *Let q be univalent in the unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set*

$$Q(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + Q(z).$$

Suppose that

1. *$Q(z)$ is starlike univalent in $\mathcal{U}$ and*
2. *$Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \mathcal{U}$.*

If p is analytic with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq \mathcal{D}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{6}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 1.2. [5, p.289, Corollary 3.2] *Let q be convex univalent in the unit disk $\mathcal{U}$ and ϑ and φ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{U})$. Suppose that*

1. *$Re\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ for $z \in \mathcal{U}$ and*
2. *$\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in $\mathcal{U}$.*

If $p(z) \in H[q(0), 1] \cap Q$*, with* $p(\mathcal{U}) \subseteq \mathcal{D}$*, and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{7}$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Subordination results

Using Lemma 1.1, we first prove the following theorem.

Theorem 2.1. *Let* $\Phi, \Psi \in \mathcal{A}$*,* $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$*,* $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$*, and* q *be convex univalent with* $q(0) = 1$*, and assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{8}$$

If $f \in \mathcal{A}$ *satisfies*

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) = \Delta(f,\, \Phi, \Psi,\, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}, \tag{9}$$

where

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) := \begin{cases} \gamma_1 + \gamma_2 \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{2\mu} \\ +\gamma_3 \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu} \\ +\gamma_4\mu \left(\frac{\alpha(\alpha_1+1)[\mathcal{H}_m^l[\alpha_1+2](f*\Phi)(z)-\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)]+\beta\alpha_1[\mathcal{H}_m^l[\alpha_1+1](f*\Psi)(z)-\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)]}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right), \end{cases} \tag{10}$$

then

$$\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu} \prec q(z)$$

and q *is the best dominant.*

Proof. Define the function p by

$$p(z) := \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu} \quad (z \in \mathcal{U}). \tag{11}$$

Then the function p is analytic in $\mathcal{U}$ and $p(0) = 1$. Therefore, by making use of (11) and (5), we obtain

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)}$$
$$= \begin{cases} \gamma_1 + \gamma_2 \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{2\mu} + \gamma_3 \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu} \\ +\gamma_4\mu \left(\frac{\alpha(\alpha_1+1)[\mathcal{H}_m^l[\alpha_1+2](f*\Phi)(z)-\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)]+\beta\alpha_1[\mathcal{H}_m^l[\alpha_1+1](f*\Psi)(z)-\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)]}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right). \end{cases} \tag{12}$$

By using (12) in (9), we have

$$\gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)} \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}. \tag{13}$$

By setting

$$\theta(w) := \gamma_1 + \gamma_2\omega^2(z) + \gamma_3\omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma_4}{w},$$

it can be easily observed that $\theta(w)$, $\phi(w)$ are analytic in $\mathcal{C} \setminus \{0\}$ and $\phi(w) \neq 0$. Also we see that

$$Q(z) := zq'(z)\phi(q(z)) = \gamma_4 \frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathcal{U}$ and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

By the hypothesis of Theorem 2.1, the result now follows by an application of Lemma 1.1.

Taking $p(z) := \left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^{\mu}$, we obtain the following theorem.

Theorem 2.2. *Let* $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ *such that* $\mu \neq 0$ *and* $\alpha + \beta \neq 0$, *and* q *be convex univalent with* $q(0) = 1$, *and assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{14}$$

If $f \in \mathcal{A}$ satisfies

$$\Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi) = \Delta_1(f,\,\Phi,\Psi,\,\gamma_1,\gamma_2,\gamma_3,\gamma_4) \prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}, \quad (15)$$

where

$$\Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi) := \begin{cases} \gamma_1 + \gamma_2 \left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^{2\mu} \\ +\gamma_3 \left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^{\mu} \\ -\gamma_4\mu \left(\frac{\alpha(\alpha_1+1)[\mathcal{H}_m^l[\alpha_1+2](f*\Phi)(z)-\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)]+\beta\alpha_1[\mathcal{H}_m^l[\alpha_1+1](f*\Psi)(z)-\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)]}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right), \end{cases} \quad (16)$$

then

$$\left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (8) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^{2\mu} + \gamma_3 \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^{\mu}$$
$$+\gamma_4\mu \left(\frac{\alpha(\alpha_1+1)[\mathcal{H}_m^l[\alpha_1+2]f(z)-\mathcal{H}_m^l[\alpha_1+1]f(z)]+\beta\alpha_1[\mathcal{H}_m^l[\alpha_1+1]f(z)-\mathcal{H}_m^l[\alpha_1]f(z)]}{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}\right)$$
$$\prec \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

By taking $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.1, we state the following corollary.

Corollary 2.2. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \dots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (8) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{\alpha z(f * \Phi)'(z) + \beta(f * \Psi)(z)}{(\alpha + \beta)z} \right)^{2\mu} + \gamma_3 \left(\frac{\alpha z(f * \Phi)'(z) + \beta(f * \Psi)(z)}{(\alpha + \beta)z} \right)^{\mu}$$
$$+\gamma_4 \mu \left(\frac{\alpha z^2(f * \Phi)''(z) + \beta[z(f * \Psi)'(z) - (f * \Psi)(z)]}{\alpha z(f * \Phi)'(z) + \beta(f * \Psi)(z)} \right)$$
$$\prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\left(\frac{\alpha z(f * \Phi)'(z) + \beta(f * \Psi)(z)}{(\alpha + \beta)z} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Corollary 2.2, we obtain the following corollary.

Corollary 2.3. *Let $\gamma_i \in \mathcal{C}$ $(i = 1, \dots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (8) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(\frac{\alpha z f'(z) + \beta f(z)}{(\alpha + \beta)z} \right)^{2\mu} + \gamma_3 \left(\frac{\alpha z f'(z) + \beta f(z)}{(\alpha + \beta)z} \right)^{\mu}$$
$$+\gamma_4 \mu \left(\frac{\alpha z^2 f''(z) + \beta[z f'(z) - f(z)]}{\alpha z f'(z) + \beta f(z)} \right)$$
$$\prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$\left(\frac{\alpha z f'(z) + \beta f(z)}{(\alpha + \beta)z} \right)^{\mu} \prec q(z)$$

and q is the best dominant.

By fixing $\alpha = 1$ and $\beta = 0$ in Corollary 2.3, we obtain the following corollary.

Corollary 2.4. *Let $\gamma_i \in \mathcal{C}$ $(i = 1, \dots, 4)(\gamma_4 \neq 0)$, $\mu \in \mathcal{C}$ such that $\mu \neq 0$ and q be convex univalent with $q(0) = 1$, and (8) holds true. If $f \in \mathcal{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(f'(z) \right)^{2\mu} + \gamma_3 \left(f'(z) \right)^{\mu} + \gamma_4 \mu \frac{z f''(z)}{f'(z)} \prec \gamma_1 + \gamma_2 p^2(z) + \gamma_3 p(z) + \gamma_4 \frac{zp'(z)}{p(z)},$$

then

$$(f'(z))^{\mu} \prec q(z)$$

and q is the best dominant.

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.1, we have the following corollary.

Corollary 2.5. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$. Assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right) + \frac{2\gamma_2}{\gamma_4}\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{1-ABz^2}{(1+Az)(1+Bz)}\right\} > 0.$$

If $f \in \mathcal{A}$ and

$$\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi) \prec \gamma_1 + \gamma_2\left(\frac{1+Az}{1+Bz}\right)^2 + \gamma_3\frac{1+Az}{1+Bz} + \gamma_4\frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z) + \beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Remark 1.1. Similarly results could be obtained for Corollaries 2.1 to 2.5 for Theorem 2.2, so we omitted the details.

3. Superordination results

Now, by applying Lemma 1.2, we prove the following theorem.

Theorem 3.1. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z)\right\} \geq 0. \tag{17}$$

If $f \in \mathcal{A}$, $\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu} \in H[q(0), 1] \cap Q$. Let $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ be univalent in $\mathcal{U}$ and

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4\frac{zq'(z)}{q(z)} \prec \Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi), \tag{18}$$

where $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *is given by (10), then*

$$q(z) \prec \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}$$

and q *is the best subordinant.*

Proof.

Define the function p by

$$p(z) := \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}. \tag{19}$$

Simple computation from (19), we get,

$$\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi) = \gamma_1+\gamma_2 p^2(z)+\gamma_3 p(z)+\gamma_4\frac{zp'(z)}{p(z)},$$

then

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4\frac{zq'(z)}{q(z)} \prec \gamma_1+\gamma_2 p^2(z)+\gamma_3 p(z)+\gamma_4\frac{zp'(z)}{p(z)}.$$

By setting $\vartheta(w)=\gamma_1+\gamma_2 w^2+\gamma_3 w$ and $\phi(w)=\frac{\gamma_4}{w}$, it is easily observed that $\vartheta(w)$ is analytic in $\mathcal{C}$. Also, $\phi(w)$ is analytic in $\mathcal{C}\setminus\{0\}$ and $\phi(w)\neq 0$. Now Theorem 3.1 follows by applying Lemma 1.2.

Theorem 3.2. *Let* $\Phi,\Psi\in\mathcal{A}$, $\gamma_i\in\mathcal{C}$ $(i=1,\ldots,4)(\gamma_4\neq 0)$, $\mu,\alpha,\beta\in\mathcal{C}$ *such that* $\mu\neq 0$ *and* $\alpha+\beta\neq 0$, *and* q *be convex univalent with* $q(0)=1$, *and assume that*

$$Re\left\{\frac{\gamma_3}{\gamma_4}q(z)+\frac{2\gamma_2}{\gamma_4}q^2(z)\right\}\geq 0. \tag{20}$$

If $f\in\mathcal{A}$, $\left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^{\mu}\in H[q(0),1]\cap Q$. *Let* $\Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *be univalent in* $\mathcal{U}$ *and*

$$\gamma_1+\gamma_2 q^2(z)+\gamma_3 q(z)+\gamma_4\frac{zq'(z)}{q(z)} \prec \Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi), \tag{21}$$

where $\Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *is given by (16), then*

$$q(z) \prec \left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^{\mu}$$

and q is the best subordinant.

For the choice of $p(z) = \left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^\mu$, the proof of Theorem 3.2 is line similar to the proof of Theorem 3.1, so we omitted the proof of Theorem 3.2.

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (17) holds true. If $f \in \mathcal{A}$, $\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^\mu \in H[q(0), 1]\cap Q$. Let $\gamma_1+\gamma_2\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^{2\mu}$ $+\gamma_3\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^\mu$ $+\gamma_4\mu\left(\frac{\alpha(\alpha_1+1)[\mathcal{H}_m^l[\alpha_1+2]f(z)-\mathcal{H}_m^l[\alpha_1+1]f(z)]+\beta\alpha_1[\mathcal{H}_m^l[\alpha_1+1]f(z)-\mathcal{H}_m^l[\alpha_1]f(z)]}{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}\right)$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec$$

$$\gamma_1 + \gamma_2\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^{2\mu} + \gamma_3\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^\mu$$

$$+\gamma_4\mu\left(\frac{\alpha(\alpha_1+1)[\mathcal{H}_m^l[\alpha_1+2]f(z)-\mathcal{H}_m^l[\alpha_1+1]f(z)]+\beta\alpha_1[\mathcal{H}_m^l[\alpha_1+1]f(z)-\mathcal{H}_m^l[\alpha_1]f(z)]}{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}\right),$$

then

$$q(z) \prec \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1]f(z)+\beta\mathcal{H}_m^l[\alpha_1]f(z)}{(\alpha+\beta)z}\right)^\mu$$

and q is the best subordinant.

When $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 3.1, we derive the following corollary.

Corollary 3.2. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha+\beta \neq 0$, and q be convex univalent with $q(0) = 1$, and (17) holds true. If $f \in \mathcal{A}$, $\left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^\mu \in H[q(0), 1]\cap Q$. Let $\gamma_1+\gamma_2\left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{2\mu}+$ $\gamma_3\left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^\mu+\gamma_4\mu\left(\frac{\alpha z^2(f*\Phi)''(z)+\beta[z(f*\Psi)'(z)-(f*\Psi)(z)]}{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}\right)$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec$$

$$\gamma_1 + \gamma_2 \left(\frac{\alpha z(f*\Phi)'(z) + \beta(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{2\mu} + \gamma_3 \left(\frac{\alpha z(f*\Phi)'(z) + \beta(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\mu}$$
$$+\gamma_4 \mu \left(\frac{\alpha z^2(f*\Phi)''(z) + \beta[z(f*\Psi)'(z) - (f*\Psi)(z)]}{\alpha z(f*\Phi)'(z) + \beta(f*\Psi)(z)} \right)$$

then

$$q(z) \prec \left(\frac{\alpha z(f*\Phi)'(z) + \beta(f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\mu}$$

and q is the best subordinant.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$, $\alpha = 1$ and $\beta = 0$ in Corollary 3.2, we obtain the following corollary.

Corollary 3.3. *Let $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $0 \neq \mu \in \mathcal{C}$ and q be convex univalent with $q(0) = 1$, and (17) holds true. If $f \in \mathcal{A}$, $(f'(z))^{\mu} \in H[q(0), 1] \cap Q$. Let $\gamma_1 + \gamma_2 (f'(z))^{2\mu} + \gamma_3 (f'(z))^{\mu} + \gamma_4 \mu \frac{zf''(z)}{f'(z)}$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 (f'(z))^{2\mu} + \gamma_3 (f'(z))^{\mu} + \gamma_4 \mu \frac{zf''(z)}{f'(z)},$$

then

$$q(z) \prec (f'(z))^{\mu}$$

and q is the best subordinant.

By taking $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Let $\Phi, \Psi \in \mathcal{A}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and q be convex univalent with $q(0) = 1$, and $Re\left\{ \frac{\gamma_3}{\gamma_4} \left(\frac{1+Az}{1+Bz} \right) + \frac{2\gamma_2}{\gamma_4} \left(\frac{1+Az}{1+Bz} \right)^2 \right\} > 0$. If $f \in \mathcal{A}$, $\left(\frac{\alpha \mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z) + \beta \mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\mu} \in H[q(0), 1] \cap Q$. Let $\Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi)$ be univalent in $\mathcal{U}$ and*

$$\gamma_1 + \gamma_2 (\frac{1+Az}{1+Bz})^2 + \gamma_3 \frac{1+Az}{1+Bz} + \gamma_4 \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \Delta^{(\gamma_i)_1^4}(f; \Phi, \Psi),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{\alpha \mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z) + \beta \mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z} \right)^{\mu}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Remark 2.1. Special cases of Theorem 3.2 are line similar to the Corollaries 3.1 to 3.4, so we omitted the details.

4. Sandwich results

There is a complete analog of Theorem 2.1, 2.2 for differential subordinations and Theorem 3.1, 3.2 for differential superordinations. We can combine the results of Theorem 2.1 with Theorem 3.1 and Theorem 2.2 with Theorem 3.2, we obtain the following sandwich theorems.

Theorem 4.1. *Let q_1 and q_2 be convex univalent in $\mathcal{U}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and let q_2 satisfy (8) and q_1 satisfy (17). For $f, \Phi, \Psi \in \mathcal{A}$, let $\left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^\mu \in H[1,1] \cap Q$ and $\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ defined by (10) be univalent in $\mathcal{U}$ satisfying*

$$\gamma_1 + \gamma_2 q_1^2(z) + \gamma_3 q_1(z) + \gamma_4 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi) \prec \gamma_1 + \gamma_2 q_2^2(z) + \gamma_3 q_2(z) + \gamma_4 \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}{(\alpha+\beta)z}\right)^\mu \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and best dominant.

Theorem 4.2. *Let q_1 and q_2 be convex univalent in $\mathcal{U}$, $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\mu, \alpha, \beta \in \mathcal{C}$ such that $\mu \neq 0$ and $\alpha + \beta \neq 0$, and let q_2 satisfy (14) and q_1 satisfy (20). For $f, \Phi, \Psi \in \mathcal{A}$, let $\left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^\mu \in H[1,1] \cap Q$ and $\Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ defined by (16) be univalent in $\mathcal{U}$ satisfying*

$$\gamma_1 + \gamma_2 q_1^2(z) + \gamma_3 q_1(z) + \gamma_4 \frac{zq_1'(z)}{q_1(z)} \prec \Delta_1^{(\gamma_i)_1^4}(f;\Phi,\Psi) \prec \gamma_1 + \gamma_2 q_2^2(z) + \gamma_3 q_2(z) + \gamma_4 \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{(\alpha+\beta)z}{\alpha\mathcal{H}_m^l[\alpha_1+1](f*\Phi)(z)+\beta\mathcal{H}_m^l[\alpha_1](f*\Psi)(z)}\right)^\mu \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and best dominant.

By taking $q_1(z) = \frac{1+A_1z}{1+B_1z}$ $(-1 \leq B_1 < A_1 \leq 1)$ and $q_2(z) = \frac{1+A_2z}{1+B_2z}$ $(-1 \leq B_2 < A_2 \leq 1)$ in Theorem 4.1, we obtain the following result.

Corollary 4.1. *For $f, \Phi, \Psi \in \mathcal{A}$, let $\left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^\mu \in H[1,1] \cap Q$ and*

$\Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$ *defined by (10) be univalent in* $\mathcal{U}$ *satisfying*

$$\gamma_1+\gamma_2(\frac{1+A_1z}{1+B_1z})^2+\gamma_3\frac{1+A_1z}{1+B_1z}+\gamma_4\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}$$
$$\prec \Delta^{(\gamma_i)_1^4}(f;\Phi,\Psi)$$
$$\prec \gamma_1+\gamma_2(\frac{1+A_2z}{1+B_2z})^2+\gamma_3\frac{1+A_2z}{1+B_2z}+\gamma_4\frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$$

then

$$\frac{1+A_1z}{1+B_1z}\prec\left(\frac{\alpha z(f*\Phi)'(z)+\beta(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

We remark that, one can easily restated Theorem 4.1, for the different choices of $\Phi(z)$, $\Psi(z)$, l, m, $\alpha_1,\alpha_2,\ldots\alpha_l$, $\beta_1,\beta_2,\ldots\beta_m$ and for $\gamma_1,\gamma_2,\gamma_3,\gamma_4$.

Remark 4.1.

1. Putting $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\alpha=1$, $\beta=0$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, $q(z)=\frac{1}{(1-z)^{2ab}}$ $(b\in\mathcal{C}\setminus\{0\})$, $\mu=a$ and $\gamma_4=\frac{1}{b}$ in Corollary 2.2, we get the result obtained by Obradovič et al., [16, Theorem 1].

2. Putting $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\alpha=0$, $\beta=1$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, $q(z)=\frac{1}{(1-z)^{2b}}$ $(b\in\mathcal{C}\setminus\{0\})$, $\mu=1$ and $\gamma_4=\frac{1}{b}$ in Corollary 2.2 and then combining this together with Lemma 1.1, we obtain the result of Srivastava and Lashin [22, Theorem 3].

3. Taking $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\alpha=0$, $\beta=1$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, $\gamma_4=\frac{e^{i\lambda}}{abcos\lambda}$ $(a,b\in\mathcal{C},|\lambda|<\frac{\pi}{2})$, $\mu=a$ and $q(z)=(1-z)^{-2abcos\lambda e^{-i\lambda}}$ in Corollary 2.2, we obtain the result of Aouf et al. [3, Theorem 1].

4. Taking $\alpha_1=1$, $\alpha_2=1$, $\beta_1=1$, $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\beta=1-\alpha$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, in Theorems 2.1, 3.1, 4.1, we obtain the results obtained by Shanmugam et al., [20, Theorem 3.4, Theorem 4.3, Theorem 5.2].

5. Putting $\alpha_1=a$, $\alpha_2=1$, $\beta_1=c$, $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\beta=1-\alpha$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, in Theorems 2.1, 3.1, 4.1, we obtain the results obtained by Shanmugam et al., [19, Theorem 3.6, Theorem 4.3, Theorem 5.2].

6. For $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=0$, $\beta=1-\alpha$, $\Phi(z)=\Psi(z)=\frac{z}{1-z}$, in Theorem 2.1, we have the results obtained by Mostafa and Aouf [12, Theorem 3].

7. By taking $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\alpha = 0$, $\beta = 1$, in Corollary 2.1, we have the result obtained by the first author [13, Theorem 3.5].

8. For $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\alpha = 0$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} [1 + (n-1)\lambda]^m \frac{(b)_{n-1}}{(c)_{n-1}} a_n z^n$, all the results in [17] are special cases of our results.

9. For $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\alpha = 0$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} [1 + (n-1)\lambda]^m b_n z^n$, all the results in [2] are special cases of our results.

10. When $l = 2$, $m = 1$, $\alpha_1 = \alpha_2 = \beta_1 = 1$ and $\alpha = \beta = 1$ in Theorems 2.1, 3.1, 4.1, we obtain the results obtained by Magesh et al., [9].

Acknowledgement: The authors would like to thank the referees for their insightful suggestions.

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Nanjundan Magesh
Department of Mathematics, Government Arts College (Men)
Krishnagiri - 635001, Tamilnadu, India.
E-mail:*nmagi_2000@yahoo.co.in*.

Niranjan Basavanthappa Gatti
P.G. Studies in Mathematics, Govt Science College,
Chitradurga - 577501, Karnataka, India.
E-mail:*nbg_71@rediffmail.com*.

Subramanian Mayilvaganan
Department of Mathematics, Adhiyamaan College of Engineering (Autonomous)
Hosur - 635109, Tamilnadu, India.
E-mail:*vaganan_mayil@yahoo.com*.